R-trivial Markov Processes

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(joint work with A. Schilling, B. Steinberg and N. Thiéry)

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Outline

1. Tsetlin Library
2. R-trivial Processes
3. Toom-Tsetlin model
4. Directed nonabelian sandpile model

A model of a library

- n books on a shelf

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}$$

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- The probability of choosing book B_i is x_i.
- Once the book is chosen, it is moved to the back.

$$\boxed{B_1}\boxed{B_2}\cdots\boxed{B_i}\cdots\boxed{B_n} \to \boxed{B_1}\boxed{B_2}\cdots\boxed{B_n}\boxed{B_i} \text{ with probability } x_i.$$

A Markov chain on permutations

- Let $\sigma \in S_n$ be a permutation.
- Steady state π: (Tsetlin '63, Hendricks '72)

$$\pi(\sigma) = \prod_{i=1}^{n} \frac{x_{\sigma_i}}{x_{\sigma_1} + \cdots + x_{\sigma_i}}.$$

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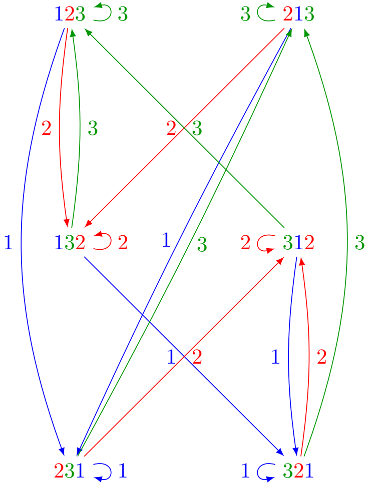
- A *derangement* is a permutation with no fixed points.
- d_m be the number of derangements in S_m.
- Let T_n be the Markov matrix. Then (Phatarfod '91)

$$\det(\lambda I - T_n) = \prod_{S \subset [n]} (\lambda + x_S)^{d_{|S|}}$$

where $x_S = \sum_{i \in S} x_i$.

Example: $n = 3$



Example: $n = 3$

- $$M_3 = \begin{pmatrix} * & x_3 & 0 & 0 & x_3 & 0 \\ x_2 & * & x_2 & 0 & 0 & 0 \\ 0 & 0 & * & x_3 & 0 & x_3 \\ x_1 & 0 & x_1 & * & 0 & 0 \\ 0 & 0 & 0 & x_2 & * & x_2 \\ 0 & x_1 & 0 & 0 & x_1 & * \end{pmatrix} \begin{bmatrix} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{bmatrix}$$
- $$\pi(231) = \frac{x_3 x_1}{(x_2 + x_3)(x_1 + x_2 + x_3)}$$
- Eigenvalues: 0, $-x_1 - x_2$, $-x_1 - x_3$, $-x_2 - x_3$ and $-x_1 - x_2 - x_3$ twice.

Out of equilibrium behaviour

- The **total variation distance** between two probability distributions P, Q on a finite set Ω is

$$||P - Q||_{\text{TV}} = \max_{A \subset \Omega} |P(A) - Q(A)|.$$

- Let π_0 be the starting distribution. The **mixing time** is the smallest t such that

$$||M_t \pi_0 - \pi||_{\text{TV}} \leq \frac{e^{-2}}{2}.$$

- For the case of equal weights $x_i = 1/n$, the mixing time is $n \log n + 2n$ (Diaconis, '93).
- Exact formulas in the general case of going from permutation σ to τ in k steps (Fill, '96).

Generalizations

- Umpteen generalizations!
- Different moves, more shelves.
- Infinite libraries.
- Hyperplane arrangements (Bidigare, Hanlon, Rockmore '99)
- Left regular bands (Brown '00)
- Linear extensions of posets (A., Klee, Schilling '14)

Underlying Philosophy

- Equilibrium/reversible Markov processes satisfy **detailed balance**.
- If the process has a generator g, then (morally) so is g^{-1}.

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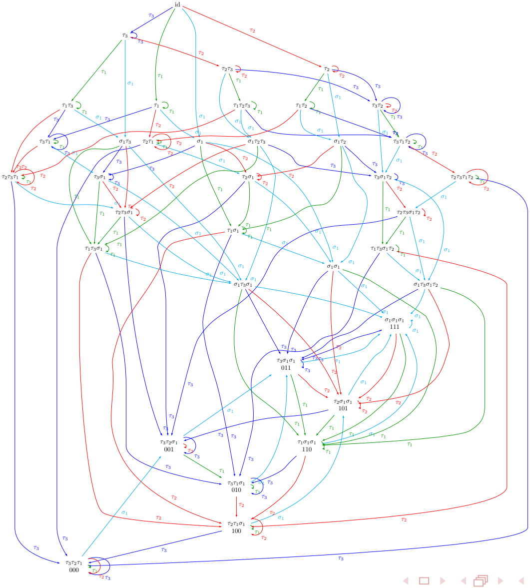
- Equilibrium/reversible Markov processes satisfy **detailed balance**.
- If the process has a generator g, then (morally) so is g^{-1}.
- The set of generators of the process form a **group**.
- The set of generators of a nonequilibrium/reversible Markov process will in general form a **monoid**.
- A **monoid** $\mathcal{M}$ is a set with an associative product and an identity.

Generators

- Each generator corresponds to a fixed action.
- For example, in an 1D exclusion process, g_i could correspond to hopping of a particle from site i to $i+1$.
- By construction, generators will be **column-monomial matrices**, i.e., matrices of 0's and 1's with a single 1 per column.

Bad example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τ_3
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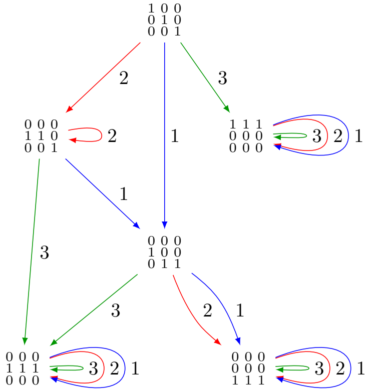
$\tau_2\sigma_1\sigma_1$ 101
$\tau_3\tau_2\sigma_1$ 001
$\tau_1\sigma_1\sigma_1$ 110
$\tau_3\tau_1\sigma_1$ 010
$\tau_2\tau_1\sigma_1$ 100
$\tau_3\tau_2\tau_1$ 000

Nice example



Order Relations

- A **partial order** is a binary relation on a set which is *reflexive*, *antisymmetric* and *transitive*.
- A **preorder** is a binary relation on a set which is *reflexive* and *transitive*.
- Natural preorders on $\mathcal{M}$:

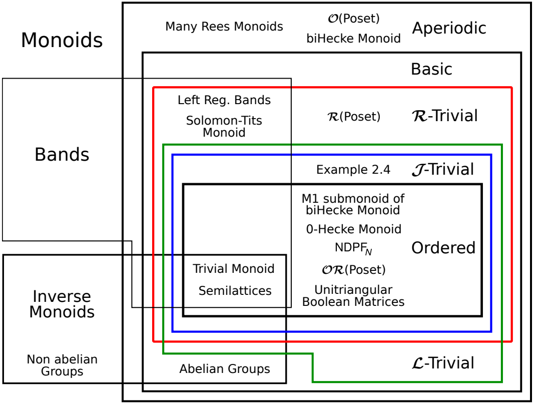
$$x \leq_R y \text{ if } y = xu \text{ for some } u \in \mathcal{M}$$
$$x \leq_L y \text{ if } y = ux \text{ for some } u \in \mathcal{M}$$

- Equivalence classes on $\mathcal{M}$:

$$xRy \text{ if } y\mathcal{M} = x\mathcal{M}$$
$$xLy \text{ if } \mathcal{M}y = \mathcal{M}x$$

- $\mathcal{M}$ is R-trivial (L-trivial) if all R-classes (L-classes) are singletons. Equivalently, if the preorders are partial orders.

Monoid Classes



Usefulness of R-trivial Monoids

- Solvable models with quenched disorder.
- Explicit formula for the steady state.
- Formulas for absorption times and mixing times.
- Guarantee that all eigenvalues are real.
- Eigenvalues are linear in the generator rates.
- Irreversible Markov processes

An anchored interface model

- (Derrida, Lebowitz, Speer, Spohn '91): model of $\pm$ spins on $\mathbb{Z}_+$
- Each $\pm$ exchanges with the first $\mp$ on its right with rate $\lambda_\pm$.
- For example,

$$+ - \underline{- - +} + - + + + + \quad \longrightarrow \quad + - \underline{+ - -} + - + + + +$$

A simplification

- (Lebowitz, Neuhauser, Ravishankar '96): simpler variant
- Rather than allowing all spins to exchange, only allow the leftmost spin in a block to exchange with the first opposite spin to its right.

$$+\underline{-\;-\;-\;+}\;+\;-\;+\;+\;+\;+ \quad \longrightarrow \quad +\underline{+\;-\;-\;-}\;+\;-\;+\;+\;+\;+$$

- Further, the first spin flips independently with rate α.

Finite exclusion process

- Replace $\pm$ by $1,0$ resp.
- Exclusion process on the closed lattice with n_0 0's and n_1 1's.

$$\underbrace{0\ldots0}_{k}1 \to 1\underbrace{0\ldots0}_{k}, \text{ with rate } \alpha,$$

$$\underbrace{1\ldots1}_{k}0 \to 0\underbrace{1\ldots1}_{k}, \text{ with rate } \beta.$$

Example: $n_0 = n_1 = 2$

- Configurations: $\{0011, 0101, 0110, 1001, 1010, 1100\}$.
- Markov matrix:

$$\begin{pmatrix} -\alpha & \beta & \beta & 0 & 0 & 0 \\ 0 & -\beta-2\alpha & 0 & \beta & 0 & 0 \\ 0 & \alpha & -\beta-\alpha & 0 & \beta & \beta \\ \alpha & \alpha & 0 & -\beta-\alpha & \beta & 0 \\ 0 & 0 & \alpha & 0 & -2\beta-\alpha & 0 \\ 0 & 0 & 0 & \alpha & \alpha & -\beta \end{pmatrix}$$

- Stationary distribution: Forward

$$\left(\frac{\beta^2}{(\beta+\alpha)^2}, \frac{\beta^3\alpha}{(\beta+\alpha)^4}, \frac{\alpha^2\beta\,(2\,\beta+\alpha)}{(\beta+\alpha)^4}, \frac{\beta^2\alpha\,(\beta+2\,\alpha)}{(\beta+\alpha)^4}, \frac{\alpha^3\beta}{(\beta+\alpha)^4}, \frac{\alpha^2}{(\beta+\alpha)^2} \right)$$

Steady state properties

- Not a product measure
- If $k \leq n_1$, then

$$\langle \eta_1 \dots \eta_k \rangle = \frac{\alpha^k}{(\alpha+\beta)^k}.$$

- If $k \leq \min(n_0, n_1)$, then the density

$$\rho_k = \langle \eta_k \rangle = \frac{\alpha}{\alpha+\beta}.$$

- Many other nice properties (A., 2015)

Tsetlin library with multiple copies of books

- We have m books – $\{b_1, \ldots, b_m\}$.
- A fixed number n_i of books b_i.
- Total number of books is $\sum_{i=1}^{m} n_i = L$.
- Configurations are words $\sigma = (\sigma_1, \ldots, \sigma_L)$.
- We will describe a multiparameter process

Dynamics

- Suppose the current state is σ.
- Choose a book b and an index j ($1 \leq j \leq \#b$'s) with rate $x_{b,j}$
- If $j = 1$, move the first copy of b to the front.
- Otherwise, move the j^{th} copy of b next to the $(j-1)^{st}$ copy of b.
- Example: $m = 4, n = (1,4,2,2)$. All moves of book b in

$$cbacbbddb \longrightarrow \begin{cases} \underline{bc}acbbddb & \text{with rate } x_{b,1} \\ cb\underline{bac}bddb & \text{with rate } x_{b,2} \\ cbacb\underline{b}ddb & \text{with rate } x_{b,3} \\ cbacbb\underline{bdd} & \text{with rate } x_{b,4} \end{cases}$$

Complete Example: $m = 2$ with 2 1's and 2 2's

- Configurations: $\{1122, 1212, 1221, 2112, 2121, 2211\}$.
- Markov matrix:

$$\begin{pmatrix} * & x_{1,2} & x_{1,2} & 0 & 0 & 0 \\ 0 & * & 0 & x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & * & 0 & x_{1,1} & x_{1,1} \\ x_{2,1} & x_{2,1} & 0 & * & x_{1,2} & 0 \\ 0 & 0 & x_{2,1} & 0 & * & 0 \\ 0 & 0 & 0 & x_{2,2} & x_{2,2} & * \end{pmatrix}$$

- Compare with the Toom model example

Complete Example: $m = 2$ with 2 1's and 2 2's

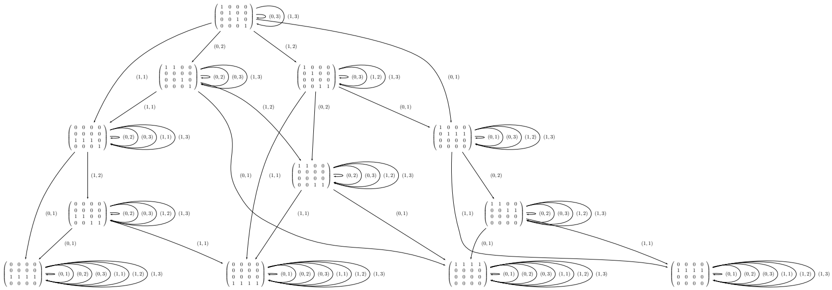
- The stationary distribution is complicated.
- But the eigenvalues are exceptionally simple

$$(0,\ -x_{1,1} - x_{2,1},\ -x_{2,1} - x_{1,2},\ -x_{2,2} - x_{1,1},\ -x_{1,2} - x_{2,2},\ -x_{2,1} - x_{2,2} - x_{1,2} - x_{1,1})$$

- Of course, this is also the case for the Toom model

$$(0,\ (-\alpha - \beta)^4,\ -2\alpha - 2\beta)$$

Cayley graph



Derangements of Words

- σ has content $\vec{n} = (n_1, \ldots, n_m)$.
- σ is a **derangement** if no letter in σ is in the same position as

$$(1, \ldots, 1, 2, \ldots, 2, \ldots, m, \ldots, m).$$

- $(3, 2, 1, 1)$ is, but $(2, 1, 3, 1)$ is not.
- Let $d_{\vec{n}}$ denote the number of derangements of words of content $\vec{n}$. (Even and Gillis '76)

$$d_{\vec{n}} = (-1)^L \int_0^\infty e^{-x} \prod_{j=1}^{m} L_{n_j}(x) \mathrm{d}x,$$

where $L_n(x)$ are Laguerre polynomials.

Main Result

For $I_j \subseteq [n_j] = \{1, 2, \ldots, n_j\}$, let $x_{b_j, I_j} = \sum_{s \in I_j} x_{b_j, s}$.

Theorem (ASST 2014)

The characteristic polynomial of the Markov matrix $T_{\vec{n}}$ is

$$|\lambda I - T_{\vec{n}}| = \prod_{I_1 \subseteq [n_1], \ldots, I_m \subseteq [n_m]} \left(\lambda + \sum_{j=1}^{m} x_{b_j, I_j} \right)^{d_{(|I_1|, \ldots, |I_m|)}} .$$

Special Cases

1. Toom model: $m = 2$

$$|\lambda I \; - \; T_{(n_1,n_2)}| = \prod_{\substack{I_1 \subseteq [n_1], I_2 \subseteq [n_2] \\ |I_1| = |I_2|}} \left(\lambda + x_{1,I_1} + x_{2,I_2}\right).$$

2. Toom model: $m = 2$, with $x_{1,j} = \beta$ and $x_{2,j} = \alpha$

$$|\lambda I - T_{(n_1,n_2)}| = \prod_{k=0}^{\min(n_1,n_2)} (\lambda + k(\alpha + \beta))^{\binom{n_1}{k}\binom{n_2}{k}}$$

3. Tsetlin library: $\vec{n} = (1, \ldots, 1)$ with $x_{i,1} = x_i$

$$\det(T_n - \lambda I) = \prod_{S \subset [n]} (\lambda + x_S)^{d_{|S|}}$$

Steady state properties

- For $k \leq n_p$, the joint correlation of p'th species is

$$\langle \eta_1^{(p)} \dots \eta_k^{(p)} \rangle = \prod_{i=1}^{k} \frac{x_{p,i}}{x_{1,1} + \cdots x_{p-1,1} + x_{p,i} + x_{p+1,1} + \cdots + x_{m,1}}.$$

- For $x_{p,i} = y_p \forall i \in [n_p]$ and $k \leq \min(n_1, \dots, n_m)$,

$$\langle \eta_k^{(p)} \rangle = \frac{y_p}{y_1 + \cdots y_m}$$

Abelian Sandpile Model

- Prototypical model for the phenomenon of self-organized criticality, like a heap of sand.
- Data: A graph, $G = (V, E)$. A subset S of V, of sinks.
- Allowed configurations: Maps $\phi : V \setminus S \to \mathbb{Z}_{\geq 0}$, such that $\phi(v) < deg(v)$, interpreted as the number of grains of sand sitting at vertex v.
- Move: Pick a random v, and add one grain to it. If $\phi(v) + 1 \geq deg(v)$, topple, giving one grain each to its neighbors, and continue. A grain given to a sink is considered lost.

Arborescences or upward rooted trees

- Arborescence $\mathcal{T}$: exactly one directed path from any vertex to the root r
- Set of leaves L: vertices with in-degree zero.

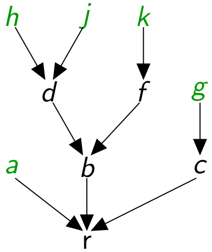


Figure: An arborescence with leaves at a, g, h, j, k.

Configurations

- Threshold T_v: maximal number of grains at vertex $v \in V$.
- Configuration space:

$$\Omega(\mathcal{T}) = \{(t_v)_{v \in V} \mid 0 \leq t_v \leq T_v\}.$$

- Variable t_v: the number of grains of sand at $v \in V$.

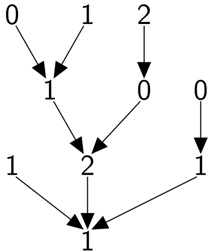


Figure: A configuration with all thresholds 2.

Sandpile dynamics

- We define a Markov chain on these arborescences.
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- We define a Markov chain on these arborescences.
- Sand grains enter at the leaves, ...
- ..., topple along the vertices, ...
- ..., and exit at the root.
- Unlike in the (usual) abelian sandpile model, sand grains only enter at leaves.

This a discrete-time process, unlike the other examples.

Source Operator

Path to root: vertex $v \in V$

$$v^{\downarrow} = (v = v_0 \to v_1 \to \cdots \to v_a = \mathrm{r}).$$

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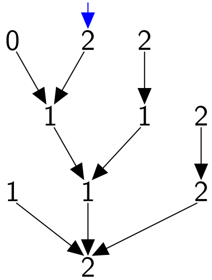
$$\sigma_\ell \colon \Omega(\mathcal{T}) \to \Omega(\mathcal{T})$$

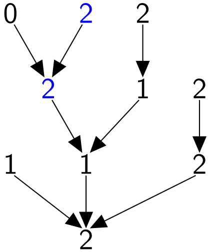
Follow the path $\ell^{\downarrow}$ from ℓ to the root r

- Add a grain to the first vertex along the way that has not yet reached its threshold, if such a vertex exists.
- If no such vertex exists, then the grain is interpreted to have left the tree at the root and $\sigma_\ell(t) = t$.

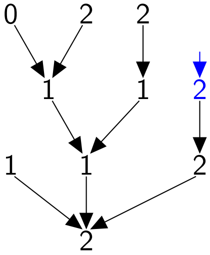
σ_j :



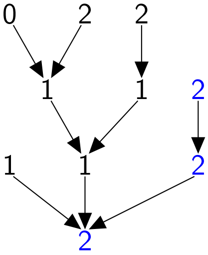
$\mapsto$



σ_g :



$\mapsto$



Topple operators

Definition (Landslide sandpile model)

$$\tau_v : \Omega(\mathcal{T}) \to \Omega(\mathcal{T})$$

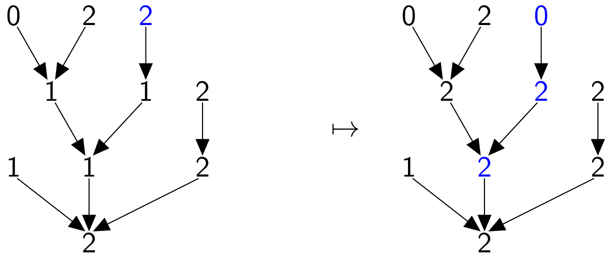
τ_v moves all grains from $v \in V$ to the first available sites along $v^{\downarrow}$. Grains remaining after the root exit the system.

Remark

If $t_v = 0$ (no grain at site v), then $\theta_v(t) = \tau_v(t) = t$.

Toppling in the Landslide sandpile model

τ_k :



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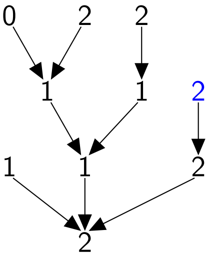
τ_g :



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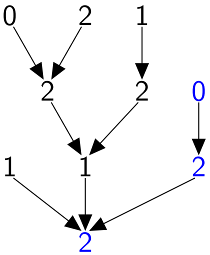

Generators

- Probabilities: $\{x_v, y_\ell \mid v \in V, \ell \in L\}$
  x_v: probability of choosing the topple operator θ_v (resp. τ_v)
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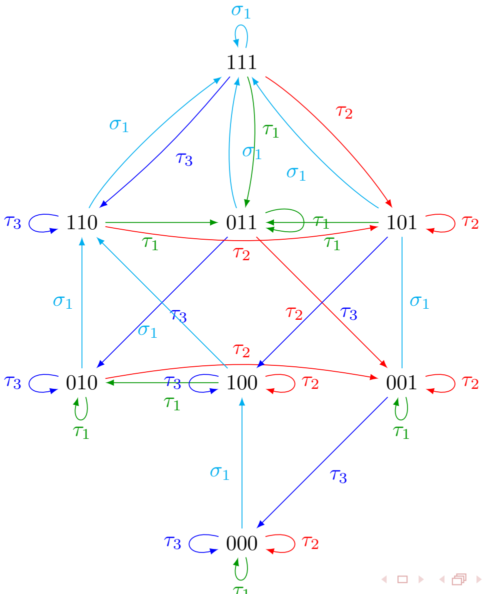
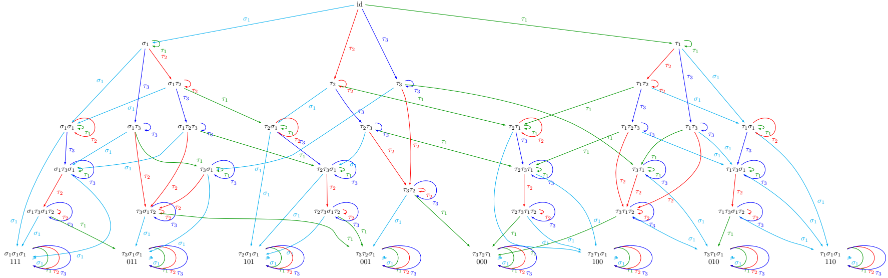
  x_v: probability of choosing the topple operator θ_v (resp. τ_v)
  y_ℓ: probability of choosing the source operator σ_ℓ
  We assume that

  1. $$x_v, y_\ell > 0, \quad \sum_{v \in V} x_v + \sum_{\ell \in L} y_\ell = 1$$

Markov chains on a line with thresholds 1



Cayley Graph



Landslide sandpile model: Stationary distribution

$$\mu_v(h) := \begin{cases} \dfrac{Y_v^h x_v}{(Y_v + x_v)^{h+1}} & \text{if } h < T_v \\ \\ \dfrac{Y_v^{T_v}}{(Y_v + x_v)^{T_v}} & \text{if } h = T_v \end{cases}$$

Theorem (ASST 2013)

Let $T_v = 1$ for all $v \in V$, $v \neq \mathrm{r}$ and $T_\mathrm{r} = m$ for some positive integer m. Then the stationary distribution of the Landslide sandpile model defined on G_τ is given by the product measure

$$\mathbb{P}(t) = \prod_{v \in V} \mu_v(t_v).$$

Landslide sandpile model: Spectrum

For subsets $S \subseteq V$ and $\ell^{\downarrow}$ the set of vertices on path from ℓ to r:

$$y_S = \sum_{\ell \in L, \ell^{\downarrow} \subseteq S} y_\ell \quad \text{and} \quad x_S = \sum_{v \in S} x_v.$$

Transition matrix for Landslide sandpile model M_τ

Theorem (ASST 2013)

The characteristic polynomial of M_τ is given by

$$\det(M_\tau - \lambda I) = \prod_{S \subseteq V} (\lambda - (y_S + x_S))^{T_{S^c}},$$

where $S^c = V \setminus S$ and $T_S = \prod_{v \in S} T_v$.

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Eigenvalues: $y_S + x_S$

Multiplicities: T_{S^c}

Landslide sandpile model: Mixing time

Define $p := \min\{y_\ell \mid \ell \in L\}$ and $n_T := \sum_{v \in V} T_v$.

Theorem (ASST 2013)

The rate of convergence is bounded by

$$||M_\mathcal{T}^k \pi_0 - \pi||_{TV} \leq \exp\left(-\frac{(kp-(n_T-1))^2}{2kp}\right)$$

as long as $k \geq (n_T - 1)/p$.

Mixing time: Mixing time is at most $\frac{2n_T}{p}$.

Other Models

- **1D Asymmetric annihilation process** (A., K. Mallick '10)

$$\begin{aligned} 10 \longrightarrow 01 &\quad \text{with rate } \alpha_i \\ 11 \longrightarrow 00 &\quad \text{with rate } \alpha_i \end{aligned}$$

- **1D Asymmetric Glauber model** (A. '10)

$$\begin{aligned} +- \longrightarrow ++, -+ \longrightarrow -- &\quad \text{with rate } \alpha_i \\ ++ \longrightarrow +-, -- \longrightarrow -+ &\quad \text{with rate } \beta_i \end{aligned}$$

- **de Bruijn process** (A., V. Strehl '11)

$$w_1 \cdots w_L \longrightarrow w_2 \cdots w_L a \quad \text{with rate } \alpha_{a,i}$$

Proof ideas

- Construct the $\leq_R$ preorder on $\mathcal{M}$ and show that it is a partial order
- Use an explicit eigenvalue formula for R-trivial monoids in general.
- Use the structure theory of R-trivial monoids to get eigenvalues and their multiplicities.

Thank you!