On Shocks in the TASEP

Mustazee Rahman (KTH)

jointly with

Jeremy Quastel (U. Toronto)

This talk is about a type of

wave motion

based on the article arXiv:1801.06143.

Recall the wave equation

$$\partial_t^2 u - \partial_x^2 u = 0.$$

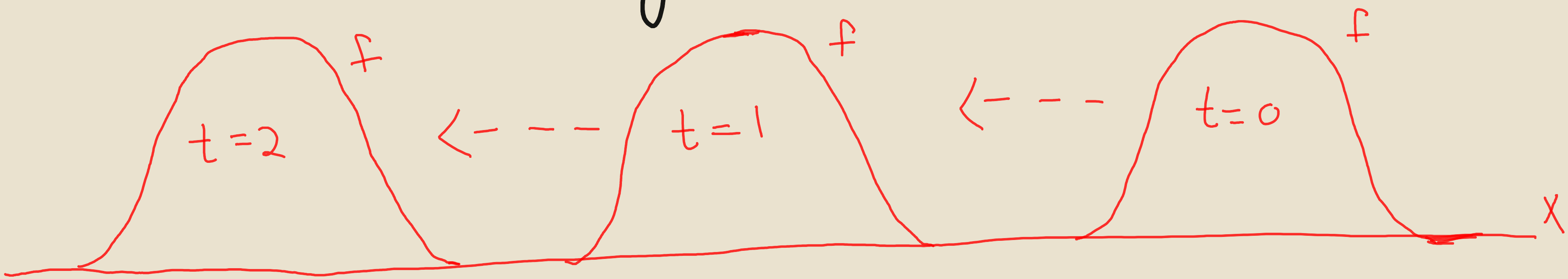
An even simpler one is

$$\partial_t u - \partial_x u = 0.$$

This is because the solution of the latter is

$$u(t,x) = f(t+x), \quad f = u(0,\cdot),$$

which is a travelling front:



The solution of the wave equation is a superposition of travelling fronts:

$$u(t,x) = f(t+x) + g(t-x).$$

But waves can easily form shocks:

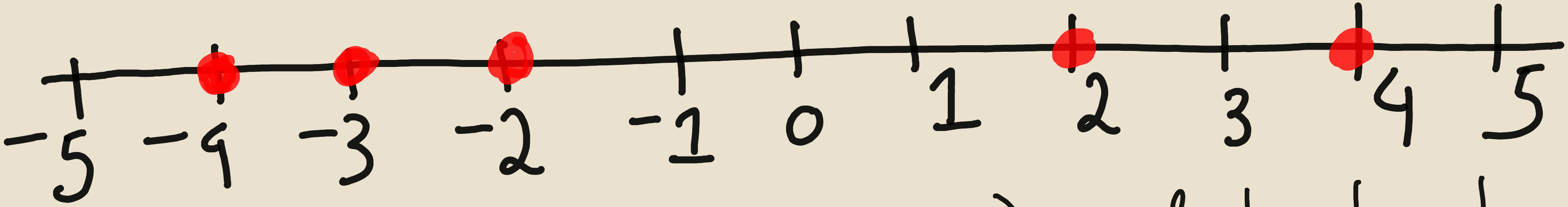
They are modelled by introducing nonlinearities in the wave equation. A prototype is Burgers' equation

$$\partial_t u - u \partial_x u = 0 .$$

Shocks form when the initial wave profile decreases in the direction of flow ($\partial_x u_0 < 0$) somewhere.

There are microscopic models to study Burgers' equation, certain interacting particle systems, which discretize it.

The most basic is the TASEP. totally asymmetric simple exclusion process



- Each ● jumps right ($x \mapsto x+1$) randomly at rate 1.
- All jumps are independent.
- But subject to exclusion rule: ● cannot jump on top another ●.

Suppose particles are initially placed such that there is a macroscopic density profile u_0:

$$\int_a^b u_0(x)\,dx = \lim_{L \to +\infty} \frac{1}{L} \cdot \# \left\{ \begin{array}{l} \text{Particles in } [aL, bL] \\ \text{at time } 0. \end{array} \right\}.$$

Then there is a macroscopic density profile $u(t,x)$ for further times:

$$\int_a^b u(t,x)\,dx = \lim_{L \to +\infty} \frac{1}{L} \cdot \# \left\{ \begin{array}{l} \text{Particles in } [aL, bL] \\ \text{at time } tL. \end{array} \right\}.$$

Furthermore, the density $u(t,x)$ satisfies Burgers' equation in the form

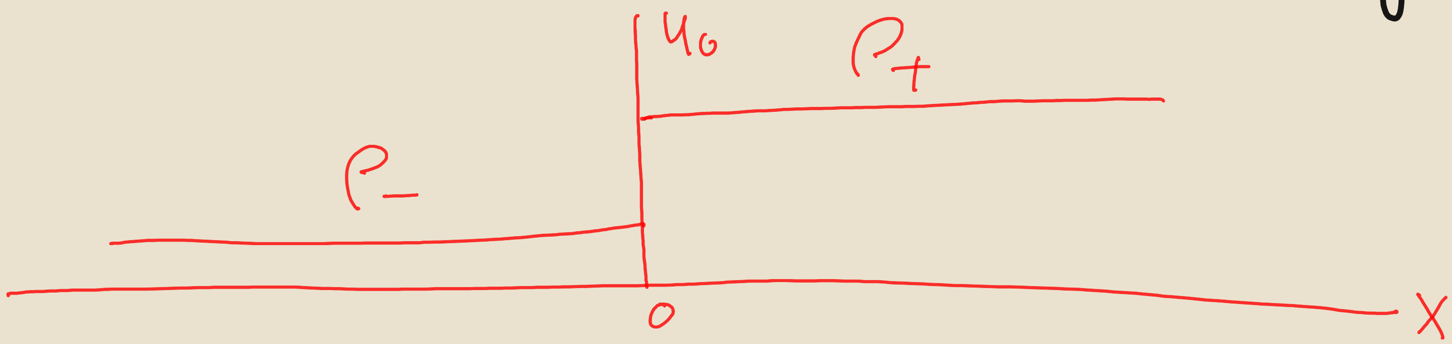
$$\circledast \qquad \partial_t u - \partial_x \left(u(1-u) \right) = 0 ,$$

$$u(0,x) = u_0(x) .$$

It is the entropic solution (the physical one).

General theorems like this in the context of hydrodynamic limits of interacting particles.

The most basic case of the shock is the increasing step:



The (entropic) solution of $*$ is the travelling front

$$u(t,x) = u_0(x - v t), \quad \text{where}$$

$$v = 1 - \rho_- - \rho_+ \quad \text{is the front speed.}$$

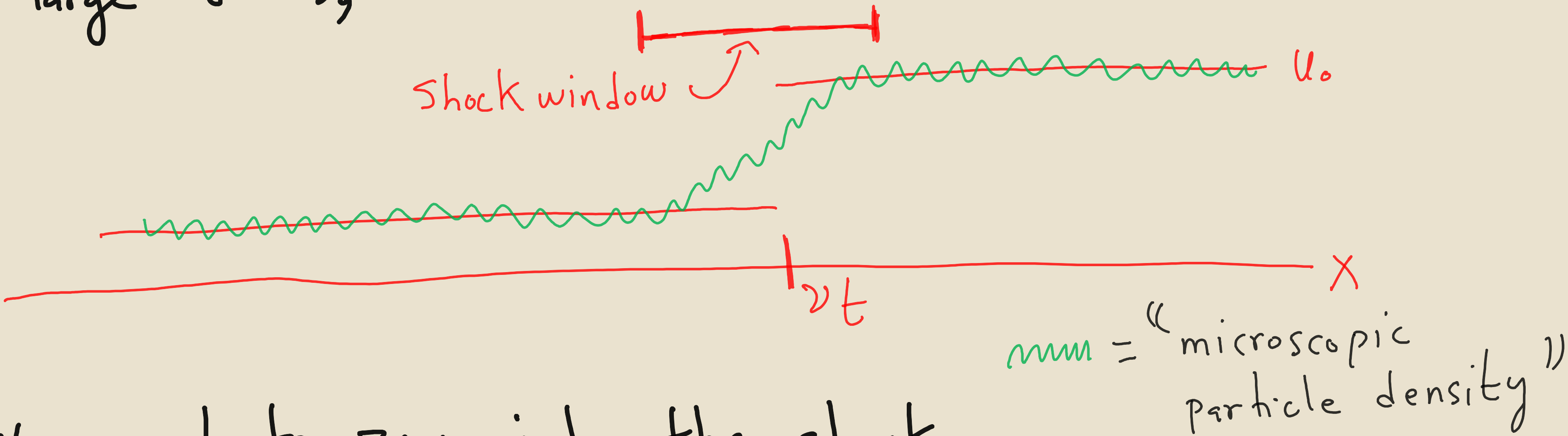
In TASEP terms:

- Particles left of origin move at speed $1-\rho_-$ and those to the right move at speed $1-\rho_+$.
- Since $1-\rho_- > 1-\rho_+$, they collide to create a jam, which moves at speed v.

Model u_0 by placing particles on $\mathbb{Z}$ in a deterministic way such that

$$\rho_t = \lim_{L\to\infty} \frac{1}{L}\cdot \#\{\text{particles in } [0, \underline{t}\, L]\}.$$

For large times, the TASEP looks like so



We want to zoom into the shock window and obseve the microscopic particle fluctuations.

We will do so by introducing a parameter to soften the shock.

For $\beta \in \mathbb{R}$ and $t \geq 2|\beta|^3$, set

$$\rho_{\pm} = \frac{1 \pm \beta (t/2)^{-1/3}}{2}.$$

- Run TASEP to time t, t as above, and observe shock window.
- Shock fluctuations are derived in the double limit $t \to \infty$ followed by $\beta \to \infty$.
- This mysterious yet judicious rescaling of densities is motivated by KPZ theory . . .

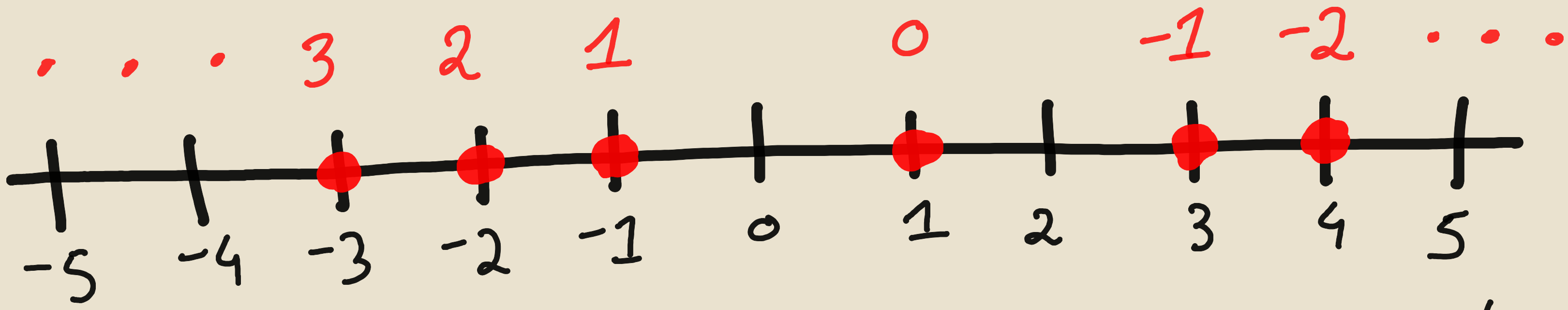
which allows for :

① Exact calculations of relevant statistics inside the shock window for large times.

② Analysis of these statistics in the large β limit, which are given by Fredholm determinants.

③ A process level description of the microscopic particle fluctuations inside the shock window by way of ① and ②.

Tag the particles in TASEP like so



$X_t(n)$ = position of particle n at time t.

The particle that is macroscopically at νt at time t has number $n_t^{shk} = (\rho_- \cdot \rho_+) t$.

- Particles within distance $t^{2/3}/\beta$ of n_t^{shk} are correlated with it.
- Their positions fluctuate to the order $t^{1/3}$.

So consider $n(t,x) = n_t^{Shk} + x\left(\frac{t^{2/3}}{2\beta}\right)$, $x \in \mathbb{R}$.

Define $\Psi_t^\beta(x) = \dfrac{X_t(n(t,x)) - (x/\beta)(t/2)^{2/3}}{-(t/2)^{1/3}}$.

macroscopic position of $n(t,x)$.

<u>Theorem.</u> In the limit $t \to \infty$ followed by $\beta \to \infty$, Ψ_t^β converges in the sense of finite dimensional laws to

$$\Psi(x) = \max\left\{ 2^{-2/3} X_{TW_1} - x,\ 2^{-2/3} X'_{TW_1} + x \right\}.$$

X_{TW_1} and X'_{TW_1} are independent GOE Tracy-Widom random variables.

Corollary: $\displaystyle \lim_{\beta\to\infty} \lim_{t\to\infty} \Pr\left(X_t(n_t^{shk}) \geq -a t^{1/3} \right) = F_1(2a)^2 .$

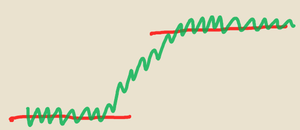
CDF of GOE TW.

The theorem should be interpreted as follows.

$$\max\left\{ 2^{-2/3} X_{TW_1} - x,\ 2^{-2/3} X'_{TW_1} + x \right\} = |x - D| + S,$$

$$D = \left(X_{TW_1} - X'_{TW_1} \right) / 2^{5/3} \quad \text{and} \quad S = \left(X_{TW_1} + X'_{TW_1} \right) / 2^{5/3}.$$

So Ψ_t^β converges to a V that is centered at D and shifted by S.

The microscopic density () is

$\frac{d}{dx} V$ = an increasing step with jump at D.

The microscopic position of the shock is at D.

This is simply because

X_t = particle position = $\int$ particle density.

More remarks.

- The soft shock setting was introduced by [Ferrari-Nejjar, 14]
  They conjectured the corollary.

- Sometimes the microscopic position of the shock is identified with the position of the 2nd class particle. It is indeed proved by [Ferrari-Ghosal-Nejjar, 17] that the rescaled position of the 2nd class particle converges to D in the setting of the increasing step.

- One can also model the increasing step with random initial data: place a particle at site x with probability ρ_- (if $x < 0$) or ρ_+ (if $x > 0$).

  In this case the shock fluctuations are of the order $t^{1/2}$. Here the randomness of the initial data overtakes the randomness inherent in TASEP, and shock fluctuations become Gaussian. A series of works by (Pablo) Ferrari, Kipnis, Liggett and many others.

I alluded to an auxillary result: large time limit of ψ^{β}_t.

Theorem. For every $\beta \in \mathbb{R}$, ψ^{β}_t converges as $t \to \infty$ in the sense of finite dimensional laws to

$$\psi^{\beta}(x) = h^{\beta}(1, x/2\beta) - \beta^2 .$$

$h^{\beta}(1,x)$ is a random function. It is given by KPZ theory or rather the KPZ fixed point.

It is a manifestation of the integrability of TASEP, and we have ...

$$\Pr\left(h^\beta(1,x)\le a\right) = \det\left(I - e^{-x\partial^2} K_\beta e^{x\partial^2}\right)_{L^2[a,\infty)}$$

for some trace class operator $e^{-x\partial^2} K_\beta e^{x\partial^2}$.

More generally, given $x_1 < x_2 < \cdots < x_k$ and $a_1, a_2, \ldots, a_k$,

$$\Pr\left(h^\beta(1,x_i)\le a_i \text{ for } i=1,\ldots,k\right) =$$

$$\det\left(I - e^{-x_k\partial^2} K_\beta e^{x_k\partial^2}\left(I - e^{(x_k-x_1)\partial^2} \cdot \chi_{a_1} e^{(x_2-x_1)\partial^2} \chi_{a_2} \cdots e^{(x_k-x_{k-1})\partial^2} \chi_{a_k}\right)\right)_{L^2(\mathbb{R})}$$

with $\chi_a =$ proj onto $L^2(-\infty,a)$.

The key object is K_β.

The operator K_β.

Introduce operators $S_x = e^{\frac{1}{3}\partial^3 + x\partial^2} \circlearrowleft L^2[a,\infty)$ for $x \in \mathbb{R}$.

Kernel: $S_x(u,v) = e^{\frac{2}{3}x^3 + x(v-u)} Ai(v-u+x^2)$.

$B(t)$ is Brownian motion with diffusivity 2.

$\tau(u) = \inf\{t \geq 0 : B(t) \leq 2\beta t - u\}$ with $u \in \mathbb{R}$.

Define operator $\mathcal{S}^{hypo}$ by its kernel

$$\mathcal{S}^{hypo}(u,v) = \mathbb{E}\, S_{-\tau(u)}(B(\tau(u)), v) \cdot \mathbb{1}_{\{\tau(u) < \infty\}}.$$

$$K_\beta = S_0^* \cdot \mathcal{S}^{hypo} + \mathcal{S}^{hypo^*} \cdot S_0 - \mathcal{S}^{hypo^*} \cdot \mathcal{S}^{hypo}.$$

- $\mathfrak{F}^{\text{hypo}}$ is introduced in generality in [Quastel-Remenik, 16].
- It is a fundamental object that describes statistics of the KPZ fixed point introduced in [Matetski-Quastel-Remenik, 17]

The latter describes the law of growth interfaces in the KPZ universality class in a continuum limit.

- $\mathfrak{F}^{\text{hypo}}$ is obtained as a (singular) limit

$$\lim_{L\to\infty} P_L^{\text{hit}} \cdot e^{-L\partial^2} \cdot K_{Ai}.$$

$\rightarrow$ Airy Kernel

$$P_L^{\text{hit}}(u,v) = \Pr\left(\begin{array}{l}\text{A Brownian bridge from } BB(0)=u \text{ to } BB(L)=v \\ \text{hits the hypograph of the curve } 2\beta t\end{array}\right).$$

A few remarks on $h^\beta(1,x)$.

- A one-parameter family of random functions that recover many of the Airy process. E.g.,

  $\beta=-\infty$: The process $\mathrm{Airy}_2(x)-x^2$.

  $\beta=0$: The process $\mathrm{Airy}_1(x)$.

  As $\beta\to-\infty$: $h^\beta(1,x+\beta)-\beta^2+2\beta(x+\beta)\to \mathrm{Airy}_{2\to1}(x)$.

- The soft-shock scaling is similar to the Baik-Ben Arous-Péché scaling from random matrices within the context of particle systems. The functions $h^\beta(1,x)$ are the corresponding distributional limits.

An idea of the proof.

Consider the probability that $\psi^{\beta}(0) \le a$, which is

$$\Pr\left(h^{\beta}(1,0) \le \beta^2 + a\right) = \det\left(I - K_\beta\right)_{L^2[a+\beta^2,\infty)}$$

$$= \det\left(I - e^{\beta^2\partial} K_\beta e^{-\beta^2\partial}\right)_{L^2[a,\infty)}$$

translation

There are two key steps:

① Calculate $\mathcal{S}^{\text{hypo}}$.

② Factorize $I - e^{\beta^2\partial} K_\beta e^{-\beta^2\partial} = (I-A)(I-B)$.

Lemma 1. Define operators M_β and R by

$$M_\beta f(x) = e^{\beta x} f(x) \quad \text{and} \quad Rf(x) = f(-x).$$

$$S^{hypo} = \chi_0 S_0 + (1-\chi_0) M_\beta R M_\beta^{-1} S_0 .$$

Lemma 2. $I - e^{\beta^2 \partial} K_\beta e^{-\beta^2 \partial} = (I-Q)^* (I-Q)$,

$Q = M_\beta (A + E_\beta) M_\beta^{-1}$, $A = S_0^* R S_0$, $E_\beta = \text{error}$.

$A(u,v) = 2^{-\frac{1}{3}} Ai(2^{-1/3}(u+v))$ and $\det(I-A)_{L^2[a,\infty)} = Pr(X_{TW_1} \le 2^{2/3} a)$.

Decompose $(I-Q)^*(I-Q) = \underbrace{(I-Q)^*\chi_a(I-Q)}_{X} + \underbrace{(I-Q)^*(1-\chi_a)(I-Q)}_{Y}$.

Since $1-\chi_a =$ proj. onto $L^2[a,\infty)$,

$$\det(Y)_{L^2[a,\infty)} = \det(I-Q)^2_{L^2[a,\infty)}.$$

The trace norms of X and E_β on $L^2[a,\infty)$ vanish as $\beta\to\infty$.

So
$$\Pr(\psi^\beta(0)\le a) = \det\big((I-Q)^*(I-Q)\big)_{L^2[a,\infty)}$$
$$\approx \det(Y)_{L^2[a,\infty)}$$
$$\approx \det(I-A)^2_{L^2[a,\infty)}$$

$\square$

How does one calculate $\mathcal{S}^{hypo}$?

- Consider $\beta = 0$. For general β, use Cameron-Martin Theorem to tilt space so that the line $2\beta t$ becomes flat. This results in the conjugation by M_β in Lemma 1.
- Only need to consider $u > 0$ in $\mathcal{S}^{hypo}_{(u,v)}$.

  $\Upsilon =$ hitting time of B to origin started from u.

$$\mathcal{S}^{hypo}_{(u,v)} = \mathbb{E}\left[e^{\frac{1}{3}\partial^3 - \Upsilon \partial^2}_{(0,v)}\right] = \lim_{T \to \infty} \mathbb{E}\left[e^{\frac{1}{3}\partial^3 - \Upsilon \partial^2}_{(0,v)} \cdot \mathbb{1}_{\{\Upsilon \leq T\}}\right].$$

- All the randomness enters from the left, so

$$\mathbb{E}\left[e^{\frac{1}{3}\partial^3 - \gamma\partial^2}(0,v)\, 1_{\{\gamma \le T\}} \right] = \mathbb{E}\left[e^{(T-\gamma)\partial^2}(0,\cdot)\, 1_{\{\gamma\le T\}}\right] \cdot e^{\frac{1}{3}\partial^3 - T\partial^2}(\cdot, v).$$

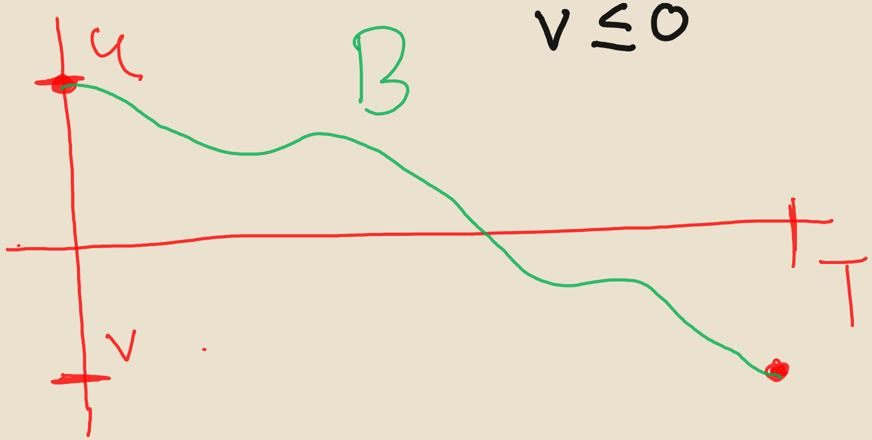
multiplication

- By strong Markov property,

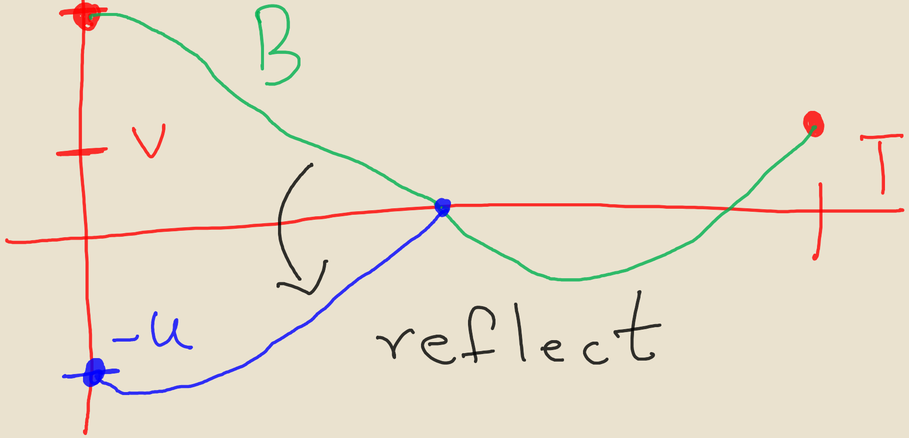
$$\mathbb{E}[\cdots] = \Pr\left(B \text{ goes from } B(0)=u \text{ to } B(T)=v \text{ while hitting origin} \right).$$

- There are 2 cases:

$v \le 0$



$v > 0$



So by the reflection principle,

$$\mathbb{E}[\cdots] = \begin{cases} e^{T\partial^2}(u,v) & \text{if } v \le 0, \\ \underbrace{e^{T\partial^2}(-u,v)}_{R e^{T\partial^2}(u,v)} & \text{if } v > 0. \end{cases}$$

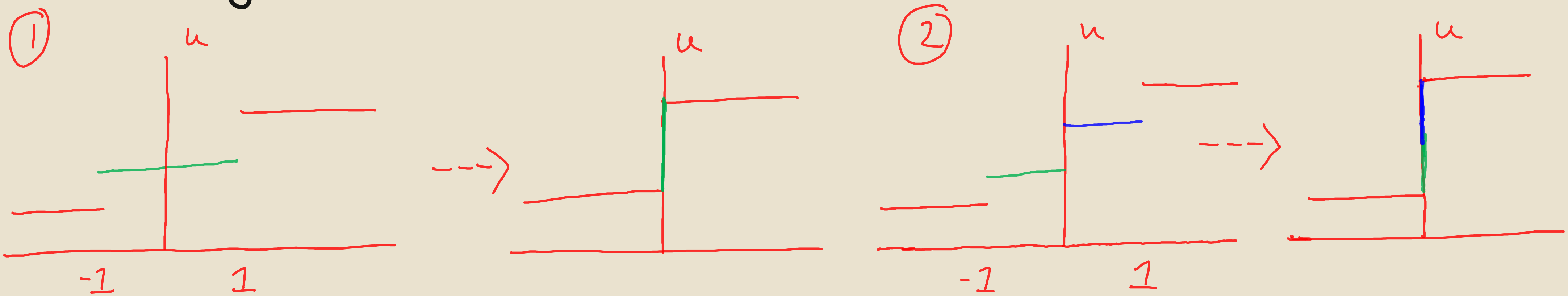
$$\mathcal{S}^{hypo}(u,v) = \lim_{T\to\infty} e^{T\partial^2} \cdot \underbrace{\chi_0}_{v\le 0} \cdot S_{-T}(u,v) + R e^{T\partial^2} \cdot \underbrace{(1-\chi_0)}_{v>0} \cdot S_{-T}(u,v)$$

$$= R S_0(u,v) + \lim_{T\to\infty} (I-R) e^{T\partial^2} \cdot \chi_0 \cdot S_{-T}(u,v).$$

The limit turns out to vanish to give

$$\mathcal{S}^{hypo} = \underbrace{\chi_0}_{u\le 0} S_0 + \underbrace{(1-\chi_0)}_{u>0} R S_0 .$$

Merger of multiple shocks



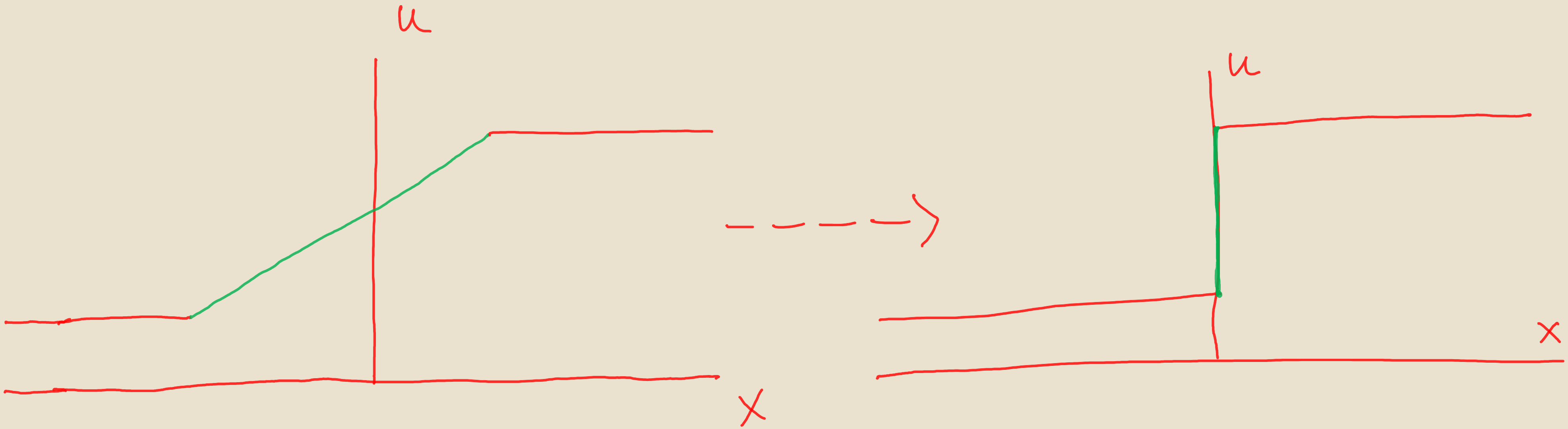
A similar approach may be used to show that at the merger of M-1 shocks the fluctuations are given by the maximum of M GOE T-Ws. The process level description is

$$x \mapsto \max\left\{ X^{(0)}_{TW_1} - x,\ X^{(1)}_{TW_1} - \left(1-\frac{2\cdot 1}{M-1}\right)x,\ X^{(2)}_{TW_1} - \left(1-\frac{2\cdot 2}{M-1}\right)x, \dots, X^{(M-1)}_{TW} + x \right\}.$$

independent

Questions remain

The emerging shock:



Fluctuations at the shock?

Thank you