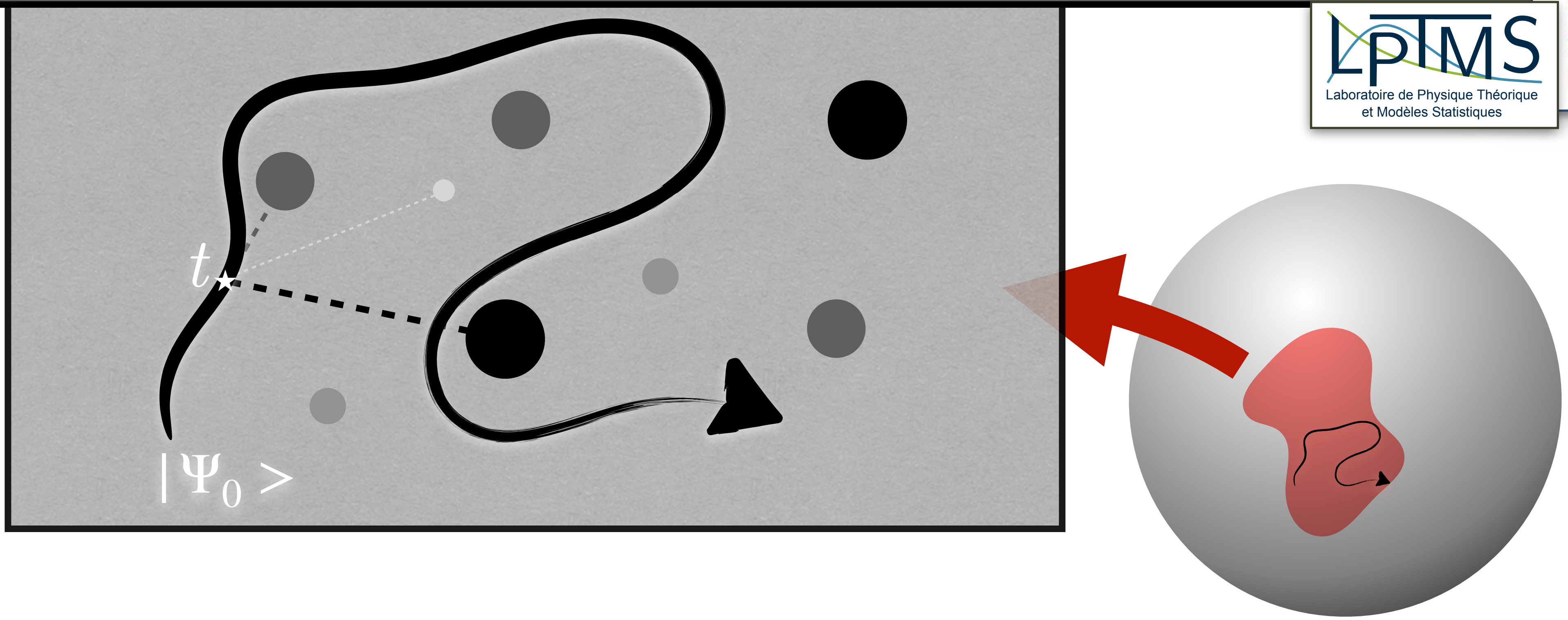
Entanglement evolution, generalized hydrodynamics, and invariant subspaces

Maurizio Fagotti



November 22 2019, Thermalization, Many Body Localization and Hydrodynamics - ICTS, Bangalore

(Entanglement evolution,)

Generalized hydrodynamics and invariant subspaces

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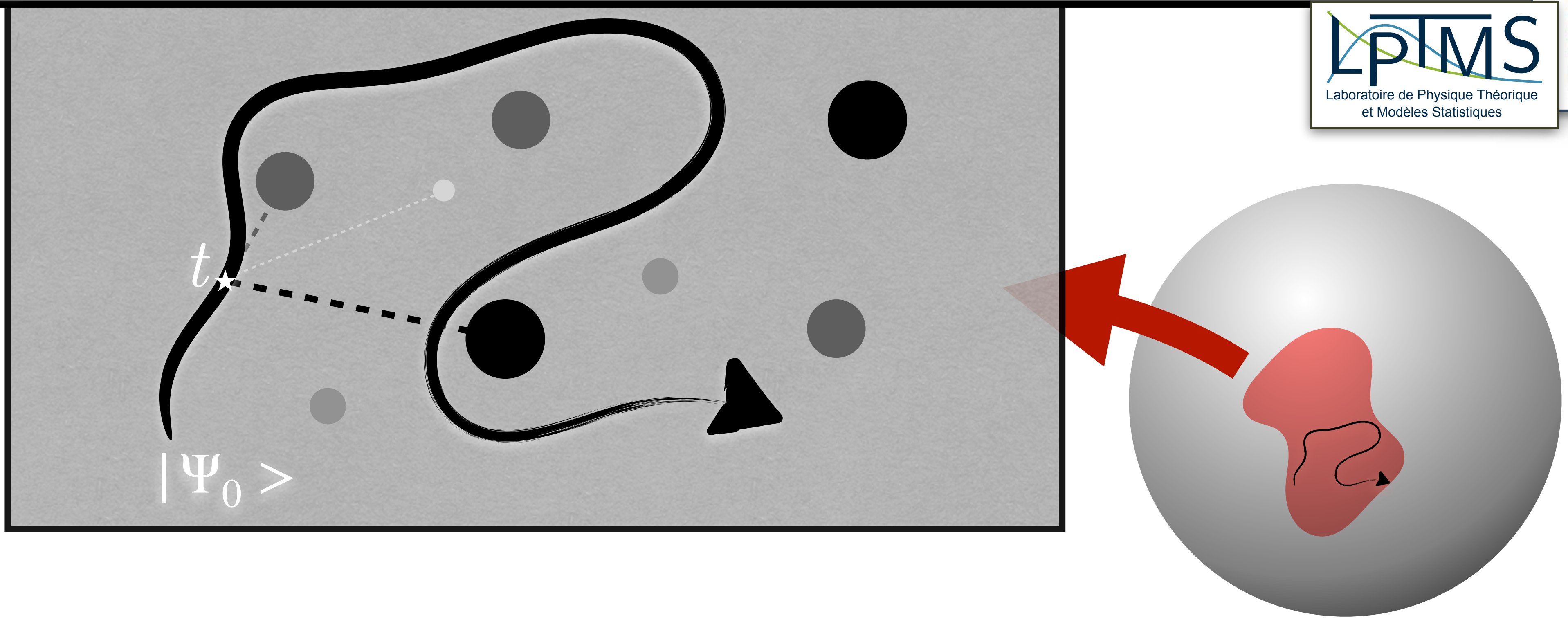

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Quench dynamics

coined by J. Cardy

QUANTUM QUENCH $g_0 \to g$

$$\boldsymbol{H}(g_0)|\Psi_0> \ = E_{\mathrm{GS}}|\Psi_0>$$
$$\boldsymbol{H} \ = \boldsymbol{H}(g)$$

a many-body system time evolves unitarily

$$|\Psi_t> \ = e^{-i\boldsymbol{H}t}|\Psi_0> \qquad (\boldsymbol{\rho} = |\Psi><\Psi|)$$
$$\boldsymbol{\rho}_t = e^{-i\boldsymbol{H}t}\boldsymbol{\rho}_0 e^{i\boldsymbol{H}t}$$

typical examples ——
- *spin lattice systems*
- *quantum field theories*

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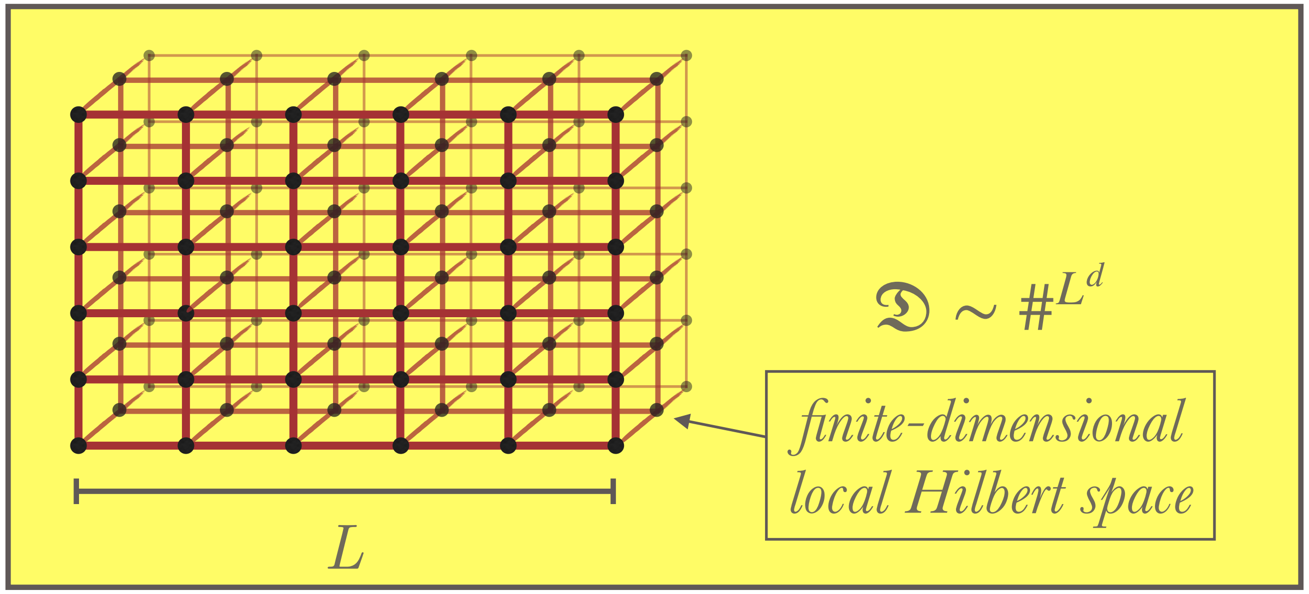
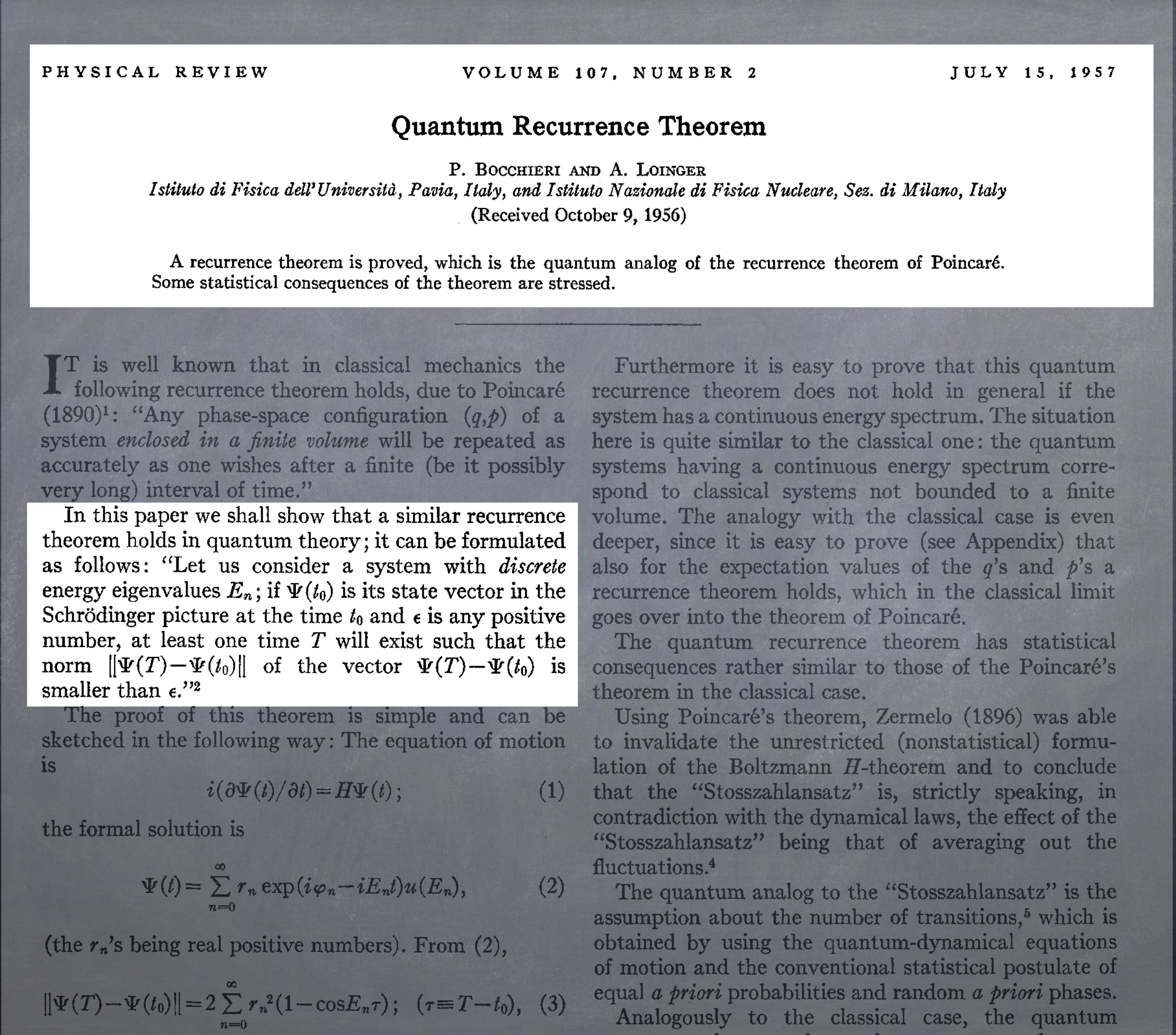




PHYSICAL REVIEW VOLUME 107, NUMBER 2 JULY 15, 1957

Quantum Recurrence Theorem

P. Bocchieri and A. Loinger
Istituto di Fisica dell'Università, Pavia, Italy, and Istituto Nazionale di Fisica Nucleare, Sez. di Milano, Italy

(Received October 9, 1956)

A recurrence theorem is proved, which is the quantum analog of the recurrence theorem of Poincaré. Some statistical consequences of the theorem are stressed.

IT is well known that in classical mechanics the following recurrence theorem holds, due to Poincaré (1890)[1]: "Any phase-space configuration (q,p) of a system *enclosed in a finite volume* will be repeated as accurately as one wishes after a finite (be it possibly very long) interval of time."

In this paper we shall show that a similar recurrence theorem holds in quantum theory; it can be formulated as follows: "Let us consider a system with *discrete* energy eigenvalues E_n; if $\Psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least one time T will exist such that the norm $\|\Psi(T)-\Psi(t_0)\|$ of the vector $\Psi(T)-\Psi(t_0)$ is smaller than ϵ."[2]

The proof of this theorem is simple and can be sketched in the following way: The equation of motion is

$$i(\partial\Psi(t)/\partial t)=H\Psi(t); \qquad (1)$$

the formal solution is

$$\Psi(t)=\sum_{n=0}^{\infty} r_n \exp(i\varphi_n-iE_nt)u(E_n), \qquad (2)$$

(the r_n's being real positive numbers). From (2),

$$\|\Psi(T)-\Psi(t_0)\|=2\sum_{n=0}^{\infty} r_n^2(1-\cos E_n\tau); \quad (\tau\equiv T-t_0), \qquad (3)$$

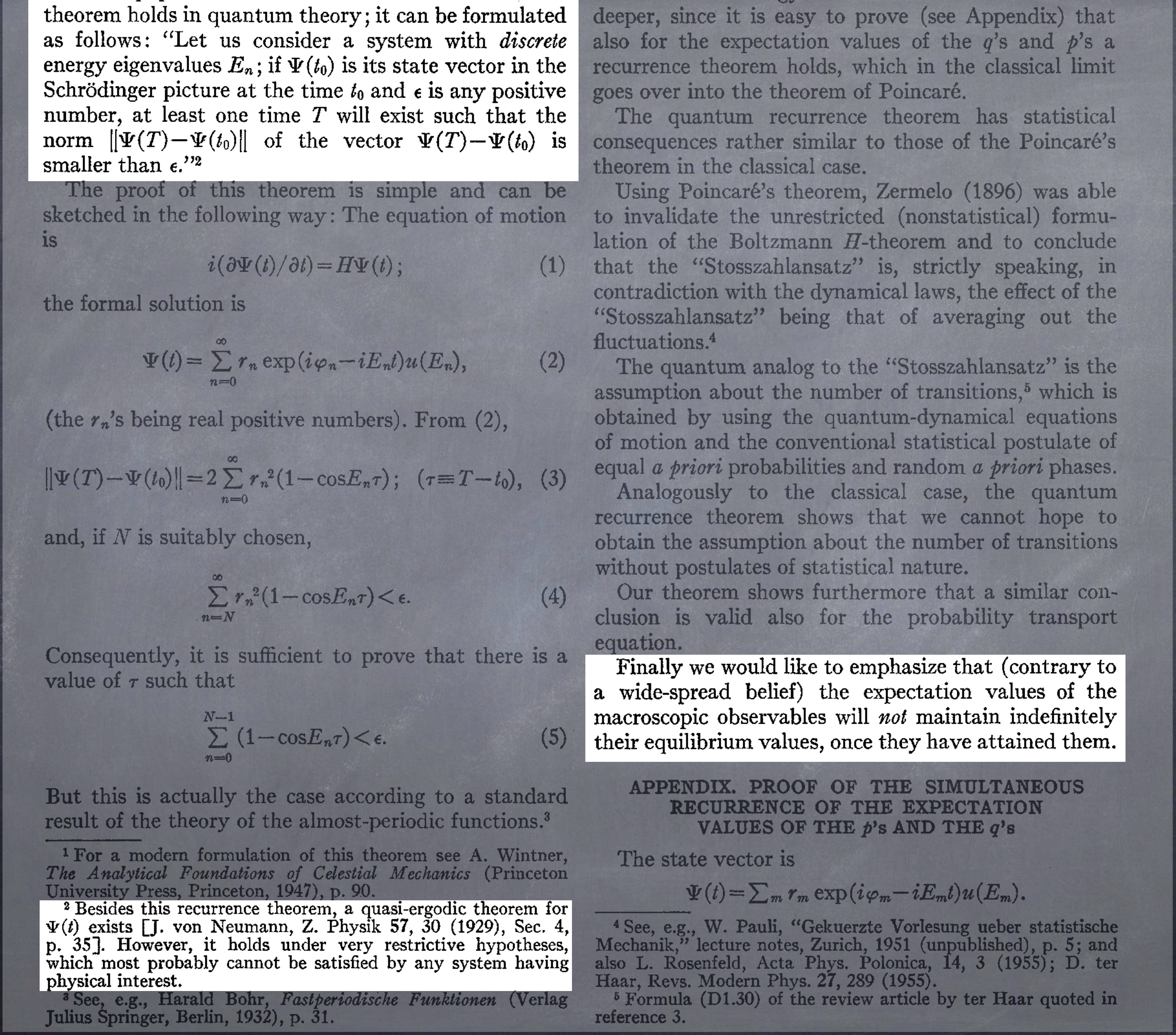
Furthermore it is easy to prove that this quantum recurrence theorem does not hold in general if the system has a continuous energy spectrum. The situation here is quite similar to the classical one: the quantum systems having a continuous energy spectrum correspond to classical systems not bounded to a finite volume. The analogy with the classical case is even deeper, since it is easy to prove (see Appendix) that also for the expectation values of the q's and p's a recurrence theorem holds, which in the classical limit goes over into the theorem of Poincaré.

The quantum recurrence theorem has statistical consequences rather similar to those of the Poincaré's theorem in the classical case.

Using Poincaré's theorem, Zermelo (1896) was able to invalidate the unrestricted (nonstatistical) formulation of the Boltzmann H-theorem and to conclude that the "Stosszahlansatz" is, strictly speaking, in contradiction with the dynamical laws, the effect of the "Stosszahlansatz" being that of averaging out the fluctuations.[4]

The quantum analog to the "Stosszahlansatz" is the assumption about the number of transitions,[5] which is obtained by using the quantum-dynamical equations of motion and the conventional statistical postulate of equal *a priori* probabilities and random *a priori* phases.

Analogously to the classical case, the quantum



theorem holds in quantum theory; it can be formulated as follows: "Let us consider a system with *discrete* energy eigenvalues E_n; if $\Psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least one time T will exist such that the norm $\|\Psi(T)-\Psi(t_0)\|$ of the vector $\Psi(T)-\Psi(t_0)$ is smaller than ϵ."[2]

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and, if N is suitably chosen,

$$\sum_{n=N}^{\infty} r_n^2(1-\cos E_n\tau)<\epsilon. \tag{4}$$

Consequently, it is sufficient to prove that there is a value of τ such that

$$\sum_{n=0}^{N-1}(1-\cos E_n\tau)<\epsilon. \tag{5}$$

But this is actually the case according to a standard result of the theory of the almost-periodic functions.[3]

deeper, since it is easy to prove (see Appendix) that also for the expectation values of the q's and p's a recurrence theorem holds, which in the classical limit goes over into the theorem of Poincaré.

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Analogously to the classical case, the quantum recurrence theorem shows that we cannot hope to obtain the assumption about the number of transitions without postulates of statistical nature.

Our theorem shows furthermore that a similar conclusion is valid also for the probability transport equation.

Finally we would like to emphasize that (contrary to a wide-spread belief) the expectation values of the macroscopic observables will *not* maintain indefinitely their equilibrium values, once they have attained them.

APPENDIX. PROOF OF THE SIMULTANEOUS RECURRENCE OF THE EXPECTATION VALUES OF THE p's AND THE q's

The state vector is

$$\Psi(t)=\sum_m r_m \exp(i\varphi_m-iE_mt)u(E_m).$$

---

[1] For a modern formulation of this theorem see A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, 1947), p. 90.

[2] Besides this recurrence theorem, a quasi-ergodic theorem for $\Psi(t)$ exists [J. von Neumann, Z. Physik **57**, 30 (1929), Sec. 4, p. 35]. However, it holds under very restrictive hypotheses, which most probably cannot be satisfied by any system having physical interest.

[3] See, e.g., Harald Bohr, *Fastperiodische Funktionen* (Verlag Julius Springer, Berlin, 1932), p. 31.

[4] See, e.g., W. Pauli, "Gekuerzte Vorlesung ueber statistische Mechanik," lecture notes, Zurich, 1951 (unpublished), p. 5; and also L. Rosenfeld, Acta Phys. Polonica, **14**, 3 (1955); D. ter Haar, Revs. Modern Phys. **27**, 289 (1955).

[5] Formula (D1.30) of the review article by ter Haar quoted in reference 3.

Quench dynamics

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ways to circumvent it
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relaxation "on average"

Rigol's lecture

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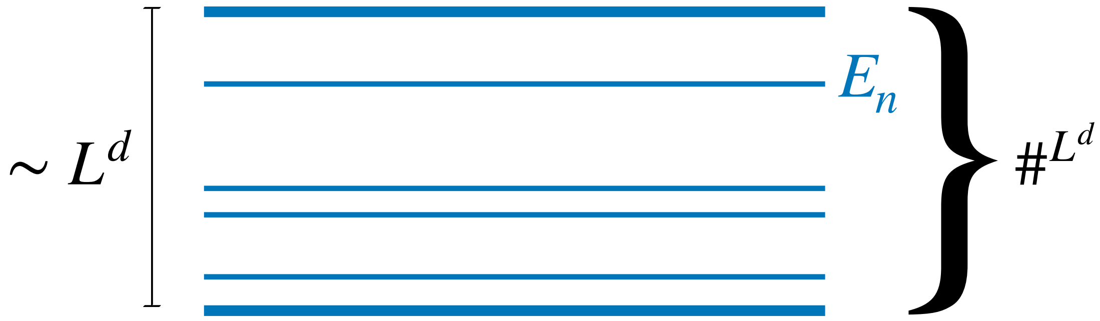
thermodynamic limit

- *thermalization in generic systems*
- *relaxation to GGEs in integrable systems*

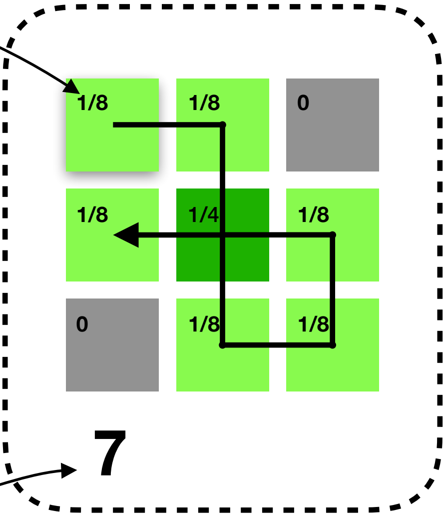


Time averaged state

$$\bar{\rho}_{t_0,t} = \int_{t_0}^{t_0+t} \frac{\mathrm{d}\tau}{t} \, |\Psi(\tau) > < \Psi(\tau)| = e^{-i\boldsymbol{H}t_0} \, \bar{\rho}_{0,t} \, e^{i\boldsymbol{H}t_0}$$

✦ if $| < \Psi_{t_1} | \Psi_{t_2} > |^2 \in \{0,1\}$:

- the eigenvalues of $\bar{\rho}_{0,t}$ would be **the fraction of time in** $[t_0, t_0 + t]$ **spent in the corresponding eigenstate**
- the number of nonzero eigenvalues would give the **size of the space visited in the interval** $[t_0, t_0 + t]$



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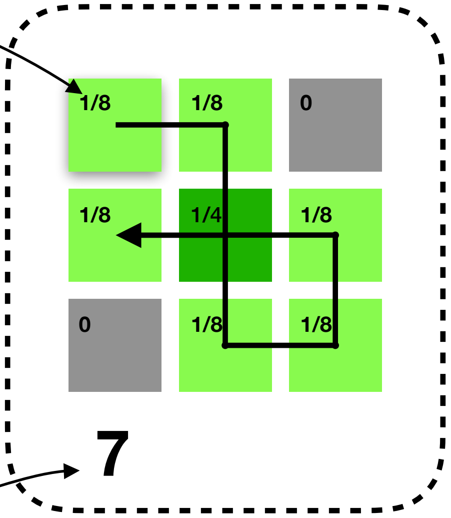

- ✦ in fact, $\bar{\rho}_{t_0,t} | \Psi_\tau > \not\propto | \Psi_\tau >$, but
  - the state at any time in $[t_0, t_0 + t]$ is a linear combination of the eigenvectors of $\bar{\rho}_{t_0,t}$ with **nonzero eigenvalues**
  - the projection onto the subspace generated by the eigenvectors with the **largest eigenvalues** gives the "best" approximation…

Time averaged state

Energy cumulants

$$\kappa_n = \partial_t^n \log \Big|_{t=0} < \Psi_0 | e^{t\boldsymbol{H}} | \Psi_0 > \qquad < \Psi_0 | \Psi_t > \overset{t \approx 0}{=} \exp\Big(\sum_{n=1}^{\infty} (-i)^n \frac{\kappa_n t^n}{n!} \Big)$$

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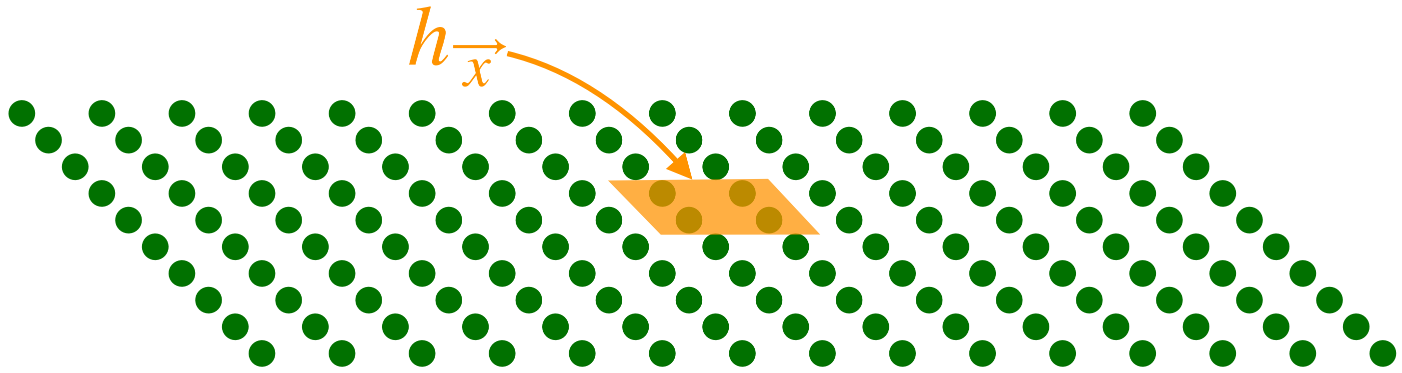
(Quasi)local Hamiltonian

$$\boldsymbol{H} = \sum_{\vec{x}} \boldsymbol{h}_{\vec{x}}$$



Ground state in the generic case *(finite correlation lengths)*

$$< \Psi_0 | \boldsymbol{O}_{\vec{x}} \boldsymbol{O}_{\vec{x}+\vec{r}} | \Psi_0 > - < \Psi_0 | \boldsymbol{O}_{\vec{x}} | \Psi_0 > < \Psi_0 | \boldsymbol{O}_{\vec{x}+\vec{r}} | \Psi_0 > \sim e^{-\frac{r}{\xi}}$$

Ground state in critical systems *(e.g. infinite correlation length in 1d)*

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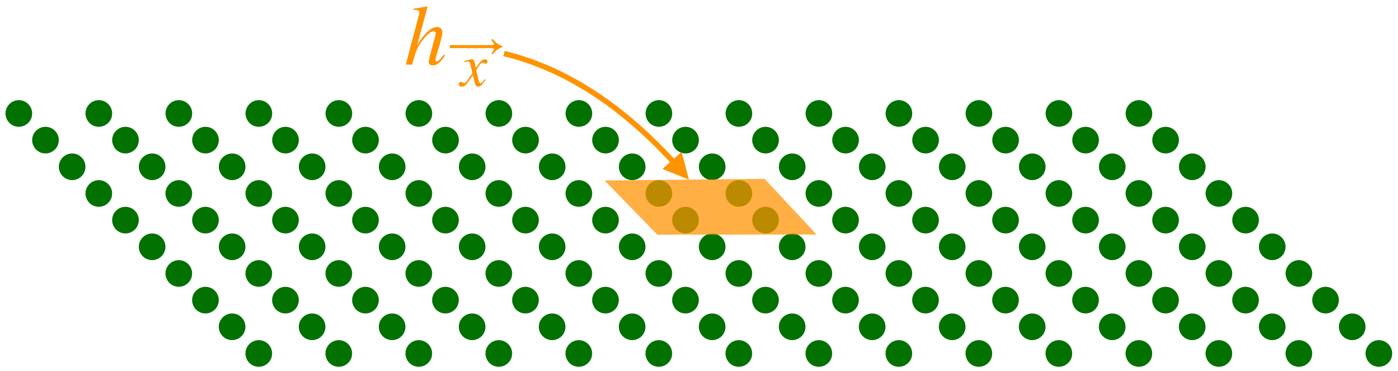

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extensive energy cumulants $\kappa_n = L^d \mathfrak{e}_n$

MF, SciPost Phys. **6**, 059 (2019)

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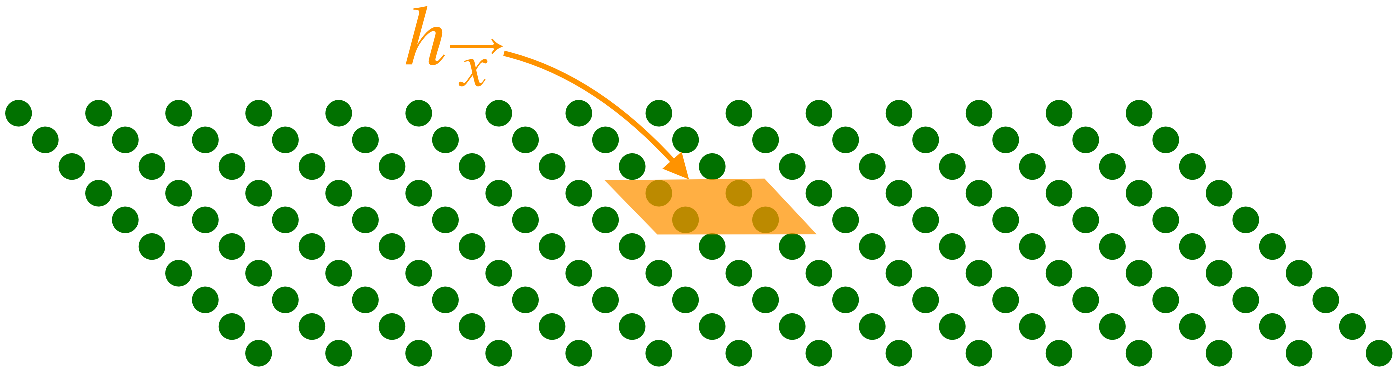

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hyperscaling $\kappa_n = L^{(2-\alpha)n} \mathfrak{e}_n \quad (d=1)$

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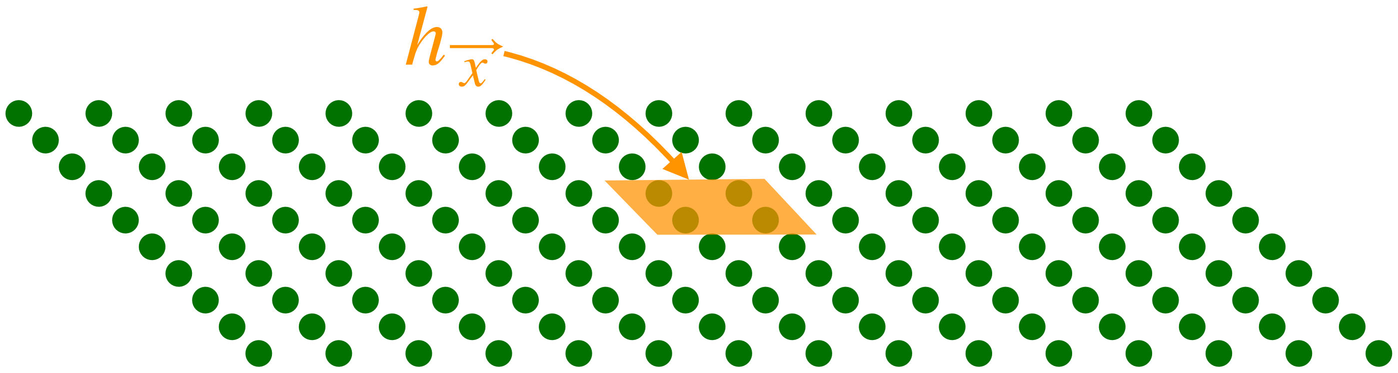

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MF, SciPost Phys. **6**, 059 (2019)

$$\mathrm{tr}[\bar{\boldsymbol{\rho}}^{\alpha}_{0,t}] = \mathrm{tr}[\cdots\bar{\boldsymbol{\rho}}^{2}_{0,t}\cdots] = \mathrm{tr}[\cdots \iint_{0}^{t} \frac{\mathrm{d}\tau}{t^{2}} \,|\,\Psi_{\tau_1} > < \Psi_{\tau_1}|\Psi_{\tau_2} > < \Psi_{\tau_2}|\cdots]$$

exponentially small in the system's size for any nonzero $\tau_1 - \tau_2$

→

the integration domain can be reduced into a region where $\log < \Psi_{\tau_1}|\Psi_{\tau_2} >$ can be series expanded about $\tau_1 \approx \tau_2$

MF, SciPost Phys. **6**, 059 (2019)

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asymptotic expansion in the limit of a large number of sites

$$\mathrm{tr}[\bar{\boldsymbol{\rho}}_t^{\alpha}] \sim \iiint_{[0,t\sqrt{L}]^{\alpha}} \frac{\mathrm{d}^{\alpha}\boldsymbol{\tau}}{t^{\alpha} L^{d\frac{\alpha}{2}}} e^{-\mathfrak{e}_2 \frac{(\tau_\alpha - \tau_1)^2 + \sum_{j=1}^{\alpha-1} (\tau_j - \tau_{j+1})^2}{2}} \sim \alpha^{-\frac{1}{2}} (\frac{\mathfrak{e}_2}{2\pi})^{\frac{1-\alpha}{2}} t^{1-\alpha} L^{d\frac{1-\alpha}{2}}$$

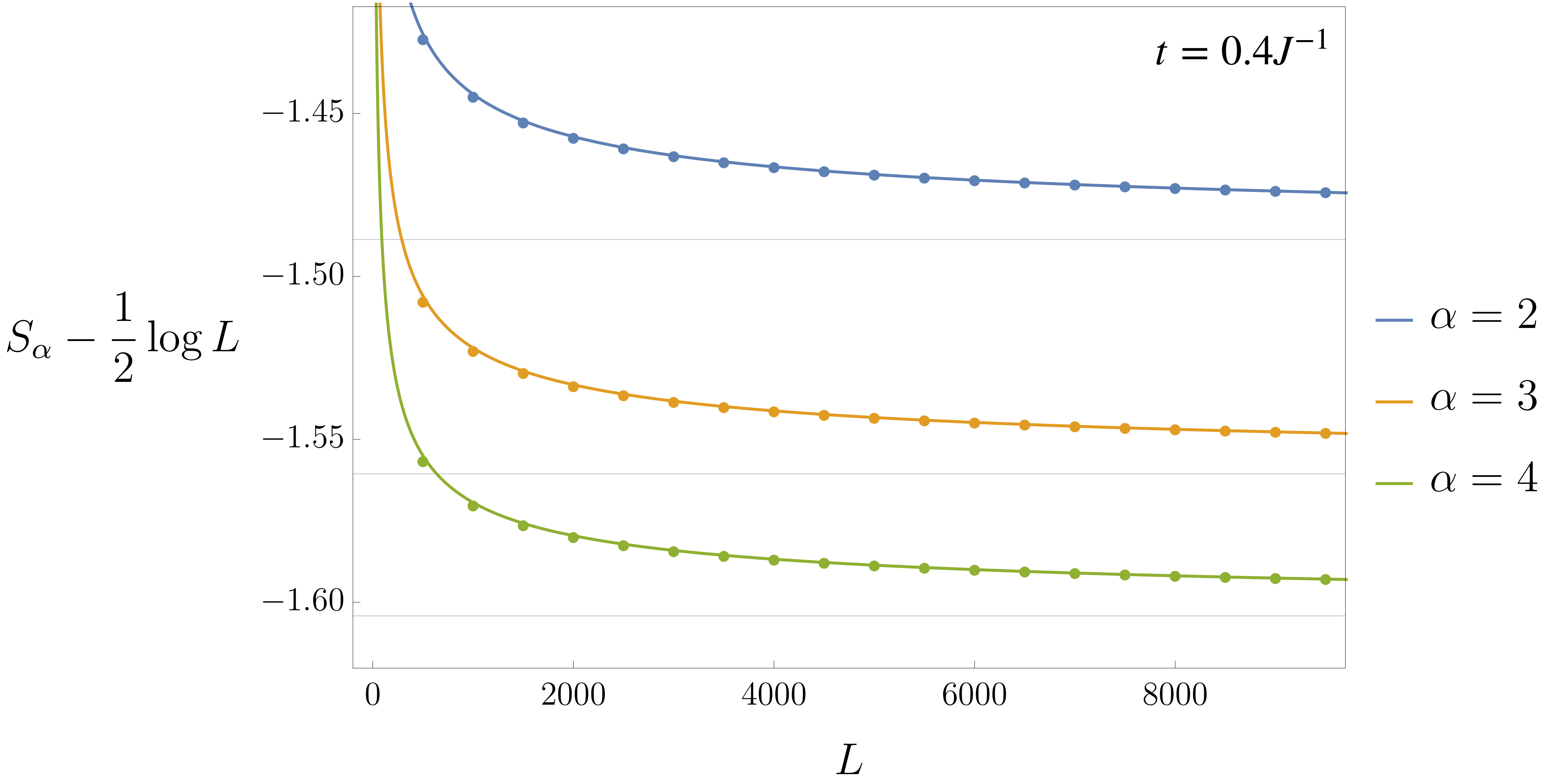
$$S_\alpha[\bar{\boldsymbol{\rho}}_t] = \frac{d}{2} \log L + \frac{1}{2} \log \frac{\mathfrak{e}_2 t^2}{2\pi} + \frac{\log \alpha}{2(\alpha - 1)} + O(L^{-\frac{d}{2}})$$

$$S_{vN}[\bar{\boldsymbol{\rho}}_t] \sim \frac{d}{2} \log L + \frac{1}{2} \log \frac{\mathfrak{e}_2 t^2}{2\pi} + \frac{1}{2}$$

MF, SciPost Phys. **6**, 059 (2019)

$$\boldsymbol{H}(h) = -J\sum_{\ell}\left(\boldsymbol{\sigma}^x_\ell\boldsymbol{\sigma}^x_{\ell+1} + h\boldsymbol{\sigma}^z_\ell\right)$$

quantum quench $h = \infty \to 1.5$



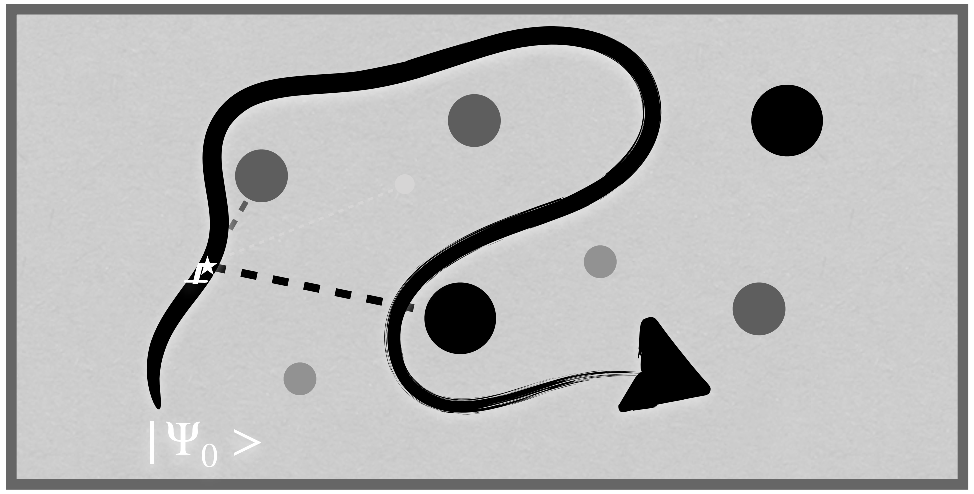
Effective Size of the Space

MF, SciPost Phys. **6**, 059 (2019)

let $\mathfrak{D}_t^{(\epsilon)}$ be the size of the space needed to approximate the time averaged state with error ϵ

$$\mathfrak{D}_t^{(\epsilon)} = \mathrm{tr}[\theta_H(\bar{\boldsymbol{\rho}}_{t_0,t} - \lambda_\epsilon)]$$

$$\epsilon = \mathrm{tr}[\bar{\boldsymbol{\rho}}_{t_0,t}\theta_H(\lambda_\epsilon - \bar{\boldsymbol{\rho}}_{t_0,t})]$$



$$\boxed{\mathfrak{D}_t^{(\epsilon)} \sim \frac{\sqrt{2\mathfrak{e}_2}}{\pi}\mathrm{erf}^{-1}(1-\epsilon)L^{\frac{d}{2}}t}$$

one can infer that ϵ is the error on the state

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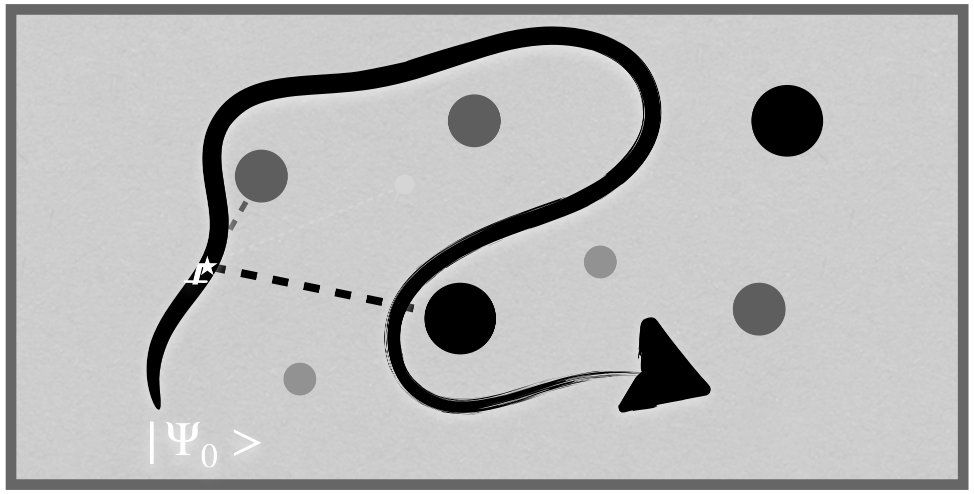

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approximate support of $e^{iHt}Oe^{-iHt}$

$\mathrm{supp}(\boldsymbol{O})$

$\ell = |\mathrm{supp}(O)|$

Lieb and Robinson, Comm. Math. Phys. **28**, 251 (1972)

Lieb-Robinson bound

L **can be replaced by** $\tilde{L} \leq \ell + 2v_{\mathrm{LR}}t + 2\xi$

correlation length in the initial state

Time averaged state

$$\boxed{\bar{\rho}_{t_0,t} = \int_{t_0}^{t_0+t} \frac{\mathrm{d}\tau}{t} \,|\,\Psi(\tau) > < \Psi(\tau)\,| = e^{-iHt_0}\,\bar{\rho}_{0,t}\,e^{iHt_0}}$$

Renyi entropies: $S_\alpha = \frac{1}{1-\alpha} \log \mathrm{tr}[\bar{\rho}_{0,t}^{\alpha}]$

(von Neumann entropy: $S_{vN} = -\,\mathrm{tr}[\bar{\rho}_{0,t} \log \bar{\rho}_{0,t}]$**)**

complete characterisation of the eigenvalue distribution *(Hausdorff moment problem)*

$$\mathrm{tr}[\bar{\boldsymbol{\rho}}_{0,t}^{\alpha}] = \mathrm{tr}[\cdots\bar{\boldsymbol{\rho}}_{0,t}^{2}\cdots] = \mathrm{tr}[\cdots \iint_0^t \frac{\mathrm{d}\tau}{t^2} \,|\,\Psi_{\tau_1} > < \Psi_{\tau_1}|\Psi_{\tau_2} > < \Psi_{\tau_2}\,|\cdots]$$

$$< \Psi_{t_1}|\Psi_{t_2} > = < \Psi_0\,|\,e^{iH(t_1-t_2)}\,|\,\Psi_0 > \sim \exp\left(\sum_{n=1}^{\infty} (-i)^n \frac{\kappa_n (t_2 - t_1)^n}{n!}\right)$$

↓

Loschmidt echo

Time averaged state

$$\bar{\rho}_{t_0,t} = \int_{t_0}^{t_0+t} \frac{\mathrm{d}\tau}{t} \,|\,\Psi(\tau) > < \Psi(\tau)\,| = e^{-iHt_0}\,\bar{\rho}_{0,t}\,e^{iHt_0}$$

Renyi entropies: $S_\alpha = \frac{1}{1-\alpha} \log \mathrm{tr}[\bar{\rho}_{0,t}^{\alpha}]$

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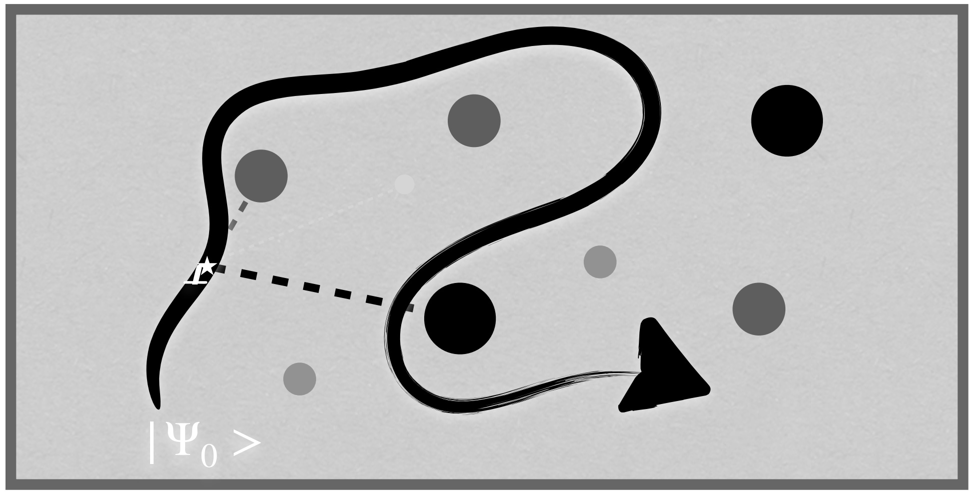
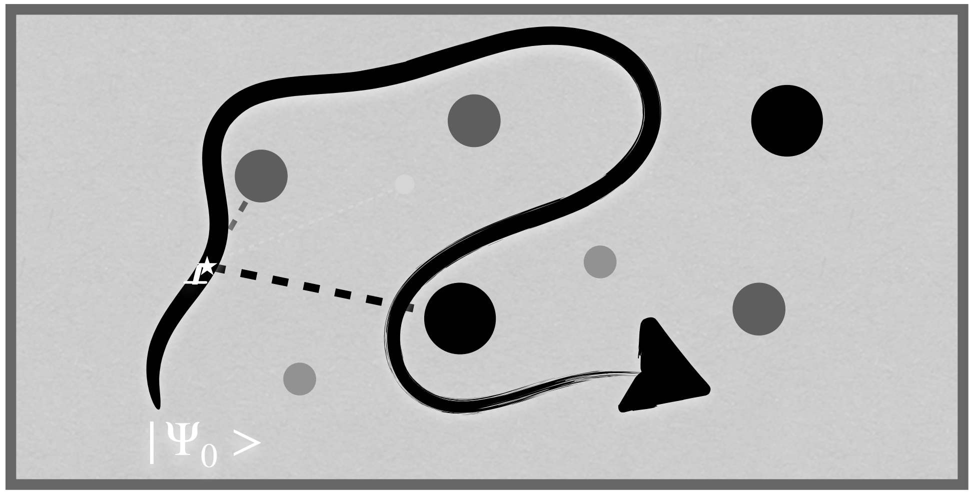
$$\big\downarrow$$

Loschmidt echo



tiny effective space ⟶ $|\Psi_t>$ characterised by a reduced number of fields

1) local relaxation	2) invariant subspaces



tiny effective space → $|\Psi_t >$ **characterised by a reduced number of fields**

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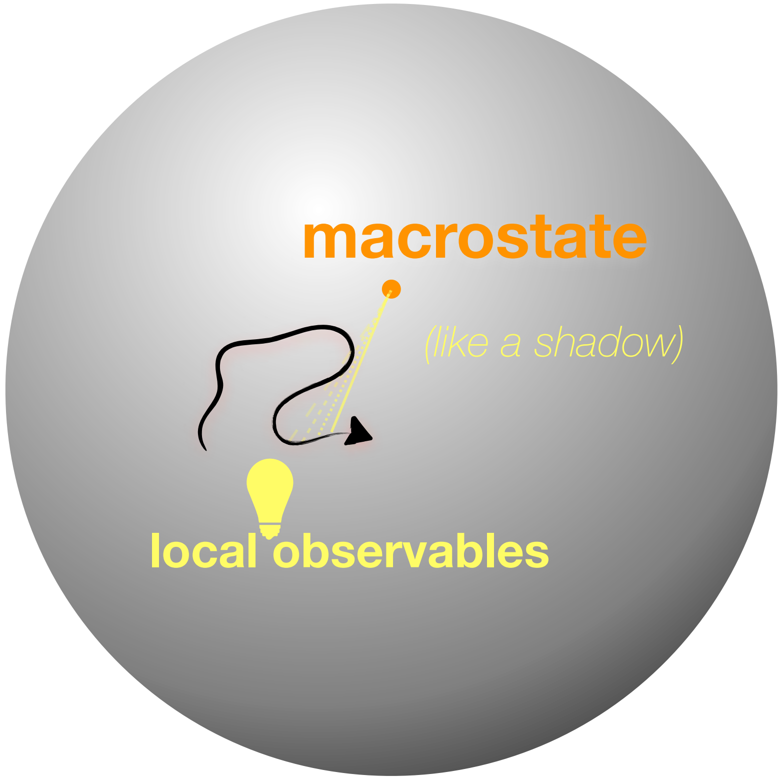
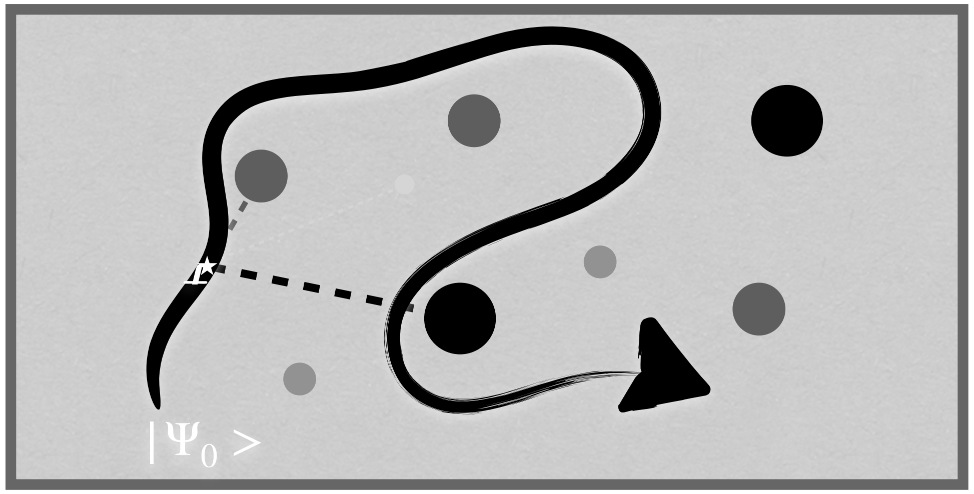
$$|\Psi_t > \xrightarrow{\overset{\overset{t \to \infty}{\dots}}{\textbf{locally}}} |\{\rho\} >$$





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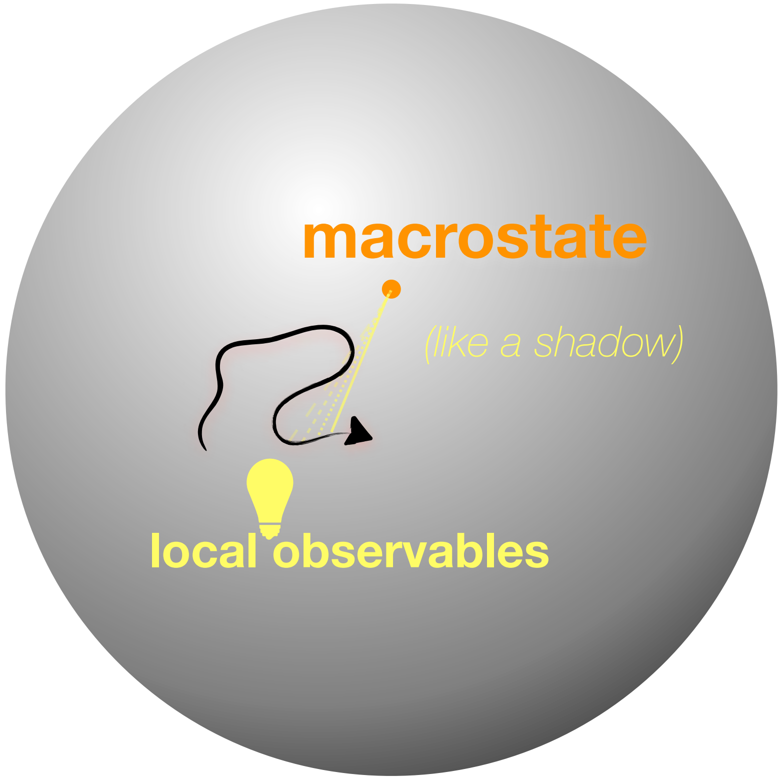
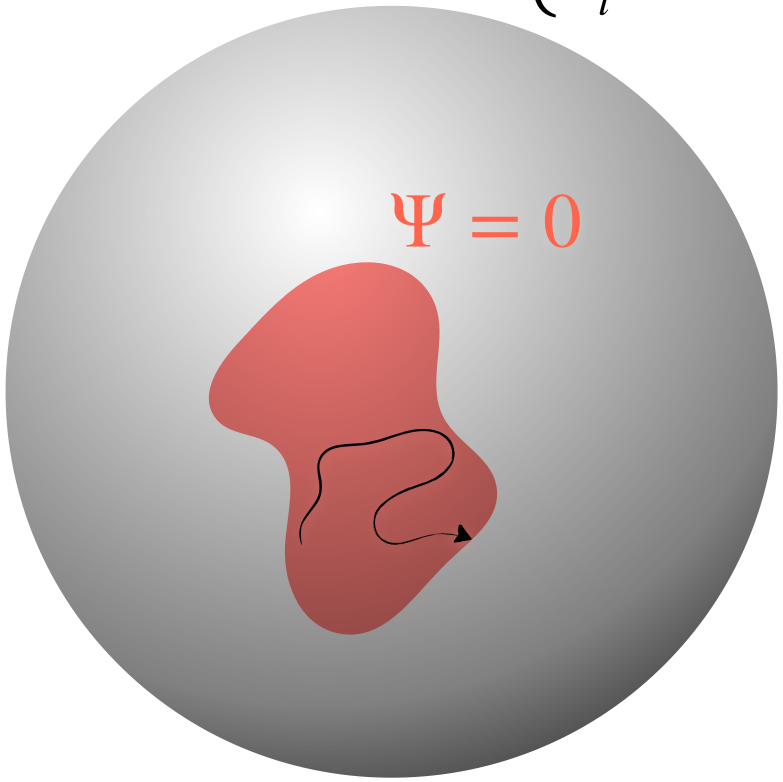
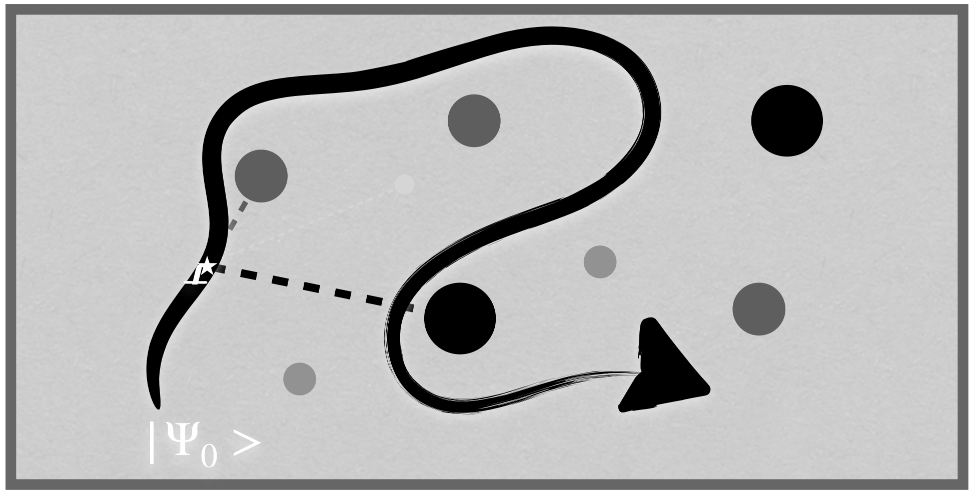


2) invariant subspaces

$$|\Psi_t > = |\{\rho_t\},\{\Psi_t\} > \Rightarrow \begin{cases} \partial_t \rho = F(\{\rho\}) \\ \partial_t \Psi = F(\{\Psi\}) \end{cases}$$



trivial example: stationary states



tiny effective space ⟶ $|\Psi_t >$ characterised by a reduced number of fields

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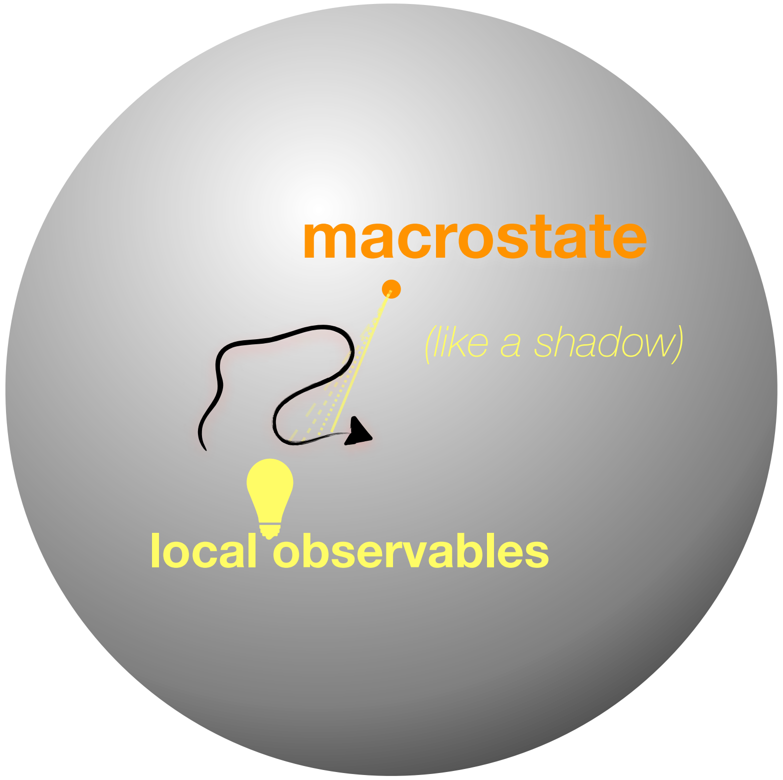

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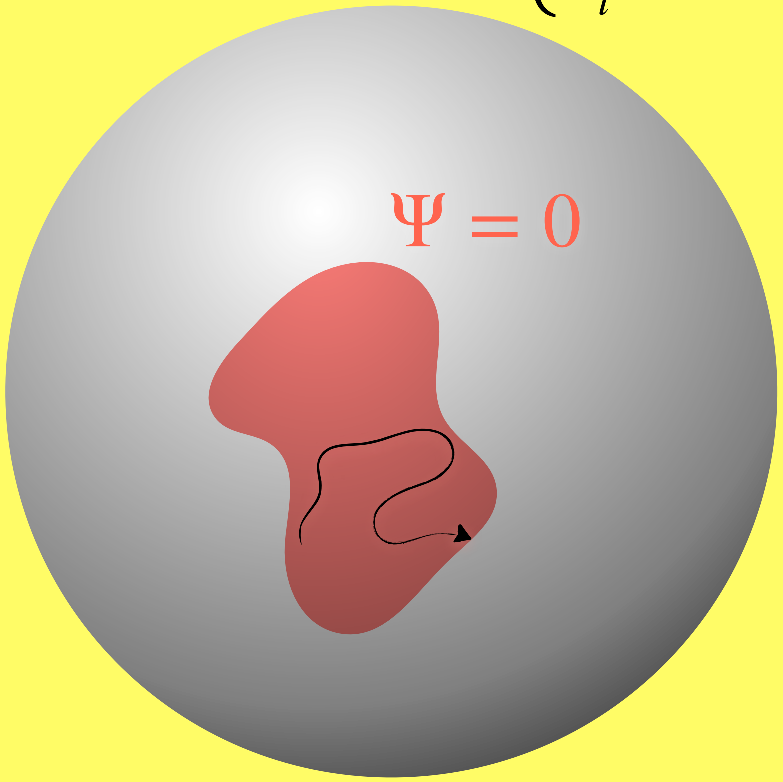
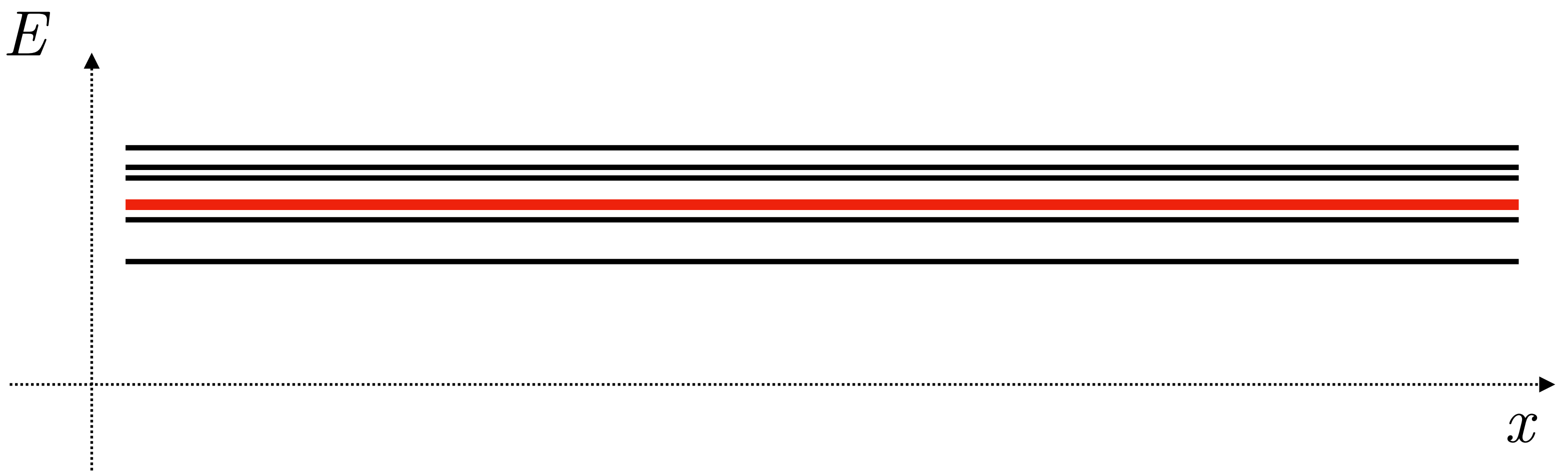


trivial example: stationary states

H translationally invariant



stationary state

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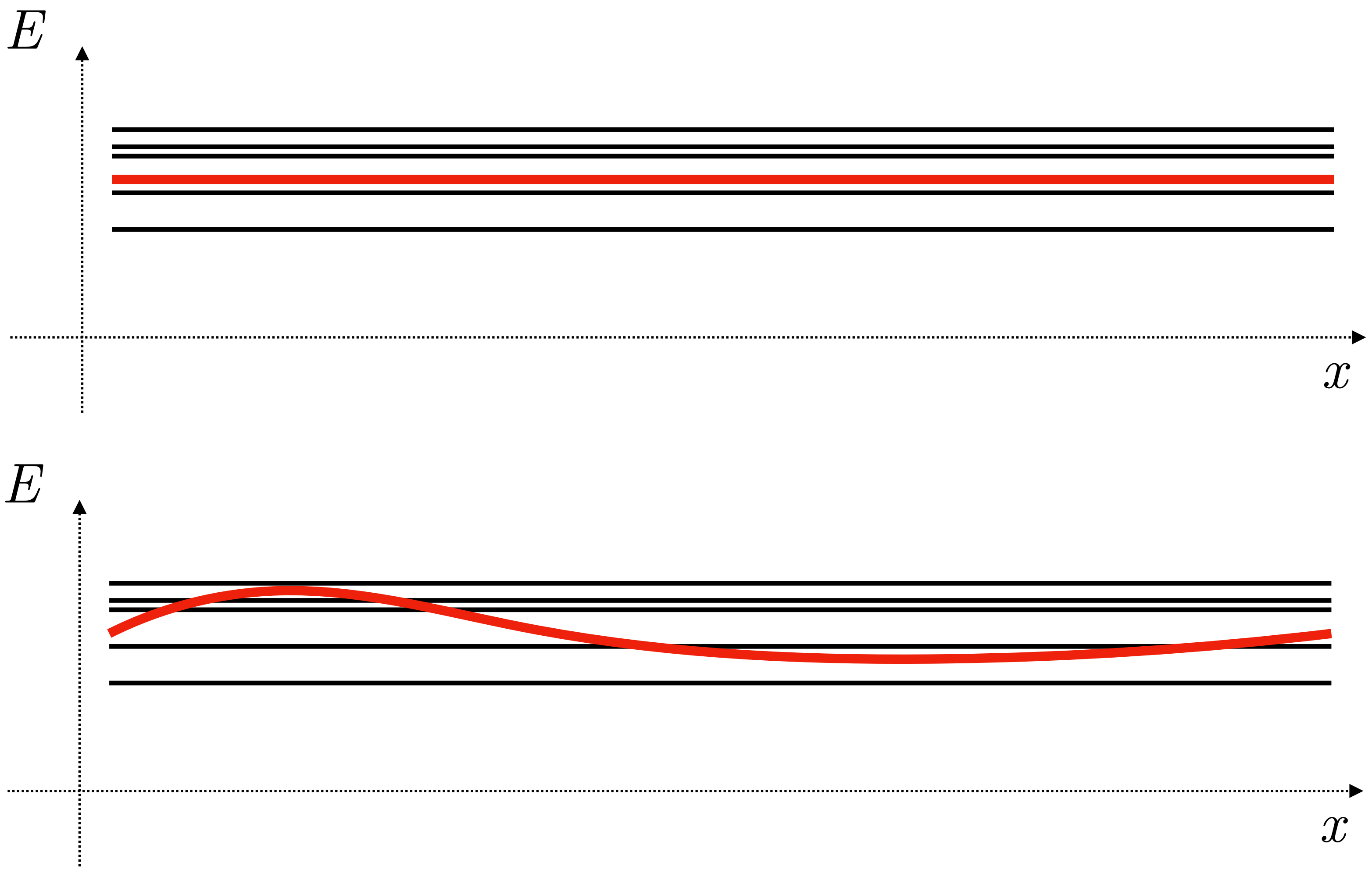

stationary state

locally quasi-stationary state

Bertini, MF, Phys. Rev. Lett. **117**, 130402 (2016)

1. Does it exist an **invariant subspace** of states that are stationary in the homogeneous limit?
2. How can the states be parametrised in that subspace?

let's start considering stationary states…

Integrable systems with a TBA description

$$H = -\sum_{\ell} s^x_\ell s^x_{\ell+1} + \frac{h}{2} s^z_\ell$$

$$H = \sum_{\ell} s^x_\ell s^x_{\ell+1} + s^y_\ell s^y_{\ell+1} + \Delta s^z_\ell s^z_{\ell+1}$$

	free-fermion system (e.g., *transverse-field Ising chain*)	**interacting integrable system** (e.g., *XXZ spin-1/2 chain*)
excited state	$\lvert \lambda_1, \lambda_2, \ldots \rangle = [b^\dagger_{\lambda_1} b^\dagger_{\lambda_2} \cdots] \lvert \varnothing \rangle$ $\{b^\dagger_\lambda, b_\mu\} = \delta_{\lambda\mu}$ $\{b^\dagger_\lambda, b^\dagger_\mu\} = 0$	$\lvert \lambda_1, \lambda_2, \ldots \rangle = [B(\lambda_1)B(\lambda_2)\cdots] \lvert \varnothing \rangle$
energy	$E = \sum_{\lambda' \in \{\lambda\}} e(\lambda')$	
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energy	$\sim L\int \mathrm{d}\lambda\rho(\lambda)e(\lambda) \quad \xleftarrow{L\to\infty}$	$E = \sum_{\lambda'\in\{\lambda\}} e(\lambda')$	$\xrightarrow{L\to\infty} \quad \sim L\sum_n \int \mathrm{d}\lambda\rho_n(\lambda)e_n(\lambda)$
momentum	$\sim L\int \mathrm{d}\lambda\rho(\lambda)p(\lambda) \quad \xleftarrow{L\to\infty}$	$P = \sum_{\lambda'\in\{\lambda\}} p(\lambda')$	$\xrightarrow{L\to\infty} \quad \sim L\sum_n \int \mathrm{d}\lambda\rho_n(\lambda)p_n(\lambda)$
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energy

$\boxed{\rho(\lambda)}$ **root densities** $\boxed{\rho_n(\lambda)}$

momentum

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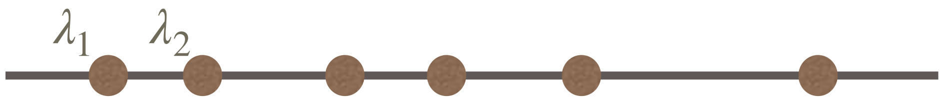
excitations



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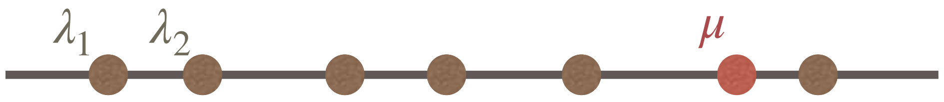
excitations	 *particle excitation*	

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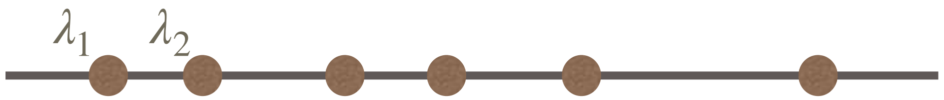
excitations		

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<table>
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(e.g., <i>transverse-field Ising chain</i>)</th><th>interacting integrable system
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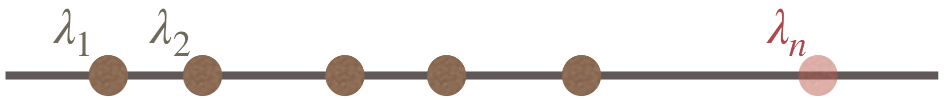
<tr><td>excitations</td><td>
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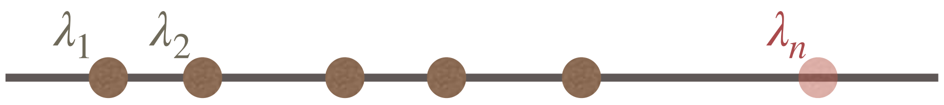

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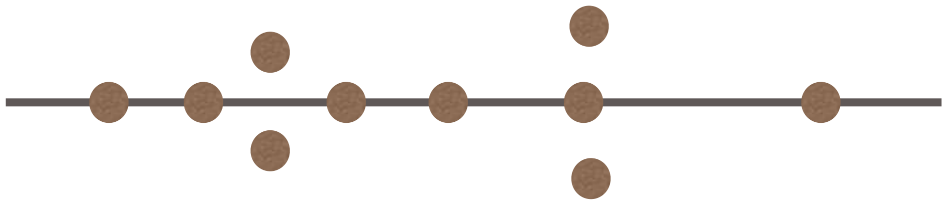

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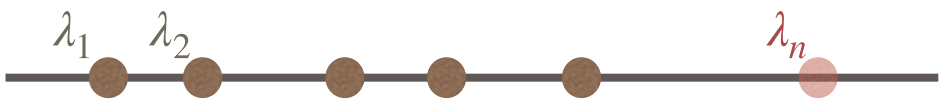
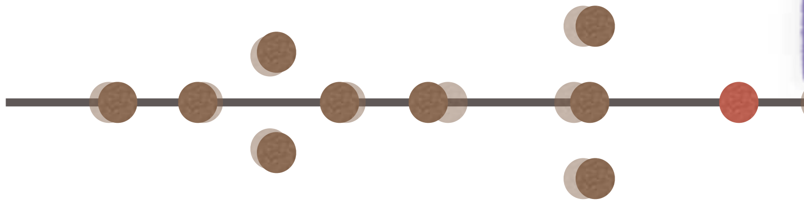
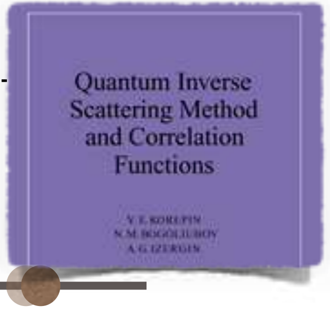
excitations	 *hole excitation*		 *particle excitation*



Quantum Inverse Scattering Method and Correlation Functions

V. E. KOREPIN
N. M. BOGOLIUBOV
A. G. IZERGIN

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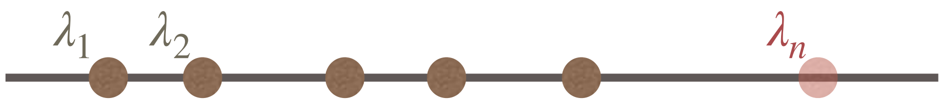
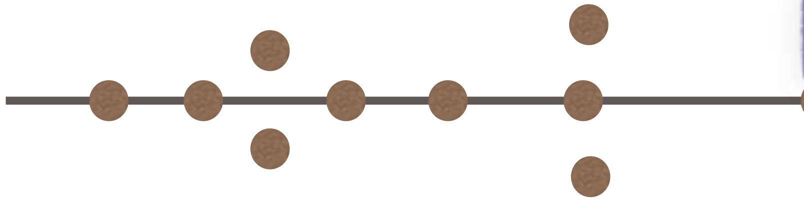
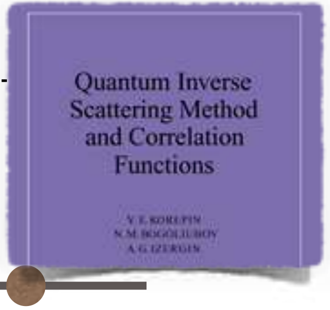
$$\sim L\int \mathrm{d}\lambda \rho(\lambda) q(\lambda) \quad \xleftarrow{L\to\infty} \quad Q = \sum_{\lambda' \in \{\lambda\}} q(\lambda') \quad \xrightarrow{L\to\infty} \quad \sim L \sum_n \int \mathrm{d}\lambda \rho_n(\lambda) q_n(\lambda)$$

excitations



hole excitation




Quantum Inverse Scattering Method and Correlation Functions

Integrable systems with a TBA description

$$H=-\sum_\ell s^x_\ell s^x_{\ell+1}+\frac{h}{2}s^z_\ell$$

$$H=\sum_\ell s^x_\ell s^x_{\ell+1}+s^y_\ell s^y_{\ell+1}+\Delta s^z_\ell s^z_{\ell+1}$$

	free-fermion system (e.g., *transverse-field Ising chain*)	**interacting integrable system** (e.g., *XXZ spin-1/2 chain*)
excited state	$\lvert\lambda_1,\lambda_2,\ldots\rangle=[b^\dagger_{\lambda_1}b^\dagger_{\lambda_2}\cdots]\lvert\varnothing\rangle$ $\{b^\dagger_\lambda,b_\mu\}=\delta_{\lambda\mu}$ $\{b^\dagger_\lambda,b^\dagger_\mu\}=0$	$\lvert\lambda_1,\lambda_2,\ldots\rangle=[B(\lambda_1)B(\lambda_2)\cdots]\lvert\varnothing\rangle$

energy

$\boxed{\rho(\lambda)}$ **root densities** $\boxed{\rho_n(\lambda)}$

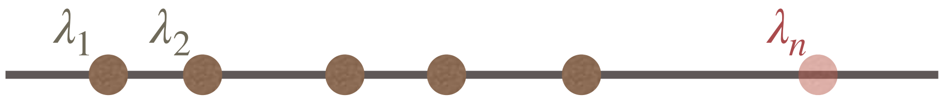
momentum

$$\sim L\int \mathrm{d}\lambda\,\rho(\lambda)p(\lambda)\ \xleftarrow{L\to\infty}\ P=\sum_{\lambda'\in\{\lambda\}}p(\lambda')\ \xrightarrow{L\to\infty}\ \sim L\sum_n\int \mathrm{d}\lambda\,\rho_n(\lambda)p_n(\lambda)$$

local charge

$$\sim L\int \mathrm{d}\lambda\,\rho(\lambda)q(\lambda)\ \xleftarrow{L\to\infty}\ Q=\sum_{\lambda'\in\{\lambda\}}q(\lambda')\ \xrightarrow{L\to\infty}\ \sim L\sum_n\int \mathrm{d}\lambda\,\rho_n(\lambda)q_n(\lambda)$$

excitations



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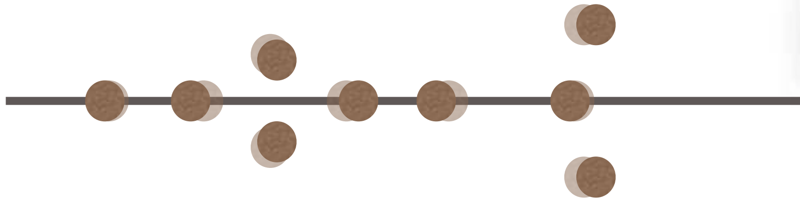

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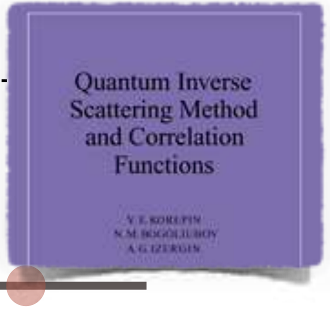
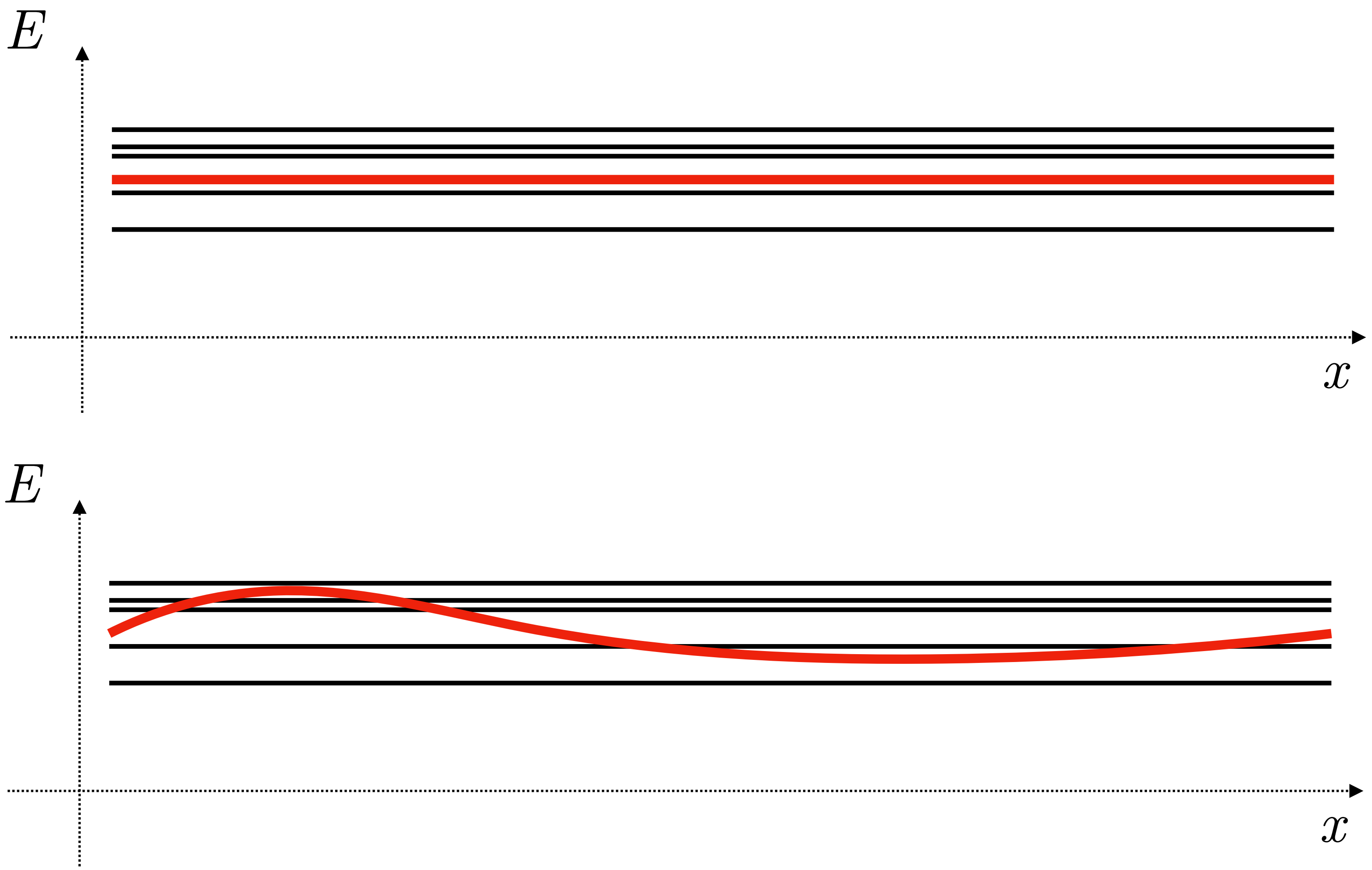


Quantum Inverse Scattering Method and Correlation Functions

V. E. KOREPIN
N.M. BOGOLIUBOV
A.G. IZERGIN

H translationally invariant



stationary state

locally quasi-stationary state

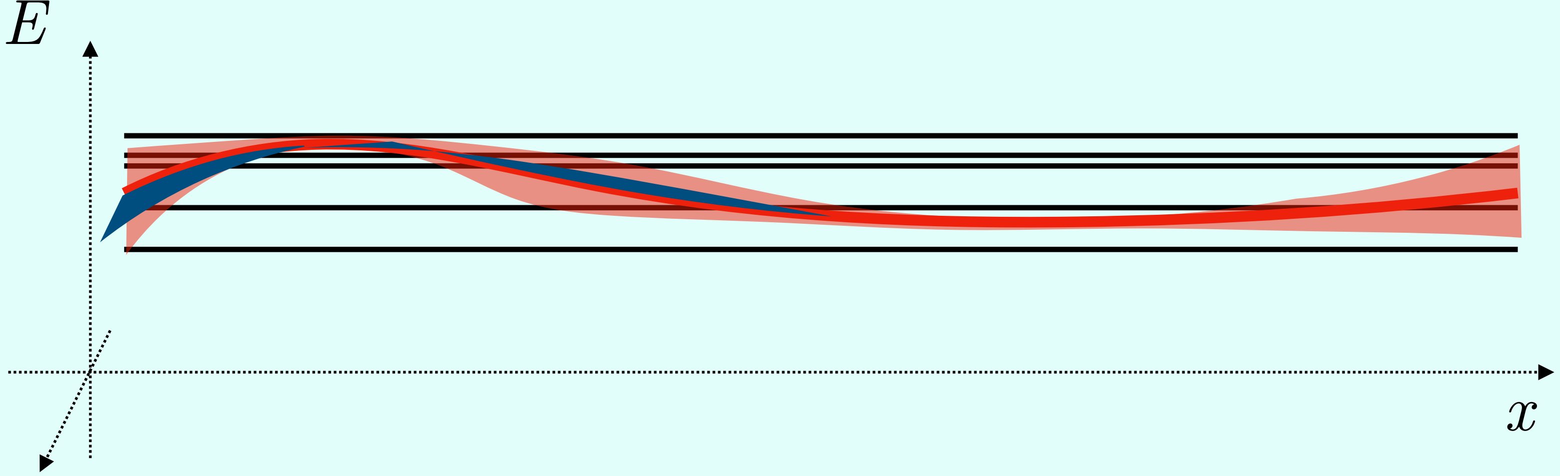
Bertini, MF, Phys. Rev. Lett. **117**, 130402 (2016)

1. Does it exist an **invariant subspace** of states that are stationary in the homogeneous limit?
2. How can the states be parametrised in that subspace?
3. Is a "space-time version" of the root densities ρ_λ^x sufficient in interacting integrable systems?
4. How does ρ_λ^x time evolve?

Euler scale vs invariant subspace

Doyon's lecture

$< \Psi_t | \boldsymbol{O}_x | \Psi_t > \sim < \{\rho^x_\lambda(t)\} | \boldsymbol{O} | \{\rho^x_\lambda(t)\} > +$ **small corrections**



$"off\text{-}diagonal"\ d.o.f$

humble/pragmatic view:

I don't know the details, but there's a limit when they don't matter

(first-order) GHD

$$\partial_t \rho^x_\lambda + \partial_x v_\lambda[\{\rho^x_\lambda\}]\rho^x_\lambda = 0$$

Castro-Alvaredo, Doyon, Yoshimura, Phys. Rev. X **6**, 041065 (2016)

Bertini, Collura, De Nardis, MF, Phys. Rev. Lett. **117**, 207201 (2016)

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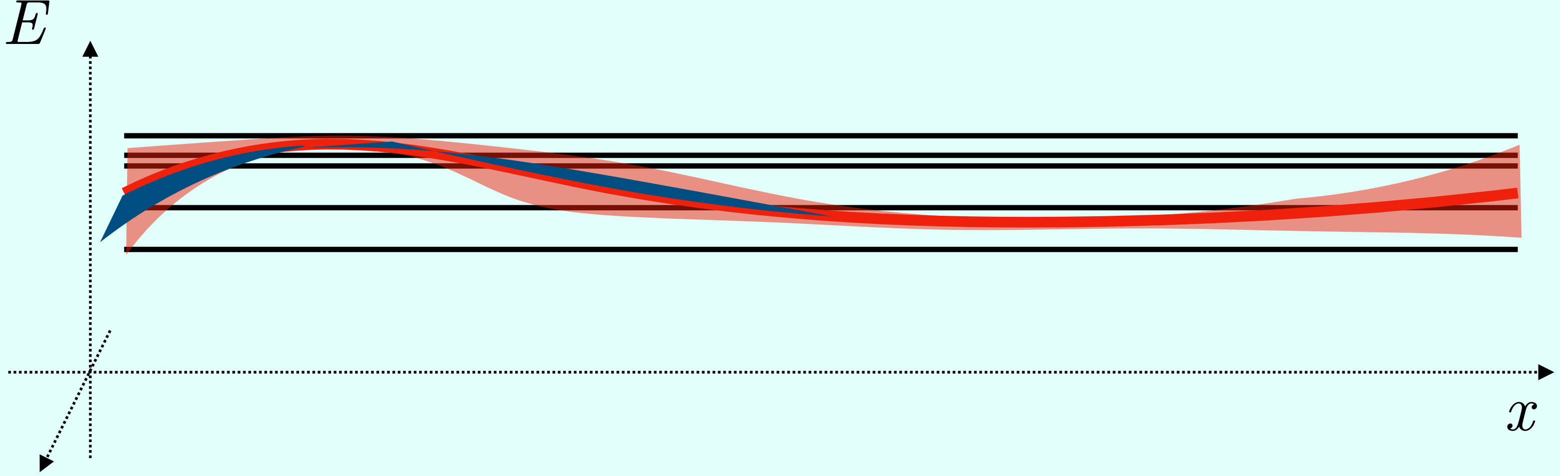

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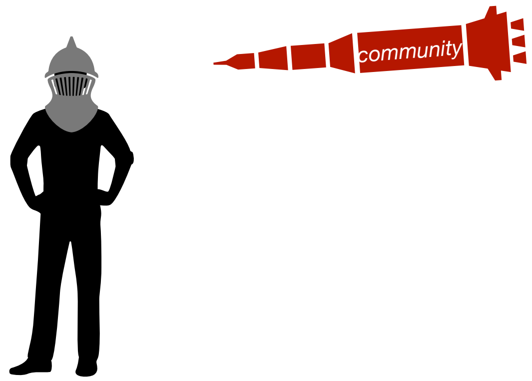
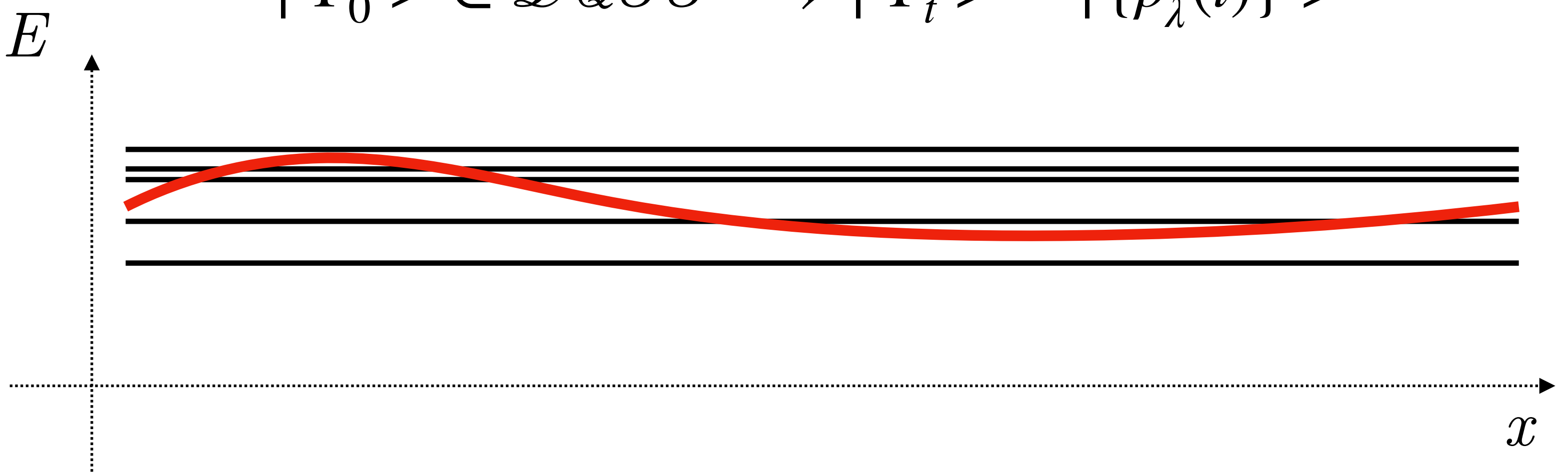


$$|\Psi_0 > \in \mathscr{LQSS} \implies |\Psi_t > = |\{\rho^x_\lambda(t)\} >$$



audacious/reckless view:

I don't believe in coincidences, the existence of the limit could indicate…

GHD ≡ Schrödinger equation

Problems to face

1. The construction of an invariant subspace requires (almost) **unambiguous** definitions of the quantities used to characterise the state

what does $\rho_{\lambda}^{x}(t)$ mean?

what's the exact relation between $\rho_{\lambda}^{x}(t)$ and expectation values?

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what's the exact relation between $\rho_\lambda^x(t)$ *and expectation values?*

2. One should be able to keep it under control degrees of freedom **orthogonal** to the invariant subspace

what's not described by $\rho_\lambda^x(t)$*?*

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is $\rho_\lambda^x(t)$ *enough to characterise an invariant subspace?*

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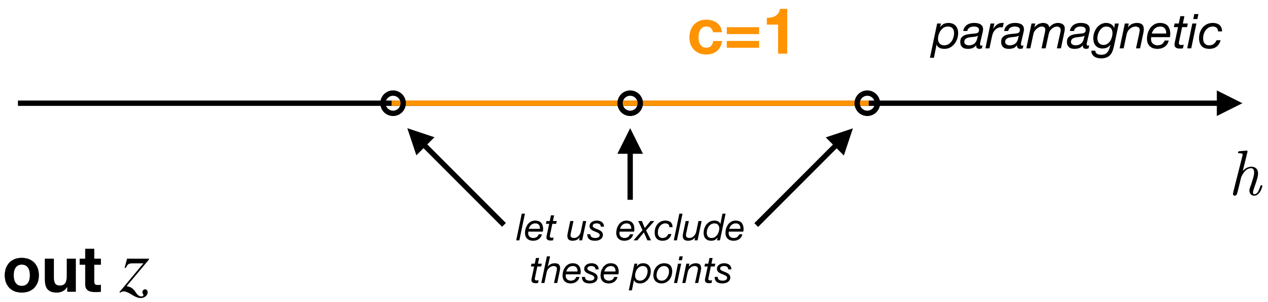
is $\rho_\lambda^x(t)$ enough to characterise an invariant subspace?

let's start with a simple example

XX model

$$\boldsymbol{H} = \sum_\ell \boldsymbol{\sigma}_\ell^x \boldsymbol{\sigma}_{\ell+1}^x + \boldsymbol{\sigma}_\ell^y \boldsymbol{\sigma}_{\ell+1}^y + h \boldsymbol{\sigma}_\ell^z$$



- $U(1)$ **symmetry of rotations about** z

- **noninteracting spin chain, mapped to a quadratic form of fermions by the Jordan-Wigner tra** $\boldsymbol{c}_\ell^\dagger = \prod_{j<\ell} \boldsymbol{\sigma}_j^z \, \boldsymbol{\sigma}_\ell^+$

Crucial Observation: **all the** $U(1)$ **and one-site shift invariant noninteracting operators commute with one another**

(it can be traced back to the fact that circulant matrices commute with one another)

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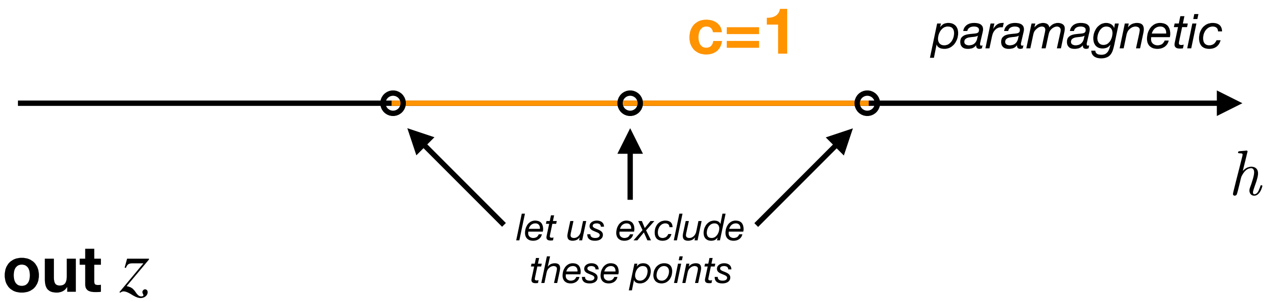

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(it can be traced back to the fact that circulant matrices commute with one another)

$$\boldsymbol{Q}^{(n)} = \sum_\ell \boldsymbol{Q}^{(n)}_\ell$$

charge densities

$$\partial_t \boldsymbol{Q}^{(n)}_\ell(t) = i[\boldsymbol{H}, \boldsymbol{Q}^{(n)}_\ell(t)] = \boldsymbol{J}^{(n)}_{\ell-1}(t) - \boldsymbol{J}^{(n)}_\ell(t)$$

currents

sensible/natural gauge

1) $[\boldsymbol{S}^z, \boldsymbol{Q}^{(n)}_\ell] = 0 \Longrightarrow [\boldsymbol{S}^z, \boldsymbol{J}^{(n)}_{\ell-1} - \boldsymbol{J}^{(n)}_\ell] = 0$

2) $[\boldsymbol{S}^z, \boldsymbol{J}^{(n)}_\ell] = 0$

the total currents are conserved!

XX model

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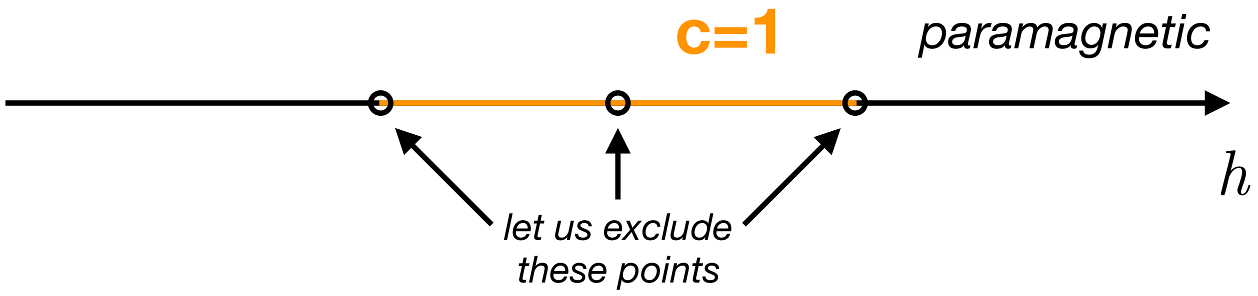

let us exclude these points

$$e^{-i\boldsymbol{H}t} \exp\Big(\sum_\ell \sum_{n=0} \lambda_\ell^n \boldsymbol{Q}_\ell^{(n)}\Big) e^{i\boldsymbol{H}t} = \exp\Big(\sum_\ell \sum_{n=0} \lambda_\ell^n(t) \boldsymbol{Q}_\ell^{(n)}\Big)$$

invariant subspace of density matrices which can be parametrised by $\lambda_\ell^n(t)$

more conveniently, it can be parametrised by a root density, as we'll see

Crucial Observation: **all the $U(1)$ and one-site shift invariant noninteracting operators commute with one another**

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sensible/natural gauge

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Noninteracting spin chains

$$\boldsymbol{H} = \sum_{\ell \in \mathbb{Z}} \sum_{\alpha,\beta \in \{x,y\}} \sum_{n \in \mathbb{N}_0} J_n^{\ell;\alpha\beta} \boldsymbol{\sigma}_\ell^\alpha \prod_{m=\ell+1}^{\ell+n-1} \boldsymbol{\sigma}_m^z \boldsymbol{\sigma}_{\ell+n}^\beta + \sum_{\ell \in \mathbb{Z}} J^{\ell;z} \boldsymbol{\sigma}_\ell^z$$

$$\boldsymbol{H}_{Ising} = -J \sum_\ell \boldsymbol{\sigma}_\ell^x \boldsymbol{\sigma}_{\ell+1}^x + h \boldsymbol{\sigma}_\ell^z$$

Problem (?): **if we impose the charge densities to be local, the total currents of the charges odd under reflections are not conserved**

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Possible Solution: we can consider a gauge with quasilocal charge densities

does it work?

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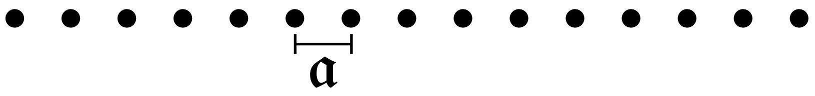
Possible Solution: **we can consider a gauge with quasilocal charge densities**

does it work?

YES

there is a quasilocal gauge in which the total currents are conserved!

Noninteracting spin chains



Jordan-Wigner Majorana fermions

$$\boldsymbol{a}_{2\ell-1} = \prod_{j<\ell} \boldsymbol{\sigma}^z_j \, \boldsymbol{\sigma}^x_\ell$$

$$\boldsymbol{a}_{2\ell} = \prod_{j<\ell} \boldsymbol{\sigma}^z_j \, \boldsymbol{\sigma}^y_\ell$$

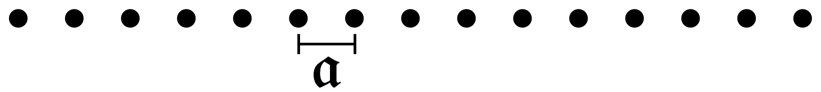
$$\Gamma^{\ell n}_{ij}(t) = \delta_{\ell n}\delta_{ij} - < a_{2\ell+i} a_{2n+j} >_t$$

$$[\hat{\Gamma}_{x,t}(z_p)]_{ij} = \sum_{\ell \in \mathbb{Z}} e^{-2i\frac{(\ell\mathfrak{a} - x)p}{\hbar}} \Gamma^{\ell, \frac{2x}{\mathfrak{a}} - \ell}_{ij}(t) + [\hat{\Gamma}^{\text{unphys}}_{x,t}(z_p)]_{ij} \qquad \frac{x}{\mathfrak{a}} \in \frac{1}{2}\mathbb{Z} \qquad z_p = e^{i\frac{\mathfrak{a}p}{\hbar}}$$

$\hat{\Gamma}^{\text{unphys}}_{x,t}(-z_p) = -(-1)^{\frac{2x}{\mathfrak{a}}} \hat{\Gamma}^{\text{unphys}}_{x,t}(z_p)$ *does not affect the correlation matrix*

↳ **irrelevant!**

Noninteracting spin chains



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homogeneous one-site shift invariant Hamiltonians

$$\hat{\Gamma}_x(z_p) = 4\pi\hbar \underbrace{e^{-i\frac{\phi(p-\frac{i\hbar\overrightarrow{\partial_x}}{2})}{2}\sigma^y} e^{-i\frac{\theta(p-\frac{i\hbar\overrightarrow{\partial_x}}{2})}{2}\sigma^z}}_{\substack{\textit{generalised Bogoliubov transformation}\\ \textit{(depends on the Hamiltonian!)}}} \Big\{ \underbrace{\mathrm{I}\rho^x_{p,\mathrm{o}}}_{\substack{\textit{odd}\\ \textit{part}}} + \sigma^y \Big[\underbrace{\rho^x_{p,\mathrm{e}}}_{\substack{\textit{even}\\ \textit{part}}} - \frac{1}{4\pi\hbar} \Big] + \sigma^z \underbrace{\Psi^x_{p,\mathrm{R}}}_{\substack{\textit{real}\\ \textit{part}}} - \sigma^x \underbrace{\Psi^x_{p,\mathrm{I}}}_{\substack{\textit{imaginary}\\ \textit{part}}} \Big\} e^{i\frac{\theta(p+\frac{i\hbar\overleftarrow{\partial_x}}{2})}{2}\sigma^z} e^{i\frac{\phi(p+\frac{i\hbar\overleftarrow{\partial_x}}{2})}{2}\sigma^y}$$

MF, arXiv:1910.01046

ρ_p^x is the inhomogeneous version of the root density

(I would have simply called it Wigner function if there were no dependence on the Hamiltonian)

Ψ_p^x is an odd complex auxiliary field

(in the homogeneous limit, it describes the off-diagonal matrix elements of the density matrix)

Noninteracting spin chains

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decoupled dynamical equations

$$i\hbar\partial_t\rho^x_p(t) = \varepsilon_p \star \rho^x_p(t) - \rho^x_p(t) \star \varepsilon_p$$

$$i\hbar\partial_t\Psi^x_p(t) = \varepsilon_p \star \Psi^x_p(t) + \Psi^x_p(t) \star \varepsilon_{-p}$$

excitation energy

Moyal star product

$$a^x_p \star b^x_p = a^x_p e^{i\hbar\frac{\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overrightarrow{\partial_x}\overleftarrow{\partial_p}}{2}} b^x_p$$

completely equivalent to the Schrödinger equation

MF, arXiv:1910.01046

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$$\partial_t\rho^x_p(t) = -\,v_p\partial_x\rho^x_p(t) + \hbar^2\frac{v''_p}{24}\partial^3_x\rho^x_p(t) - \hbar^4\frac{v''''_p}{1920}\partial^5_x\rho^x_p(t) + O(\hbar^6\partial^7_x)$$

$$i\hbar\partial_t\Psi^x_p(t) = 2\varepsilon_{p,\mathrm{e}}\Psi^x_p(t) - i\hbar v_{p,\mathrm{e}}\partial_x\Psi^x_p(t) - \hbar^2\frac{v'_{p,\mathrm{e}}(t)}{4}\partial^2_x\Psi^x_p(t) + i\hbar^3\frac{v''_{p,\mathrm{e}}}{24}\partial^3_x\Psi^x_p(t) + O(\hbar^4\partial^4_x)$$

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$\partial_t\rho^x_p(t) = -v_p\partial_x\rho^x_p(t)$ **first-order GHD (no $\hbar$)**

$$i\hbar\partial_t\Psi^x_p(t) = 2\varepsilon_{p,\mathrm{e}}\Psi^x_p(t) - i\hbar v_{p,\mathrm{e}}\partial_x\Psi^x_p(t) - \hbar^2\frac{v'_{p,\mathrm{e}}(t)}{4}\partial_x^2\Psi^x_p(t) + i\hbar^3\frac{v''_{p,\mathrm{e}}}{24}\partial_x^3\Psi^x_p(t) + O(\hbar^4\partial_x^4)$$

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Noninteracting spin chains

$$\hat{\Gamma}_x(z_p) = 4\pi\hbar\, e^{-i\frac{\phi(p-\frac{i\hbar\overrightarrow{\partial_x}}{2})}{2}\sigma^y} e^{-i\frac{\theta(p-\frac{i\hbar\overrightarrow{\partial_x}}{2})}{2}\sigma^z}\left\{\mathrm{I}\rho^x_{p,\mathrm{o}} + \sigma^y\left[\rho^x_{p,\mathrm{e}} - \frac{1}{4\pi\hbar}\right] + \sigma^z\Psi^x_{p,\mathrm{R}} - \sigma^x\Psi^x_{p,\mathrm{I}}\right\} e^{i\frac{\theta(p+\frac{i\hbar\overleftarrow{\partial_x}}{2})}{2}\sigma^z} e^{i\frac{\phi(p+\frac{i\hbar\overleftarrow{\partial_x}}{2})}{2}\sigma^y}$$

decoupled dynamical equations

$$i\hbar\partial_t\rho^x_p(t) = \varepsilon_p \star \rho^x_p(t) - \rho^x_p(t) \star \varepsilon_p$$

$$i\hbar\partial_t\Psi^x_p(t) = \varepsilon_p \star \Psi^x_p(t) + \Psi^x_p(t) \star \varepsilon_{-p}$$

completely equivalent to the Schrödinger equation

excitation energy

Moyal star product

$$a^x_p \star b^x_p = a^x_p e^{i\hbar\frac{\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overrightarrow{\partial_x}\overleftarrow{\partial_p}}{2}} b^x_p$$

$$\partial_t\rho^x_p(t) = -v_p\partial_x\rho^x_p(t) + \hbar^2\frac{v''_p}{24}\partial^3_x\rho^x_p(t) - \hbar^4\frac{v''''_p}{1920}\partial^5_x\rho^x_p(t) + O(\hbar^6\partial^7_x)$$

$$i\hbar\partial_t\Psi^x_p(t) = 2\varepsilon_p\Psi^x_p(t) - i\hbar v_p\partial_x\Psi^x_p(t) - \hbar^2\frac{v'_{p,\mathrm{e}}(t)}{4}\partial^2_x\Psi^x_p(t) + i\hbar^3\frac{v''_{p,\mathrm{e}}}{24}\partial^3_x\Psi^x_p(t) + O(\hbar^4\partial^4_x)$$

third-order GHD

(it gives rise to KPZ universal behaviour about the light-cone)

ρ^x_p is the inhomogeneous version of the root density

(I would have simply called it Wigner function if there were no dependence on the Hamiltonian)

Ψ^x_p is an odd complex auxiliary field

(n the homogeneous limit, describes the off-diagonal matrix elements of the density matrix)

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the auxiliary field is quantum

(assuming that the excitation energy is not odd)

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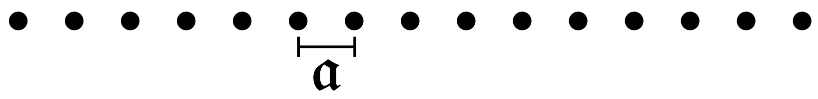


$$\Gamma^{\ell n}_{ij}(t) = \delta_{\ell n}\delta_{ij} - < a_{2\ell+i}a_{2n+j} >_t$$

Jordan-Wigner Majorana fermions

$$\boldsymbol{a}_{2\ell-1} = \prod_{j<\ell}\boldsymbol{\sigma}^z_j\,\boldsymbol{\sigma}^x_\ell$$

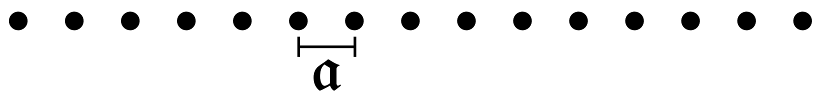
$$\boldsymbol{a}_{2\ell} = \prod_{j<\ell}\boldsymbol{\sigma}^z_j\,\boldsymbol{\sigma}^y_\ell$$

inhomogeneous Hamiltonians

$$\boldsymbol{H} = \frac{1}{4}\sum_{\ell\in\frac{1}{2}\mathbb{Z}}\sum_{n\in\mathbb{Z}}\sum_{i,j=1}^{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}[\frac{p\mathfrak{a}}{\hbar}]}{2\pi}e^{2i\frac{(\ell-n)\mathfrak{a}p}{\hbar}}[\hat{h}_{\ell\mathfrak{a}}(e^{\frac{i\mathfrak{a}p}{\hbar}})]_{ij}\boldsymbol{a}_{2(2\ell-n)+i}\boldsymbol{a}_{2n+j}$$

$$\hat{h}_x(z_p) = \underbrace{e_\star^{-i\frac{\hat{\Theta}_x(z_p)}{2}}}_{\substack{\textit{generalised space-dependent}\\ \textit{Bogoliubov transformation}}} \star \left[\varepsilon^x_{p,\mathrm{o}}\mathrm{I} + \varepsilon^x_{p,\mathrm{e}}\sigma^y\right] \star e_\star^{i\frac{\hat{\Theta}_x(z_p)}{2}}$$

Noninteracting spin chains



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within this representation, the equations of motion have the same structure:

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$$\boxed{\partial_t\rho^x_p(t) = -\partial_x[v^x_p\rho^x_p(t)] + \partial_p[(\partial_x\varepsilon^x_p)\rho^x_p(t)] + O(\hbar^2\partial^3_x)}$$

MF, arXiv:1910.01046

Summary

☑ Time evolution occurs in a tiny part of the Hilbert space

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Thank you for your attention