Dynamical Many-body Freezing and Statistical Mechanics of Periodically Driven "Closed" Quantum System

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Plan of the Talk

Periodically Driven Integrable (free-fermionic) Systems

(A) Clean System

* Dynamical Many-body Freezing (DMF)
* Asymptotic Dynamics of the States
* The Ensemble Picture

(B) Quenched Disorder: Fate of DMF and Emergent "Periodic pre-thermalization" at the Freezing Points.

Periodically Driven Non-Integrable Systems

* The Generic Scenario
* Fate of DMF under weak and strong Non-integrable perturbation:

Strong Prethermalization only at "Freezing" points

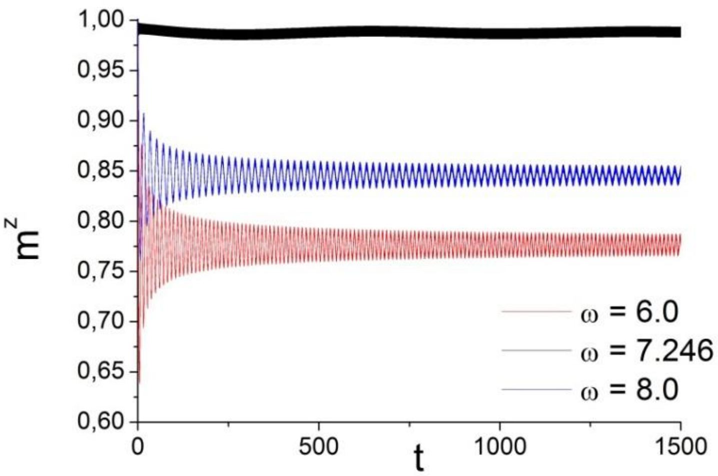
Conclusions and Outlook

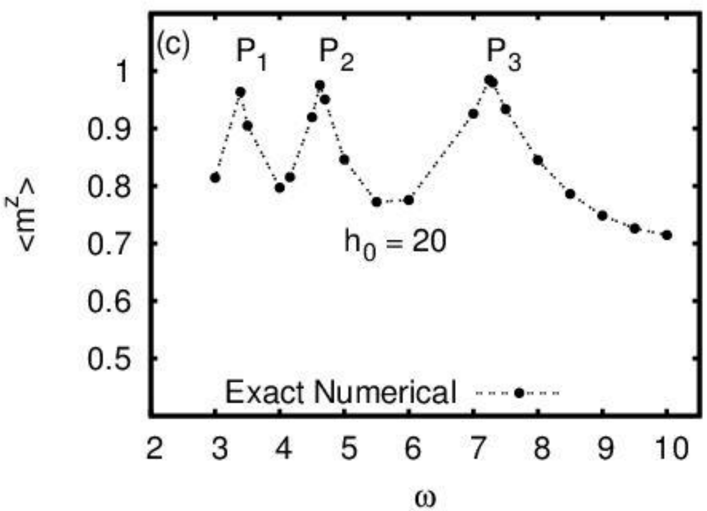
Dynamical Freezing in Transverse Ising Chain Under **Strong** and **Rapid** Periodic Drive

AD, PRB **82** (2010)

$$H = -\frac{J}{2}\left[\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x + h_0\cos(\omega t)\sum_i \sigma_i^z\right]$$

Strong and Fast Drive: $h_0, \omega \gg J$ Initial State → Ground State : $m^z \sim 1$





- ❑ Observables attain non-zero steady average values depending on the drive (ω,h_0) in the long time (no thermalization, no indefinite heating up).

- ❑ Strong Freezing (almost no dynamics of m^z) under certain freezing Condition

Mapping to 2-level Dynamics

$$H = -\frac{1}{2}\left[J\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x + h_z(t)\sum_{i=1}^{L}\sigma_i^z\right]; \quad h_z(t) = h_0\cos(\omega t)$$

$$\Downarrow$$

$$|\psi(t)\rangle = \bigotimes_{k>0}|\psi_k(t)\rangle; \quad |\psi_k(t)\rangle = u_k|0_k, 0_{-k}\rangle + v_k|1_k, 1_{-k}\rangle.$$

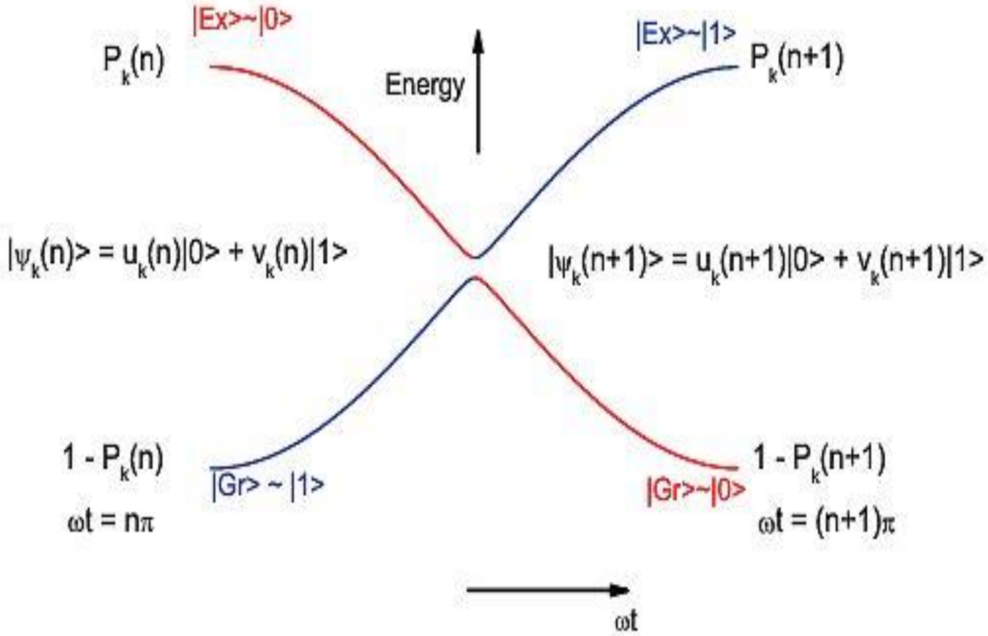
$$i\frac{d}{dt}\begin{pmatrix}u_k\\ v_k\end{pmatrix} = \begin{pmatrix}E_k & i\Delta_k\\ -i\Delta_k & -E_k\end{pmatrix}\begin{pmatrix}u_k\\ v_k\end{pmatrix}$$

$$E_k = h_0\cos(\omega t) + J\cos k; \quad \Delta_k = J\sin k$$

$$m_z(t) = \frac{4}{L}\left(\sum|v_k(t)|^2\right) - 1.$$

$$|v_k| = 1 \leftrightarrow m_z = 1, \quad v_k = 0 \leftrightarrow m_z = -1$$

Probability vs Amplitude …



Repeated **interference** of the phases between $|0_k, 0_{-k}>, |1_k, 1_{-k}>$

determines the **Asymptotic Behaviour** of the state and the **Steady Averages** !

$$|v_k(n+1)|^2 = (2P_{ex} - 1)|v_k(n)|^2 + (1 - P_{ex})$$

$$\Rightarrow$$

$$|v_k(n+1)|^2 = \frac{1}{2} + |v_k(0)|^2(2P_{ex} - 1)^{n+1} - (2P_{ex} - 1)^n$$

$$m^z \to 0 \text{ as } n \to \infty \quad \text{independent of } \omega$$

Cherng and Levitov, PRA, (2006);
V. Mukherjee, A. Dutta and D. Sen, PRB ***77****, 214427 (2008).*

The Rotating Wave Approximation: ω >> J (J = 1)

- **Unitary Transformation:** $$U_k = \exp\left[-\frac{i}{2}\left(2Jt\cos k + \frac{2h_0}{\omega}\sin(\omega t)\right)\sigma^z\right]$$
- **Transformed Wave-Function:**

$$|\psi_k'\rangle = u_k'|0_k\rangle + v_k'|1_k\rangle; \quad |u_k'|^2 = |u_k|^2, \quad |v_k'|^2 = |v_k|^2$$

- **Transformed Hamiltonian:**

$$H_k' = \Delta_k \begin{pmatrix} 0 & i\sum_{n=-\infty}^{\infty} R_n \\ -i\sum_{n=-\infty}^{\infty} R_n & 0 \end{pmatrix}$$

$$R_n = J_n\left(\frac{2h_0}{\omega}\right) e^{i(n\omega + J\cos k)t}$$

The Rotating Wave Approx (RWA): Keep only n = 0 term

$$H'(k) \approx -J_0(2h_0/\omega)\sin(k)\hat{\sigma}_k^y$$

$$|v_k(t)|^2 = \left[\frac{J_0^2(2h_0/\omega)\Delta_k^2}{4\phi_k^2}\sin^2(\phi_k t)\right]|u_k(0)|^2 + \left[\cos^2(\phi_k t) + \frac{(J\cos k)^2}{\phi_k^2}\sin^2(\phi_k t)\right]|v_k(0)|^2$$

*S. Ashhab et. al., Phys. Rev. A **75** 063414 (2007);*

The Freezing of the Response

$$\text{For} \quad |\psi(0)\rangle = |ground\rangle; \quad (|v_k(0)| \sim 1):$$

*AD, PRB **82** (2010).*

$$|v_k(t)|^2 = 1 - A_k^2 \sin^2(\phi_k t)$$

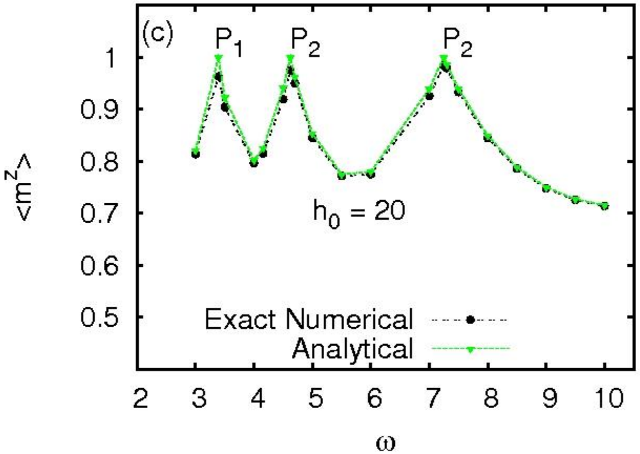
$$A_k^2 = \frac{J_0^2(2h_0/\omega)\sin^2 k}{J_0^2(2h_0/\omega)\sin^2 k + \cos^2 k}$$

$$\phi_k = \sqrt{J_0^2(2h_0/\omega)\sin^2 k + \cos^2 k}$$

→ Doesn't heat up indefinitely even at infinite time

$$Q = \langle m^z \rangle = 1/[1 + J_0(2h_0/\omega)]$$

$$(\text{Exact as } \omega \to \infty)$$



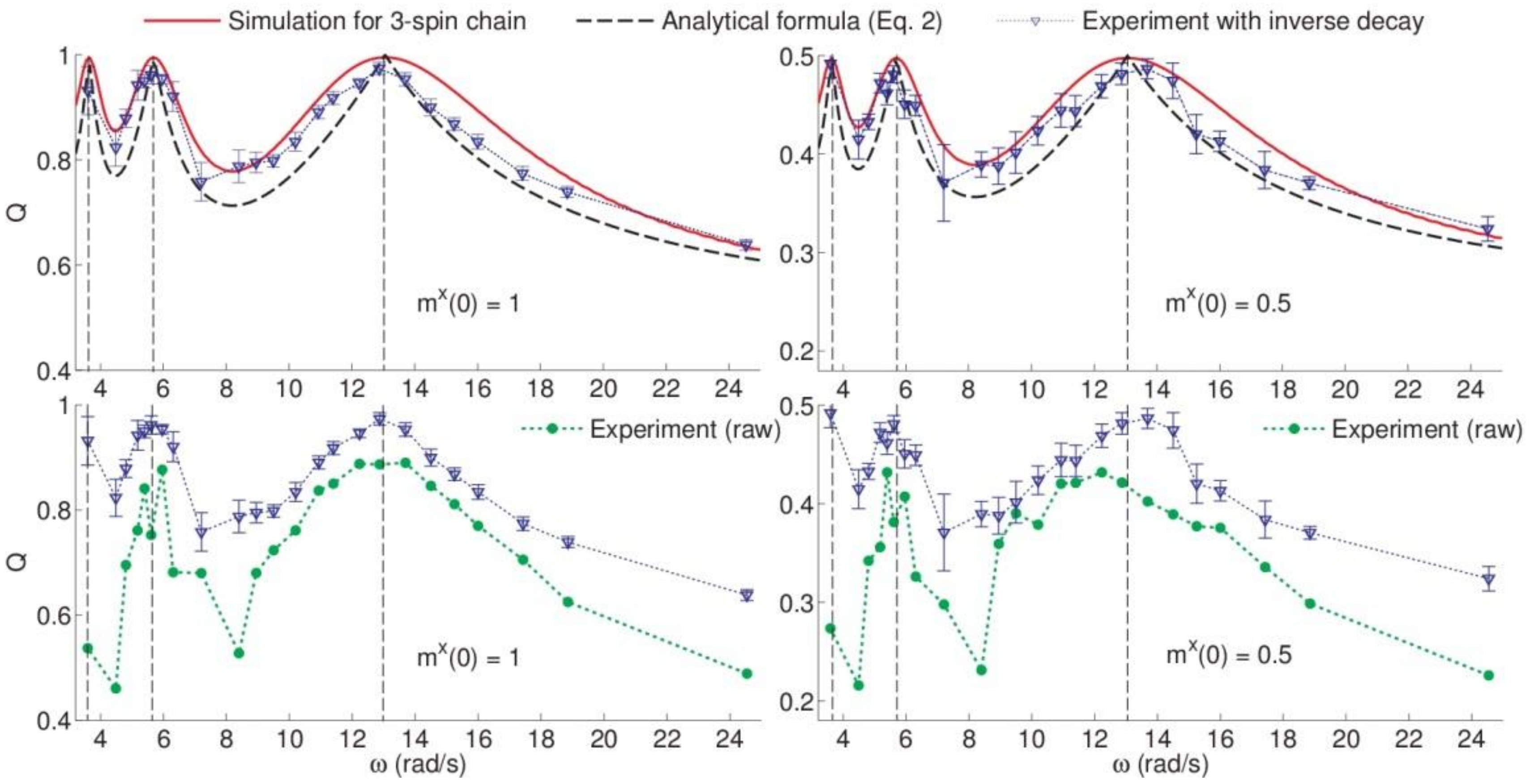
❑ Absolute Freezing (under RWA) for $J_0(2h_0/\omega) = 0$

→The effective Hamiltonian vanishes in the rotating frame.

→The Freezing is Independent of the Initial Condition.

→ All modes are frozen in the rotating frame (Experiment in small systems).

NMR Experiment



S. Hegde, et. al. PRB 90, 174407 (2014).

Freezing in Lab frame: Rectangular Pulse

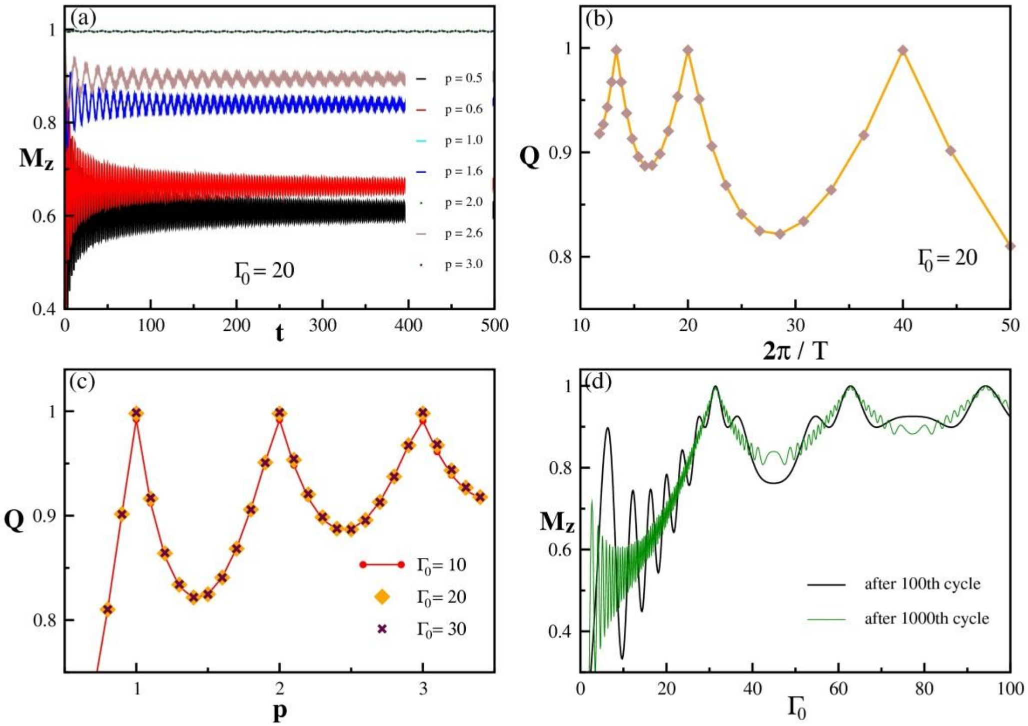
S. Bhattacharyya, AD and S. Dasgupta, PRB **86** (2012).

$$\mathcal{H} = -J\sum_{j=1}^{N} s_j^x s_{j+1}^x - h(t)\sum_{j=1}^{N} s_j^z$$

$$\Gamma(t) = \begin{cases} \Gamma_0 & \text{for } nT < t < (n+\frac{1}{2})T \\ -\Gamma_0 & \text{for } (n+\frac{1}{2})T < t < (n+1)T \end{cases}$$

Freezing Condition:

$$\Gamma_0 T = n\pi$$



Theme Questions:

(A) Generic:

- Can we *understand* the steady averages (or better, the asymptotic states themselves) in a periodically driven many-body system?

- At least frameworks and constructs (like entropy etc), as we have for understanding equilibrium states?

(B) DMF:

- How robust is this freezing under randomness and non-integrable perturbations?

T. Prosen and E. Ilievski, PRL 107, 060403 (2011) and subsequent works.

A: Generic
Understanding by mapping to an effective Time-Independent Problem

T-Periodic Hamiltonian: $H(t+T) = H(t)$

Time-Evolution Operator: $\hat{U}(0,t)|\psi(0)\rangle = |\psi(t)\rangle$

Effective Hamiltonian $H_{eff} : \hat{U}(0,T) = e^{-iH_{eff}T}$

Observation Time: $t = \epsilon + nT; \quad 0 \leq \varepsilon \leq T$

Stroboscopic Wave-function:

$$|\psi(nT)\rangle = e^{-iH_{eff}nT}|\psi(0)\rangle; \quad \left[\hat{U}(0,T)\right]^n = e^{-iH_{eff}nT}$$

$$|\psi(\varepsilon+nT)\rangle = e^{-iH_{eff}(\varepsilon)nT}|\psi(0)\rangle = \hat{U}(0,\varepsilon)e^{-iH_{eff}nT}|\psi(0)\rangle$$

Stroboscopic Observations ↔ Dynamics driven by ***time-independent*** Hamiltonian H_{eff} (of course H_{eff} is not unique for a given $\hat{U}(\varepsilon,\varepsilon+T)$).

Reaching Steady State of a **Reasonable** ***Observable*** starting from a ***Generic Initial State*** under H_{eff}

➢ **Starting from a "generic" initial state and for a "reasonable" time-independent operator $\hat{O}$ do we reach a steady value i.e., the sequence :**

$\langle\hat{O}(\varepsilon+T)\rangle, \quad \langle\hat{O}(\varepsilon+2T)\rangle, \quad \dots \quad \langle\hat{O}(\varepsilon+nT)\rangle$ **converges as** $n\to\infty$ **?**

Or equivalently, does the limit $\lim_{t\to\infty}\langle\mathcal{O}\rangle(t)$ exist under evolution with H_{eff} ?

Let $H_{eff}|n\rangle = \epsilon_n|n\rangle$

We can decompose Any Observable as: $\hat{\mathcal{O}} = \sum_{m,n}\mathcal{O}_{mn}|m\rangle\langle n|$

And an Arbitrary Initial State as: $|\psi(0)\rangle = \sum_k C_k|k\rangle$ then

$$\langle\hat{\mathcal{O}}(t)\rangle = \sum_{m,n} C_m^* C_n \mathcal{O}_{mn} e^{-i(\epsilon_n-\epsilon_m)t} \Rightarrow \lim_{t\gg 1}\langle\hat{\mathcal{O}}(t)\rangle \to \sum_m |C_m|^2\mathcal{O}_{mm} = \langle\hat{\mathcal{O}}\rangle_{DE}$$

Effective "Diagonal Ensemble":

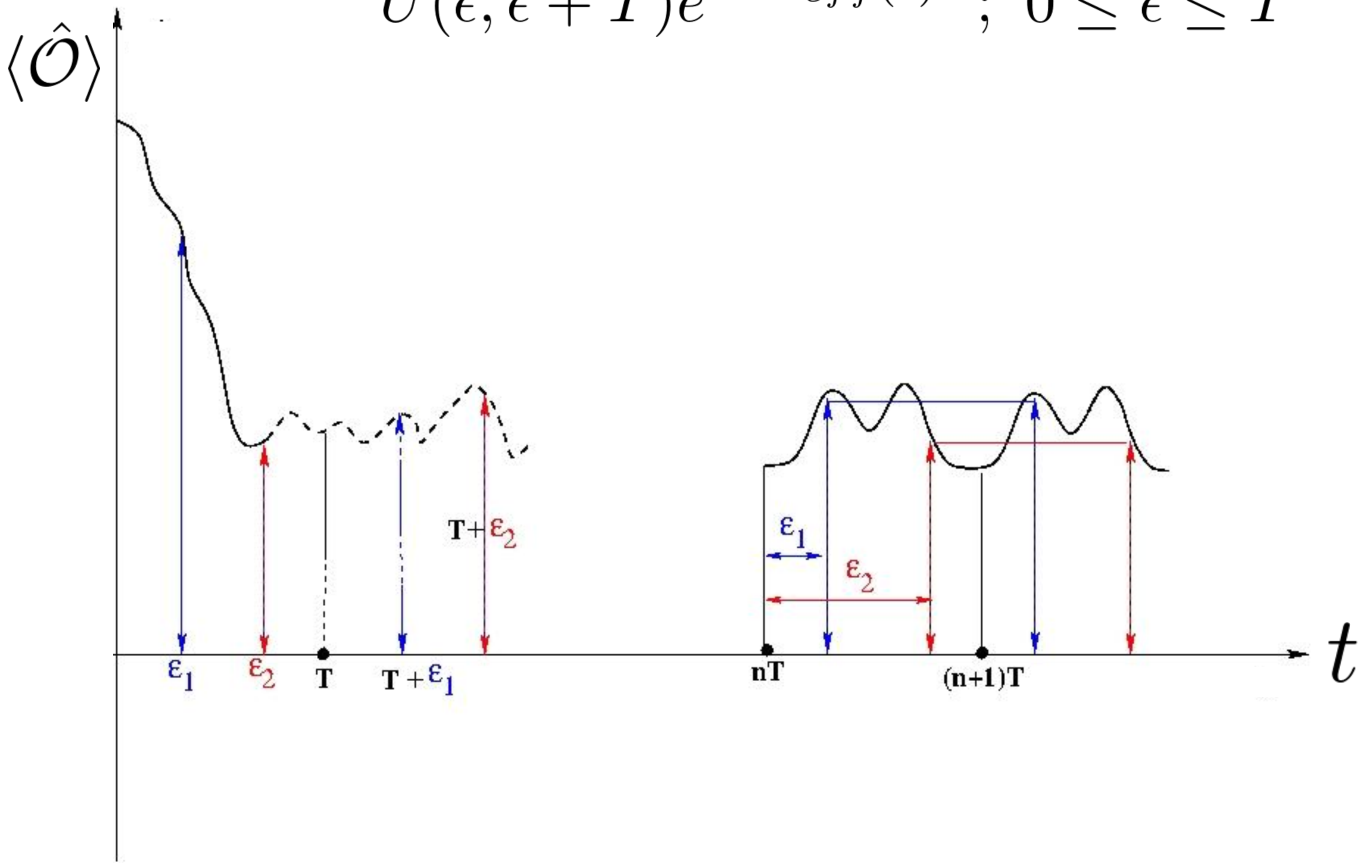
$$\hat{\rho}_{DE} = \sum_m |C_m|^2|m\rangle\langle m|$$

Iff all time-dependent contributions become negligible, and we can drop the off-diagonal terms.

(M . Rigol. et. al. Nature)

Reaching Stroboscopic Steady State Under $H_{eff}(\varepsilon) \Longrightarrow$ Synchronization under $H(t)$

$$\hat{U}(\epsilon, \epsilon + T)e^{-iH_{eff}(\epsilon)T};\ 0 \leq \epsilon \leq T$$



Numerical Demonstration for Ising Chain: A. Russomanno *et. al, PRL **109** 257201 (2012)*

Sufficient Condition for convergence of time-averages to DE: ***Reasonable Observable*** and ***Generic Initial State***

$\hat{O} = \sum_{m,n} \mathcal{O}_{mn} |m\rangle\langle n|$ $\quad |\psi(0)\rangle = \sum_k C_k |k\rangle$ $\quad \langle \hat{O}(t) \rangle = \sum_{m,n} C_m^* C_n \mathcal{O}_{mn} e^{-i(\epsilon_n - \epsilon_m)t}$

(1) $|\psi(0)\rangle$ should be well delocalized over the eigen-basis of H_{eff} :

Inverse Participation Ratio: $IPR = \sum_m |C_m|^4 \ll 1$

(2) The observable $\hat{O}$ must also connect substantial number of eigenstates, and have a finite range $\Delta_{\mathcal{O}}$

The Bound on Fluctuations from DE:

*P. Riemann, PRL **101** 190403 (2008).*

$$\sigma_{\mathcal{O}}^2 := \overline{[\langle \hat{O} \rangle(t) - Tr\{\hat{\rho}_{DE} \hat{O}\}]^2} \leq \Delta_{\mathcal{O}}^2 \times IPR$$

Overbar = Avg over all $t > 0$

Note that the above statement is essentially a statement regarding "typicality" of states through which the system passes during its evolution. Irrespective of how DE is realized dynamically, if there is a generic asymptotic behaviour, it should depend on some generic aspects of Heff. (Otherwise NO STAT-MECH).

Nature of $H_{eff}(\varepsilon)$: A glimpse through Magnus Expansion (assuming ME converges!)

$$H_{eff}(\varepsilon) : \hat{U}(\varepsilon, \varepsilon+T) = e^{-iH_{eff}(\varepsilon)T},$$

$$\text{where } \hat{U}(\varepsilon, \varepsilon+T) = \hat{U}(0,\varepsilon)\hat{U}(0,T)\hat{U}(0,\varepsilon)^{\dagger}$$

$$\begin{aligned}
H_{eff}(\varepsilon) &= \sum_{n=0}^{\infty} H^{(n)}_{eff} \quad \text{where} \\
H^{(0)} &= \frac{1}{T}\int_0^T H(t)dt \\
H^{(1)} &= \frac{1}{2!Ti\hbar}\int_{\varepsilon}^{\varepsilon+T} dt_1 \int_{\varepsilon}^{t_1} dt_2 [H(t_1), H(t_2)] \\
H^{(2)} &= \frac{1}{3!T(i\hbar)^2}\int_{\varepsilon}^{\varepsilon+T} dt_1 \int_{\varepsilon}^{t_1} dt_2 \int_{\varepsilon}^{t_2} dt_3 \left([H(t_1),[H(t_2),H(t_3)]] + [H(t_3),[H(t_2),H(t_1)]]\right)
\end{aligned}$$

$$\Longrightarrow H_{eff}(\varepsilon)$$

(a) Can be long-ranged in general

(b) but **bilinear** (local in general) in creation/annihilation operators if *H(t)* is.

Nontrivial Steady State under time-indep H_{eff} says Look for Conserved Quantities!

➢ How to obtain the most unbiased statistical distribution for a quantum system given a set of conserved quantities?

We construct a density operator $\hat{\rho}$ which minimizes the Von Neumann Entropy $S = -tr\left[\hat{\rho}\ln\hat{\rho}\right]$ subject to the constraints of conservations. This is the quantum version of maximizing Shanon Entropy.

Let's suppose we are given with n "conserved" quantities:

$$[\mathcal{I}_p, H_{eff}] = 0;\; p = 1, 2, ...n$$

Then the most unbiased distribution obeying this conservation, is

$$\hat{\rho}_0 = \frac{1}{\mathcal{Z}}\exp\left[-\sum_p \lambda_p \mathcal{I}_p\right];\; \mathcal{Z} = tr[\hat{\rho}_0]$$

E. T. Jaynes, Phys. Rev. **106** 620 (1957)

*M. Rigol. et. al. PRL **98** 050405 (2007)*

If for each $H_{eff}(\varepsilon)$ one is given with such $\mathcal{I}_p(\epsilon)$s, then we can construct the ensemble for the **synchronized asymptotic states** consistent with the stroboscopic conservations.

But which conserved quantities do we consider?

Periodic Gibbs Ensemble: The Integrable bilinear Case

$$H(t) = \sum_{i,j} [\hat{a}_i^\dagger \mathcal{M}_{ij}(t) \hat{a}_j + \hat{a}_i^\dagger \mathcal{N}_{ij}(t) \hat{a}_j^\dagger + h.c.]$$

A. Lazarides, AD, R. Moessner, PRL (2014)

Magnus Expansion $\Rightarrow$ H_{eff} is bilinear in (fermionic) $\hat{a}_i, \hat{a}_i^\dagger$ $\Rightarrow$

$$\hat{H}_{eff} = \sum_{p=1}^{L} \omega_p \tilde{a}_p^\dagger \tilde{a}_p \text{ with } \mathcal{I}_p = \tilde{a}_p^\dagger \tilde{a}_p$$

(a) Time-periodic & (b) Conserved Quantities: $\mathcal{I}_p(t) = \hat{U}(0,t)\mathcal{I}_p \hat{U}^\dagger(0,t)$

(a) $\mathcal{I}_p(t = nT + \epsilon) = \hat{U}(0,\epsilon)\mathcal{I}_p \hat{U}^\dagger(0,\epsilon)$

(b) $\langle \psi(t) | \mathcal{I}_p(t) | \psi(t) \rangle = \langle \psi(0) | \mathcal{I}_p | \psi(0) \rangle$

$$\hat{\rho}_{PGE}(t) = \mathcal{Z}^{-1} \exp\left[-\sum_p \lambda_p \mathcal{I}_p(t) \right]; \ \mathcal{Z}(t) = tr[\hat{\rho}_{PGE}(t)]$$

The Lagrange Multipliers $\{\lambda_p\}$ are determined from the initial condition:

$$\langle \psi(0) | \mathcal{I}_p | \psi(0) \rangle = tr[\hat{\rho}(0) \mathcal{I}_p]$$

How good is our choice of Conserved Quantities ?

- It can be proved, $\hat{\rho}_{PGE}(\epsilon)$ correctly reproduces the diagonal ensemble average for operators bilinear in $(\tilde{a}_p^\dagger, \tilde{a}_p)$

- For operators involving higher order products like $\tilde{a}_{p1}^\dagger \tilde{a}_{p2}^\dagger ... \tilde{a}_{pM}^\dagger \tilde{a}_{p1} \tilde{a}_{p2} ... \tilde{a}_{pM}$ can also be reproduced iff:

the state is such that Wick's theorem allows writing the expectation values of the products of bilinear operators as a linear combination of the product of the expectation values of those (i.e., all the Wick contractions of creation and annihilation operators are c-numbers times unity). For example, any state that is obtained by periodic drive from the ground state.

Model for the Numerical Study:

$$H_b(t) = -\frac{1}{2}\sum_i J_i b_i^\dagger b_{i+1} + h.c. + \sum_i V_i(t) b_i^\dagger b_i$$

HCBs

$$V_i(t) = \frac{1}{2}[(i - L/2)/l_{ho}]^2 + (-1)^i \Delta \cos(\omega t)$$

Trap

Modulated Super-lattice

$$J_i(t) = J + \delta J \cos(\omega t)$$

Initial State: Ground State of the Harmonic Trap

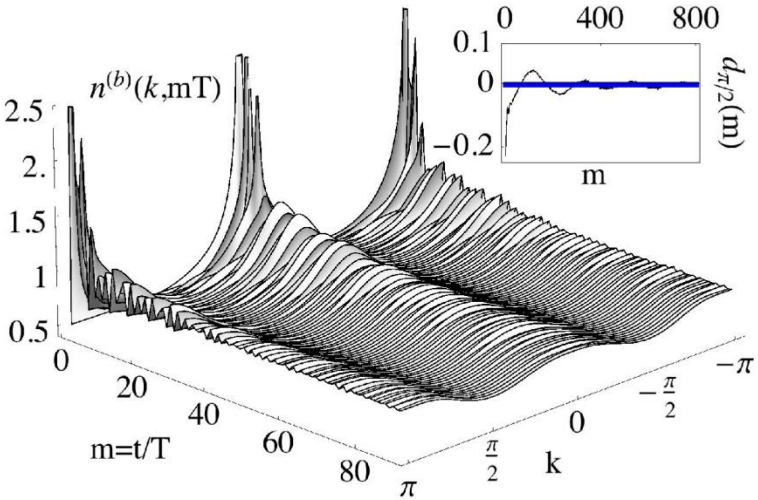
HCB -> Fermions: $b_i = a_i \prod_{j<i} (-1)^{\hat{n}_j}; \hat{n}_j = b_j^\dagger b_j = a_j^\dagger a_j$

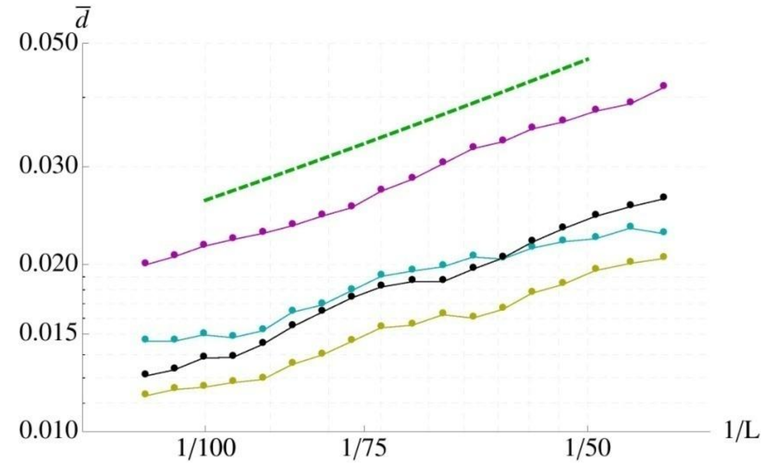
$$\begin{aligned} H_b(t) &= H(t) \text{ with } \mathcal{N}_{i,j} = 0, \\ \mathcal{M}_{i,j} &= J_i(t)(\delta_{i+1,j} + \delta_{i-1,j}) + \delta_{i,j} V_i(t) \end{aligned}$$

Numerical Results:

Observable → Momentum Distribution of Bosons (experimentally accessible):

$$\hat{n}^{(b)}(k) = \frac{1}{L}\sum_{i,j} b_i^{\dagger} b_j \exp[-2\pi k(i-j)/L]$$





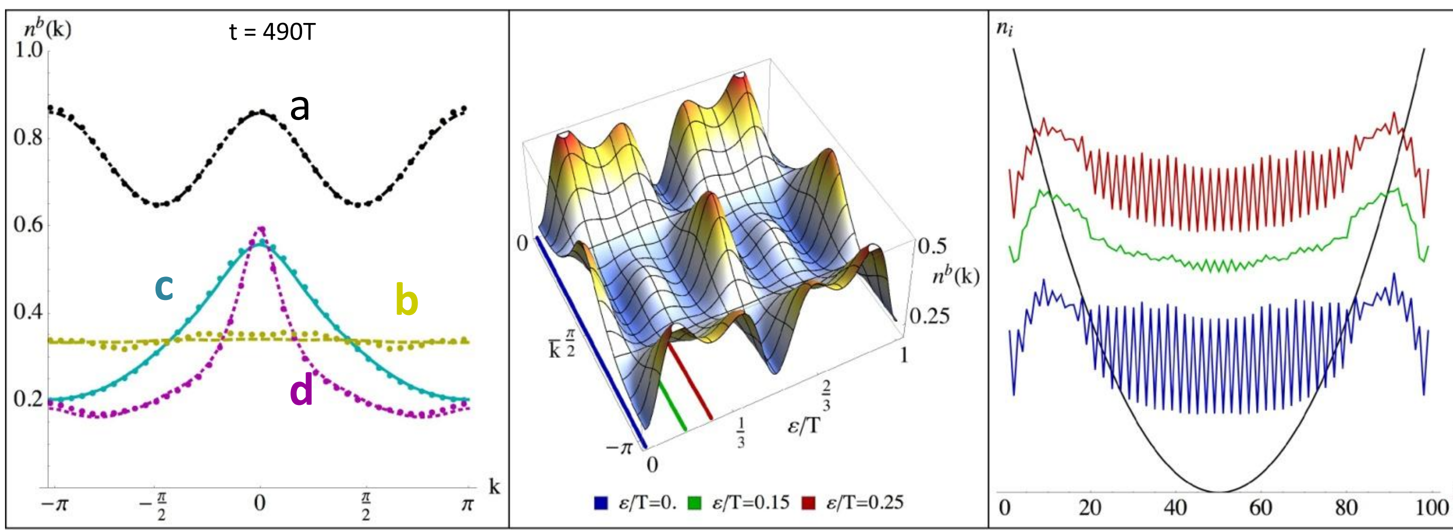
$\Delta, \delta J, \omega, \nu$:

a: 0.6, 0.5, 1.6, 3/4

b: 4.0, 0.5, 1.5, 1/3

c: 4.0, 0.75, 2.0, 1/3

d: 0.6, 0.5, 2.0, 1/4



(B) DMF:
Fate of Dynamical Freezing in Presence of Disorder: Acting Against Disorder Induced Decay

The Hamiltonian:

$$H(t) = -\alpha J \sum_{i}^{L-1} J_i \sigma_i^x \sigma_{i+1}^x - \sum_{i}^{L} \{h_0 \sin(\omega t) + \alpha h_i\} \sigma_i^z$$

$\alpha = 0.3;$ J_{ij} *Uniformly Distriuted over* $[-1, +1]$

JW +

Floquet Flow Eq. Formalism:
(Verdeny *et al.)*

to first order in 1/ω

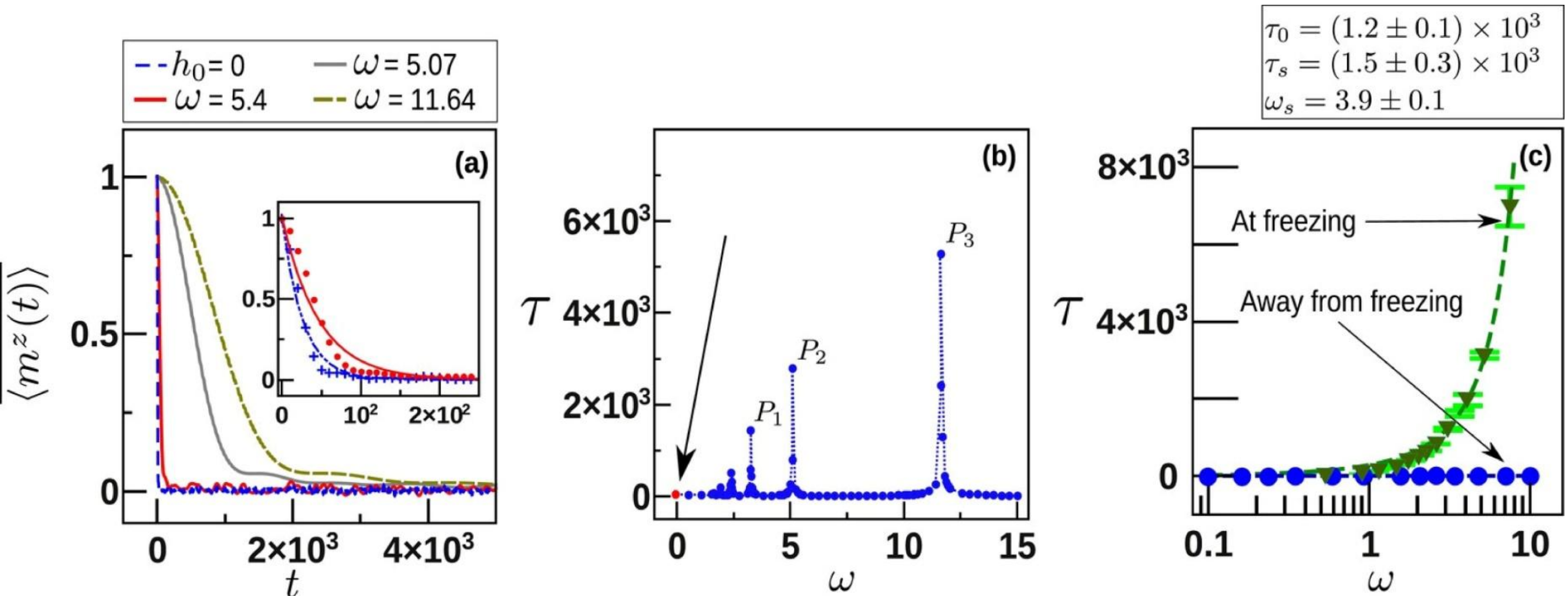
$$H_{\text{eff}} = -J \sum_i j_i^{(0)} \left(c_i^\dagger c_{i+1}^\dagger + \text{h.c.}\right) - J \sum_i j_i^{(1)} \left(c_i^\dagger c_{i+1} + \text{h.c}\right) - \mu \sum_i j_i^{(2)} c_i^\dagger c_i,$$

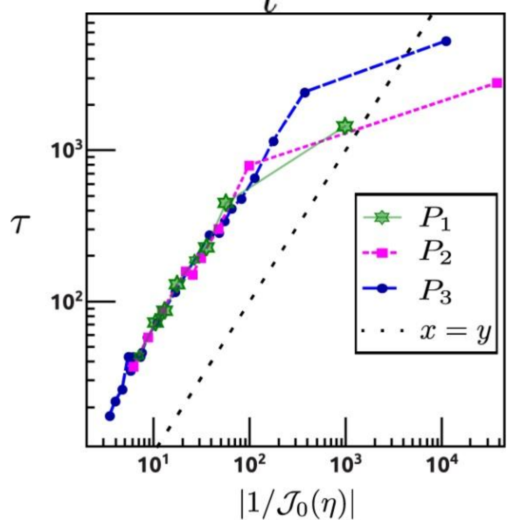
$$j_i^{(0)} \equiv \alpha J_i \left\{\mathcal{J}_0(\eta) - \frac{4\alpha h_i}{\omega}\beta(\eta)\right\} \; j_i^{(1)} \equiv \alpha J_i \; \mathcal{J}_0(\eta) \; j_i^{(2)} \equiv \frac{h_i}{J}, \quad \mu \equiv 2\alpha J.$$

$$H_{eff} \propto J_0(2h_0/\omega)$$ **(for Randomness in Interactions)**

A. Roy, AD (PRB, Rap. Comm., 2015)

Fate in Presence of Disorder …





❖ Orders of Magnitude Enhancement of τ of relaxation of $\langle m_z \rangle$

❖ Exponential enhancement of τ with ω *only* under the “Freezing” condition: ***Stabilizes Local observables*** .

Summary of What We Got So Far
(Periodically Driven Integrable Systems)

❖ We understand why *certain class* of integrable systems don’t thermalize even under external periodic drive.

❖We know how to define relevant (periodic) conserved quantities for those and construct the relevant (non-thermal) periodic ensemble (PGGE).

❖We do not in general understand yet whether or not an **Integrable** system driven periodically in time **Thermalizes** (*even if the Hamiltonian is Integrable at every instant*) .

❖ We do not yet understand the physical mechanism behind the Dynamical Many-body Freezing phenomenon (though we know how to put it trivially in terms of above construct).

❖ How robust this freezing effect is under non-integrable perturbation.

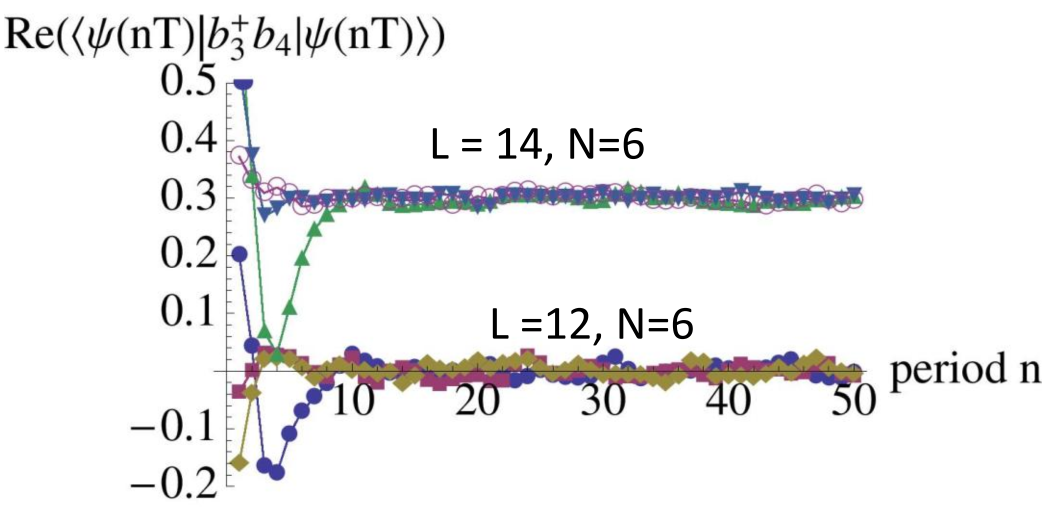
Non-integrable Systems (Without Disorder):
A: Generic What Ensemble?

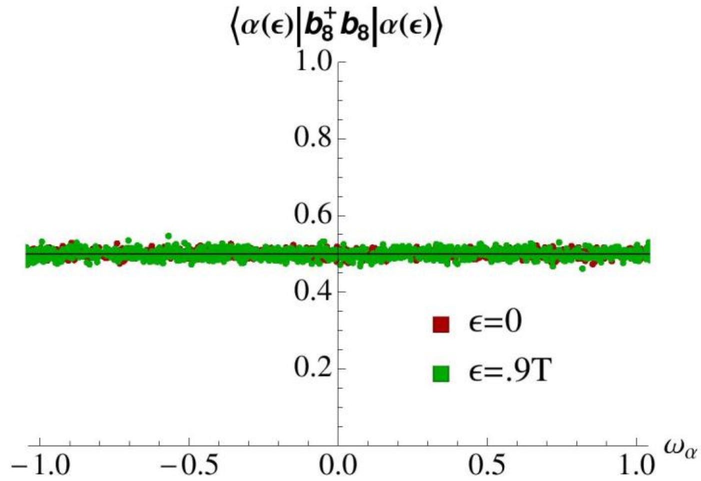
A. Lazarides, AD, R. Moessner, PRE (2014)

Given the Hilbert space, we always reach the *infinite temperature ensemble* regardless of the drive, and other details of the Hamiltonian.

$$H(t) = -\frac{1}{2}\sum_i b_i^\dagger b_{i+1} + h.c. + \sum_i n_i n_{i+1} + \sum_i n_i n_{i+2} + u\sum_i V_i(t) n_i$$

$$V_i(t) = (-1)^i \text{ for } 0 \le t < T/2$$
$$= -(-1)^i \text{ for } T/2 \le t < T$$





Every Eigenstate is thermalized to infinite temperature.

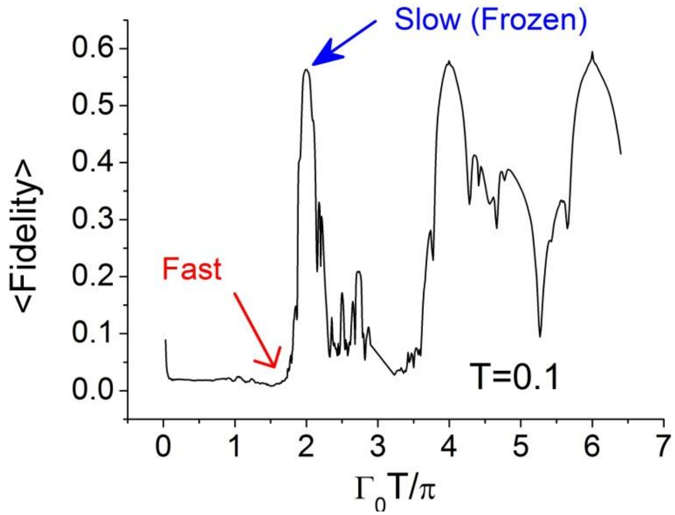
L. D'Alessio, M. Rigol, PRX (2014) draws similar conclusions

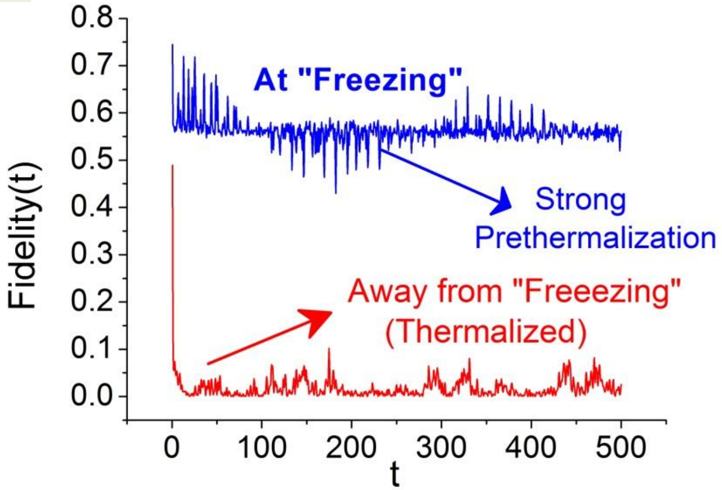
Non-integrable Systems (without disorder):
B: DMF What Happens to it ?

$$\mathcal{H} = -J \sum_{j=1}^{N} s_j^x s_{j+1}^x - h_z(t) \sum_{j=1}^{N} s_j^z - h_x \sum_i \sigma_i^x$$

$$h_z(t) = \begin{cases} \Gamma_0 & \text{for } nT < t < (n+\frac{1}{2})T \\ -\Gamma_0 & \text{for } (n+\frac{1}{2})T < t < (n+1)T \end{cases}$$

N=14, h_x = 2J



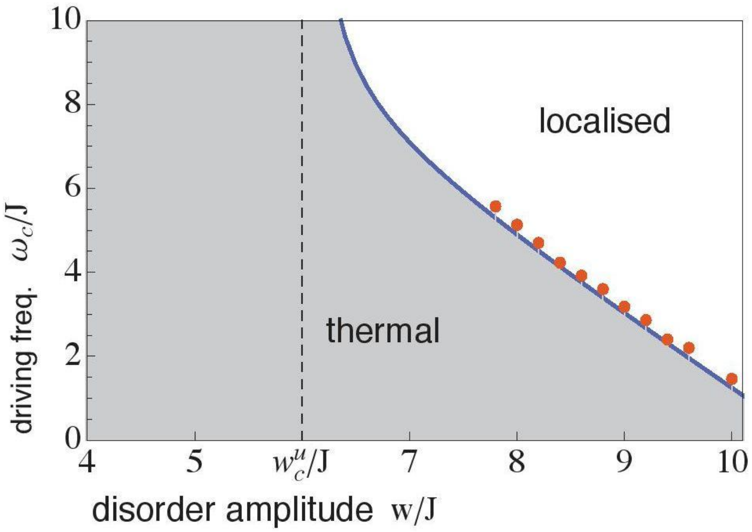


(A. Haldar *et al*., Work in progress ...)

Non-integrable Systems (with disorder):
MBL under Periodic Drive

Mobility edge	low frequency	high frequency
present	delocalised	delocalised
absent	delocalised	**localised**

A. Lazarides, AD, R. Moessner, PRL (2015)



See also , P. Ponte et. al., PRL (2015)

Open Questions/Issues

- Dynamics of Integrable Quantum Systems under (Integrable) Periodic Drive – ergodic or not?

- Does DMF work in presence of Disorder + Interaction = MBL?

Type of questions to be asked:

If an extended state is allowed to evolve under MBL Hamiltonian
can it's delocalization (say, momentum peak) be maintained using DMF?

- "Understanding" of periodic steady states are still lacking. $H_{eff}(\varepsilon)$ can be very long-ranged and even non-local in general, so exotic steady states can be engineered.

- DMF can possibly used to freeze unknown quantum states with high fidelity for quantum information processing and computation.

Thanks!

(1876)

(1907-1930)