Nodal sets of eigenfunctions of the Laplacian, with randomness

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28/March/2018

Part-1

Length of the nodal set of eigenfunctions on the torus

Joint work with Pär Kurlberg and Igor Wigman

Eigenfunctions of Laplacian on the torus

- Torus: $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 = [0,1]^2$ with opp. edges identified
- Laplacian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
- For $\lambda = (p, q) \in \mathbb{Z}^2$, writing $|\lambda|^2 = p^2 + q^2$,

$$-\Delta e^{2\pi i \lambda . x} = 4\pi^2 |\lambda|^2 e^{2\pi i \lambda . x} \quad \text{for } x = (x_1, x_2) \in \mathbb{T}^2.$$

- Eigenvalues: $4\pi^2 E$ where E is a positive integer representable as a sum of two squares. Its multiplicity?

$$\mathcal{N}_E = \#\Lambda_E, \qquad \Lambda_E := \{(p,q) \in \mathbb{Z}^2 : p^2 + q^2 = E\}.$$

  Eg., $\mathcal{N}_2 = 4$, $\mathcal{N}_5 = 8$, $\mathcal{N}_7 = 0$.

- Eigenfunctions: $\cos(2\pi\lambda.x)$ and $\sin(2\pi\lambda.x)$, $\lambda \in \Lambda_E$.

Random wave: definition

The eigenspace $\mathcal{H}_E$ for the eigenvalue $4\pi^2 E$ consists of linear combinations of $\cos(2\pi\lambda.x), \sin(2\pi\lambda.x)$, $\lambda \in \Lambda_E$.

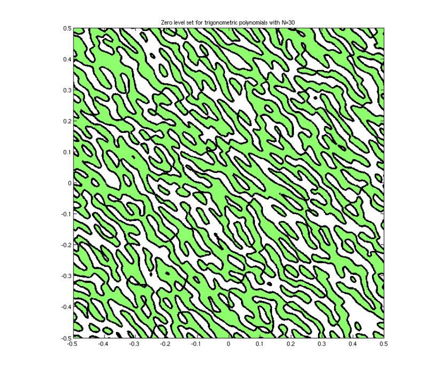
Gaussian random eigenfunction

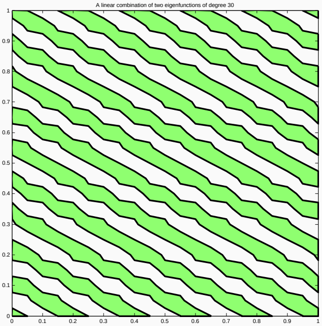
$$f_E(x) = \sqrt{\frac{2}{\mathcal{N}_E}} \sum_{\lambda \in \Lambda_E/\pm} X_\lambda \cos(2\pi\lambda.x) + Y_\lambda \sin(2\pi\lambda.x)$$

where X_λ, Y_λ are i.i.d. $N(0,1)$.

- Why i.i.d. Gaussian? So that the distribution of f_E does not depend on the choice of basis functions for $\mathcal{H}_E$.
- *Equivalent formulation:* Pick a random element on the unit sphere of the finite dimensional Hilbert space $\mathcal{H}_E$.

Picture of nodal set of the random wave





Random wave: basic properties

- f is a Gaussian process with mean and covariance

$$\mathbb{E}[f(x)] = 0, \qquad \mathbb{E}[f(x)f(y)] = \frac{1}{\mathcal{N}_E} \sum_{\lambda \in \Lambda_E} \cos(2\pi\lambda.(x-y)).$$

  In particular, $\mathrm{Var}(f(x)) = 1$ for all $x \in \mathbb{T}^2$.

- f is stationary: $f(\cdot - u) \overset{d}{=} f(\cdot)$ for any $u \in \mathbb{T}^2$.

- Nodal set: $\mathcal{Z}_f := \{x \in \mathbb{T}^2 : f(x) = 0\}$ is union of smooth closed curves (w.p.1) and hence we may define its length $\mathcal{L}_f$ ("nodal length").

Questions

What is $\mathbb{E}[\mathcal{L}_f]$? $\mathrm{Var}(\mathcal{L}_f)$? Other statistical properties of the nodal set? For example, the number of nodal domains? The topology of nodal domains? (Local/Non-local)

Prior results about the nodal length $\mathcal{L}_f$

Theorem (Rudnick, Wigman)

For any eigenvalue E,

1. $\mathbb{E}[\mathcal{L}_f] = \frac{1}{2\sqrt{2}}\sqrt{4\pi^2 E}$.
2. $Var(\mathcal{L}_f) \lesssim \frac{E}{\sqrt{\mathcal{N}_E}}$. *Conjectured:* $Var(\mathcal{L}_f) \lesssim \frac{E}{\mathcal{N}_E}$.

Some notation to state our results on fluctuations:

- Angular distribution of points in Λ_E:

$$\mu_E = \frac{1}{\mathcal{N}_E} \sum_{\lambda \in \Lambda_E} \delta_{\frac{\lambda}{\sqrt{E}}}. \quad \text{(a probability measure on the unit circle)}$$

- For a probability measure μ on the unit circle,

$$\gamma(\mu) = \frac{1 + \hat{\mu}(4)^2}{512}, \text{ where } \hat{\mu}(4) = \int_0^{2\pi} e^{-i4t} d\mu(t).$$

Fluctuations of the nodal length $\mathcal{L}_f$

Theorem (M.K., P. Kurlberg, I. Wigman)

- *If $E \to \infty$ in a way that $\mathcal{N}_E \to \infty$, then*

$$Var(\mathcal{L}_E) \sim \gamma(\mu_E) \frac{E}{\mathcal{N}_E^2}.$$

- *There exists a sequence E_k with $\mathcal{N}_{E_k} \to \infty$ and satisfying $\gamma(\mu_{E_k}) \to \gamma$ if and only if $\frac{1}{512} \leq \gamma \leq \frac{1}{256}$.*

- Variance smaller than was conjectured.
- Constant in asymptotics depends on the distribution of lattice points on the circle $|\lambda|^2 = E$.

More about the fluctuations result

- Implies that $\frac{1}{\sqrt{E}}\mathcal{L}_{f_E} \xrightarrow{P} \frac{\pi}{\sqrt{2}}$ (Concentration near the mean since $\sqrt{\text{Var}(\mathcal{L}_f)} \ll \mathbb{E}[\mathcal{L}_f]$).

- The behaviour of $\mathcal{N}_{E_k}$ as well as μ_{E_k} along sequences of integers can be quite wild.
  1. For example, if $E_k = 5^k$, then $\mathcal{N}_{E_k} = 4(k+1)$ and $\mu_{E_k} \to \text{unif}(S^1)$. This is the "generic behaviour" and in this case, $\gamma = 1/512$.
  2. However, there exist sequences E_k for which $\mu_{E_k} \to \frac{1}{4}\delta_{\pm 1, \pm i}$ (Cilleruelo). In this case, $\gamma = 1/256$.

- Calculations give the variance as $A_1/\mathcal{N} + A_2/\mathcal{N}^2 + \ldots$ but "magically" A_1 turns out to be zero - we have no real understanding why!

- An analogous cancellation first observed by Michael Berry in random plane waves. Similar results due to Igor Wigman for random spherical harmonics.

The proof: In two steps

1. Kac-Rice formula reduces $\mathbb{E}[\mathcal{L}_f]$ to calculating certain 3-dimensional Gaussian integrals and $\text{Var}(\mathcal{L}_f)$ to calculating certain 6-dimensional Gaussian integrals. The latter is often difficult to manipulate and squeeze out useful information. In our case, the end result of a long calculation/estimation is

$$\text{Var}(\mathcal{L}_E) = \gamma(\mu_E)\frac{E}{\mathcal{N}^2} + O\left(\frac{E}{\mathcal{N}^4}\sqrt{\mathcal{R}_6(E)}\right),$$

   where $\mathcal{R}_6(E) = \#\left\{(\lambda_1, \ldots, \lambda_6) : \lambda_i \in \Lambda_E,\ \sum_{i=1}^{6} \lambda_i = 0\right\}$.

2. The number theoretic problem of showing that $\mathcal{R}_6(E) = o(\mathcal{N}_E^4)$. Using techniques of additive combinatorics Bourgain gave us a proof (it improved our partial results).

Step 1: The formula of Kac and Rice (1940s)

The nodal length may be written as

$$\mathcal{L}_f = \int_{\mathbb{T}^2} \delta_{f(x)} \|\nabla f(x)\| dx.$$

Therefore, if f is random, we get

$$\mathbb{E}\left[\mathcal{L}_f\right] = \int\limits_{\mathbb{T}^2} \mathbb{E}\left[\|\nabla f(x)\| \;\middle|\; f(x) = 0\right] p_{f(x)}(0) dx.$$

$$\mathbb{E}\left[\mathcal{L}_f^2\right] = \int\limits_{\mathbb{T}^2}\int\limits_{\mathbb{T}^2} \mathbb{E}\left[\|\nabla f(x)\| \cdot \|\nabla f(y)\| \;\middle|\; \begin{array}{c} f(x) = 0 \\ f(y) = 0 \end{array}\right] p_{(f(x), f(y))}(0,0) dx dy.$$

Here p denotes the probability density of the random variable or random vector in the subscript.

Step 1: Kac-Rice to variance of nodal length

- From the formula, it is easy to get $\mathbb{E}[\mathcal{L}_E] = \frac{1}{2\sqrt{2}}\sqrt{4\pi^2 E}$.

- The computation of second moment is much more complicated and one arrives at

$$\mathrm{Var}(\mathcal{L}_E) = \gamma(\mu_E)\frac{E}{\mathcal{N}^2} + O\left(\frac{E}{\mathcal{N}^4}\sqrt{\mathcal{R}_6(E)}\right),$$

where $\mathcal{R}_6(E) = \#\left\{(\lambda_1, \ldots, \lambda_6) : \lambda_j \in \Lambda_E, \sum_{j=1}^{6} \lambda_j = 0\right\}$.

Two key ideas:

1. is to see that unless x, y are very close, conditionally $\nabla f(x), \nabla f(y)$ are nearly independent.
2. Berry's trick to get rid of absolute values by writing

$$|\alpha| = \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(1 - e^{-\frac{1}{2}\alpha^2 t}\right)\frac{1}{t^{3/2}}dt.$$

Step 2: The number theoretic part of the proof

To show: $\mathcal{R}_6(E) := o(\mathcal{N}^4)$. If not, $\mathcal{R}_6(E) \gtrsim \mathcal{N}^4$.
Three results from additive combinatorics kick in.

- $(\Lambda_E + \Lambda_E)\backslash\{0\}$ has a subset A_1 of size $\gtrsim \mathcal{N}^2$ but having $|A_1 + A_1| \lesssim \mathcal{N}^2$. (Balog-Szemeredi-Gowers)
- A_1 is contained in a generalized arithmetic progression (of complex numbers) $P = \{\ell_0 + \sum_{i=1}^{d} \ell_i z_i : 0 \leq \ell_i \leq L_i\}$ of bounded *rank d* and *volume* $\prod_{i=1}^{d} L_i \lesssim \mathcal{N}^2$. (Freiman's theorem).
- Number of ways any number can be written as product of two elements of the GAP P is $\lesssim \exp\{\kappa_d \log L_{\max} / \log\log L_{\max}\}$ (Chang's theorem).
- For any $\lambda \in \Lambda_E$, we have $\lambda\overline{\lambda} = E$, hence $4E = (\lambda + \lambda)(\overline{\lambda} + \overline{\lambda})$ can be written as a product of two elements of the GAP P in $\gtrsim \mathcal{N}$ ways.
- Last two statements contradict each other as $L_{\max} \lesssim \mathcal{N}^2$.

Part-2

Structure of nodal sets of eigenfunctions on the randomly weighted line graph

Joint work with Arvind Ayyer

Courant's nodal domain theorem on graphs

- $G = (V, E)$ a finite graph. Edge-weights $w : E \mapsto \mathbb{R}_+$.
- Laplacian: The $V \times V$ symmetric matrix,

$$L(i,j) = \begin{cases} -w(i,j) & \text{if } i \sim j, \\ \sum_{k \,:\, k \sim i} w(i,k) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

- For $\mathbf{u} \in \mathbb{R}^V$, we have $\mathbf{u}^t L \mathbf{u} = \sum_{i \sim j} w(i,j)(\mathbf{u}(i) - \mathbf{u}(j))^2$.
- Eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Eg., $\mathbf{v}_1 = \mathbf{1}$.

Courant's nodal domain theorem

The number of nodal components of $\mathbf{v}_k$ is at most k. In particular, $\mathbf{v}_2$ has exactly one positive and one negative nodal components. Fine print: Is nodal domain $\mathbf{v}_k > 0$ or $\mathbf{v}_k \geq 0$? Need proper interpretation...

Our question

If $w(e)$, $e \in E$, are random variables, the nodal domains of $\mathbf{v}_2$ are random. What is their geometry? Size?

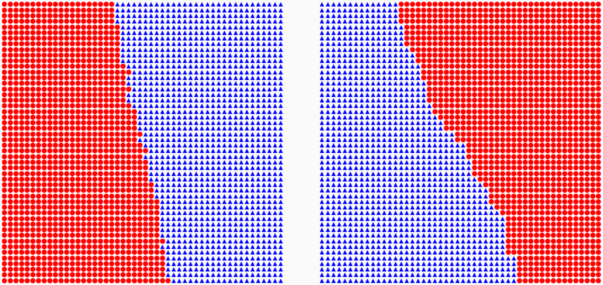


Figure: Two realizations of second eigenvector with i.i.d. uniform edge-weights on 50x50 grid

Simpler example: Path graph

$V = \{1, 2, \ldots, n\}$, edges between i and $i + 1$ for $1 \leq i \leq n - 1$. Then, the Laplacian is the tridiagonal matrix

$$L = \begin{bmatrix} w_1 & -w_1 & 0 & \ldots & \ldots & 0 \\ -w_1 & w_1 + w_2 & -w_2 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & -w_{n-2} & w_{n-2} + w_{n-1} & -w_{n-1} \\ 0 & \ldots & \ldots & 0 & -w_{n-1} & w_{n-1} \end{bmatrix}$$

Statistic of interest: point of sign-change, $1 \leq M \leq n - 1$, such that $\mathbf{v}_2(i) > 0$ for $i \leq M$ and $\mathbf{v}_2(i) < 0$ for $i > M$.

Path graph: an observation

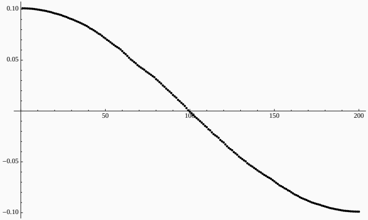


Figure: Pathgraph: Second eigenvector for uniform[1, 2] edge-weights.

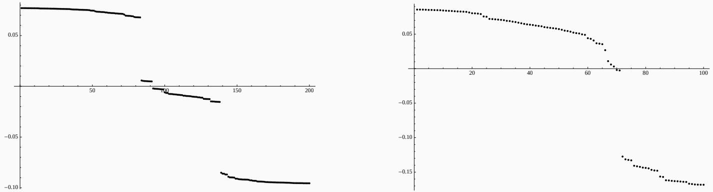


Figure: Pathgraph: Second eigenvector for uniform[0, 1] edge-weights. Two realizations.

Studied before?

Dekel-Lee-Linial studied the number of nodal domains of the Erdos-Renyi random graph $G(n, p)$ with fixed p as $n \to \infty$.

They found that the number of nodal domains of even very high energy eigenfunctions is very small (like Lewy's example) and that the largest two nodal domains cover all but $O(1)$ of the vertices...

Pathgraph: A qualitative explanation

G_n = pathgraph with n vertices.

Our observation: If w_i are i.i.d. uniform$[1,2]$, the positive nodal domain of $\mathbf{v}_2$ is of size close to $n/2$. For uniform[0, 1] edgeweights, the number has standard deviation n.

- $\mathbf{v}_2$ minimizes $\mathbf{v}^t L\mathbf{v} = \sum_{i=1}^{n-1} w_i(\mathbf{v}(i+1) - \mathbf{v}(i))^2$ subject to $\sum_i \mathbf{v}(i) = 0$ and $\sum_i \mathbf{v}(i)^2 = 1$.
- If $w_i \sim$ uniform[0, 1], there are edges of small weight $\approx \frac{1}{n}$. Makes sense for $\mathbf{v}$ to make a jump there and stay flat where w_i are not small.
- If $w_i \sim$ uniform$[1,2]$, all edges are of the same order, so $\mathbf{v}_2$ has no big jumps. Looks like a continuous curve.
- Direct methods (recursion) can be used to prove that $\mathbf{v}_2(k) \approx \cos(\pi k/n)$ in the latter case.

A better approach?

Endow [0, 1] with metric d_n such that $[\frac{i}{n}, \frac{i+1}{n}]$ has length proportional to $1/w_i$ and [0, 1] has length 1.

Case 1: $\mathbb{E}[1/w_1] < \infty$. Then $d_n(s,t) \to t - s$ for $s < t$.

Case 2: $\mathbb{E}[1/w_1] = \infty$ and some regularity of tails. Then $d_n(t,s)$ converges to a random metric of the form

$$d(t,s) = F(t) - F(s)$$

where F is the CDF of $\sum_{k=1}^{\infty} p_k \delta_{U_k}$ where U_k are i.i.d. uniform[0, 1] and p is a random probability vector independent of U_ks.

If $\mathbb{P}\{w_1 < x\} \sim C.x^{-\alpha}$ with $\alpha < 1$, then $p_k = \frac{G_k^{-1/\alpha}}{\sum_j G_j^{-1/\alpha}}$ where $G_1, G_2, G_3, \ldots$ is a Poisson process on $\mathbb{R}_+$ with constant intensity.

A better approach?

Plausible explanation for the observation:

Case-1: The limit metric is the standard $|t - s|$ on $[0, 1]$. And the second eigenfunction of the Laplacian on $[0, 1]$ (with Neumann boundary condition) is $\cos(\pi t)$.

Case-2: The limit is a random (singular) metric $|F(t) - F(s)|$ on $[0, 1]$. It "must have" a Laplacian whose second eigenfunction is the (random) discontinuous functions that the pictures show.

Thus, our approach is to

1. Define Laplacian for a metric of the form $|F(t) - F(s)|$ where F is a fixed increasing function on $[0, 1]$.
2. Show that if two metrics are close, the eigenfunctions are close.
3. Add randomness at the end to figure out what F is.

Defining a Laplacian

Instead of metric, consider a probability measure μ on [0, 1] (the metric is $\mu[s, t)$). Let m be the Lebesgue measure. Instead of Laplacian, we define a quadratic form.

1. $\mathcal{H}_\mu := \{f : f(x) = \int \varphi(t)\mathbf{1}_{[0,x]}(t)d\mu(t)$ for some $\varphi \in L^2(\mu)\}$. $D_\mu f := \varphi$ (Radon-Nikodym derivative $df(x)/d\mu(x)$). $\mathcal{H}_\mu$ is a dense subspace of $L^2(m)$.
2. $Q_\mu[f, g] := \langle D_\mu f, D_\mu g\rangle_{L^2(\mu)}$ for $f, g \in \mathcal{H}_\mu$. This is a "Dirichlet form": a symmetric, positive semi-definite, closed, Markovian, bi-linear, densely defined form on $L^2(m)$.

Defining a Laplacian: examples

Example

If $\mu = m$, then $\mathcal{H}_m = H^1$ (Solbolev space), $Q_\mu[f,g] = \int_0^1 f' g' dm$.
Relationship to Laplacian: If $f \in H^2$, $f'(0) = f'(1) = 0$, $g \in H^1$,

$$Q_m[f,g] = -\langle \Delta f, g \rangle$$

Example

If $\mu = p_1 \delta_{a_1} + \ldots + p_k \delta_{a_k}$. Then $\mathcal{H}_\mu$ contains piecewise constant functions with jumps at a_is. $Q_\mu[f,g] = \sum_{i=1}^k f_i g_i$ where $f_i = f(a_i+) - f(a_i-)$.

In particular, if $\mu = \sum_{k=0}^n \frac{c}{w_k} \delta_{k/n}$ with $c^{-1} = \sum_{j=1}^n \frac{1}{w_j}$, then $Q_\mu[f,g] = \mathbf{v}^t L \mathbf{u}$ where $\mathbf{v}(i) = f(i/n)$ and $\mathbf{u}(i) = g(i/n)$.

Eigenfunctions and eigenvalues

Given μ, we define the eigenfunctions of Q_μ as follows:

1. $f_1 = \mathbf{1}$ minimizes $Q_\mu[f, f]$ subject to $\|f\|^2_{L^2(m)} = 1$. Set $\lambda_1 = Q_\mu[f_1, f_1] = 0$.
2. Having defined $f_1, \ldots, f_{k-1}$, minimize $Q_\mu[f, f]$ subject to $\|f\|^2_{L^2(m)} = 1$ and $f \perp f_j$, $1 \le j \le k-1$. Choose a minimizer f_k and set $\lambda_k := Q_\mu[f_k, f_k]$.

Facts:

1. Minimizers exist, and form a finite dimensional space. Consequently, eigenvalues and eigenspaces are well-defined and $\lambda_k \to \infty$ (unless μ has finite support).
2. Min-max formulas are valid: For any $k-1$ dimensional space $W \subseteq \mathcal{H}_\mu$,

$$\min_{f \perp W} \frac{Q_\mu[f, f]}{\|f\|^2_{L^2(m)}} \le \lambda_k.$$

Closeness of eigenfunctions

Theorem

$\mu \mapsto f_k^\mu$ is continuous.

Caveat: As of now we have this in L^2 metric on the functions. Want to strengthen it to $D[0, 1]$ metric or better.

Adding on randomness

Corollary

Let w_i be i.i.d. positive random variables.

1. *If $\mathbb{E}[1/w_1] < \infty$, then the kth eigenfunction of L is close to $\mathbf{v}_k(j) = \cos(\pi kj/n)$. Eg., $w_1 \sim$ uniform$[1, 2]$.*
2. *If $\mathbb{P}\{w_1 \leq x\} = x^\alpha L(x)$ where L is slowly varying and $0 < \alpha < 1$, then the kth eigenfunction of L is close to the (random) kth eigenfunction for the random measure $\mu_\alpha = \sum_{k=1}^{\infty} p_k \delta_{U_k}$ where U_k are i.i.d. uniform$[0, 1]$ and $p_k = C.G_k^{-1/\alpha}$ where $G_1, G_2, \ldots$ is unit intensity Poisson process on $\mathbb{R}_+$. Eg., $w_1 = V^\alpha$, $V \sim$ uniform$[0, 1]$.*

Remark: If $w_1 \sim$ uniform[0, 1], then it falls into the first case!

Closing remarks

- Improve the approximation theorem to a stronger metric on functions.
- Can carry out the program as written for trees converging to $\mathbb{R}$-trees. Eg., uniform random tree converging to Aldous' Brownian CRT (needs checking). Relationship to Brownian motion (Aldous, Croyden, Athreya-Lohr-Winter) on $\mathbb{R}$-trees to be investigated.
- Explicit calculation of eigenvalues for special measures like Cantor measure? Weyl asymptotics for fixed μ?
- Higher dimensions? On complete graph?
- How does the string sound?