# On Hamiltonian and Action Principle formulations of plasma fluid models

ICTS Seminar

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# Outline

1. Why use Hamiltonian and Lagrangian methods?
2. On the ideal MHD Lagrangian
3. On the ideal MHD Hamiltonian
4. A unified action for extended MHD models
5. A unified Hamiltonian formulation for extended MHD models
6. Conclusions

# A reason for studying Hamiltonian and Action Principle formulations

# A reason for studying Hamiltonian and Action Principle formulations

Because we can . . . (to paraphrase Mallory's famous comment about climbing Mt. Everest: "Because it's there")

# Advantages of Hamiltonian and Action Principle formulations for plasmas

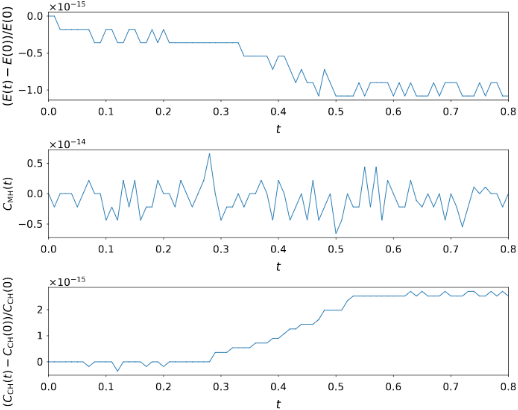
Action principles:

- The construction of reduced models from a “parent” model - introduce new terms/impose orderings directly.
- Eliminate ‘fake’ dissipation, and associated spurious instabilities.
- Many plasma models do not even conserve energy.
- The action can be suitably discretized to construct variational integrators.

Hamiltonian formulations:

- Well-suited for calculating plasma equilibria as well as analyzing their stability via the Energy-Casimir method.
- Can be used to establish underlying connections between outwardly different models.

# Advantages of Hamiltonian and Action Principle formulations for plasmas




Credit: Kraus & Maj (2017); arXiv:1707.03227

# Eulerian and Lagrangian "viewpoints" for magnetofluids

Lagrangian viewpoint:

- Fluid - continuum of particles; each 'particle' described by the coordinate $q(a, t)$, where '$a$' is the label.
- *Attributes:* Properties attached to the particle before it commences its trajectory; depend only on the label $a$ and denoted by subscript '0'. Examples: $\rho_0(a)$, $s_0(a)$, etc.

Eulerian viewpoint:

- Fluid variables are functions of $r$ and $t$. These serve as *observables* since they can tracked. Examples: $\rho(r, t)$, $s(r, t)$, etc.

Require a means of transitioning between these two viewpoints. This is accomplished via Lagrange-Euler maps.

# Lagrange - Euler maps

- The position and velocity in the two descriptions are identical, i.e. $r = q(a,t)$ and $v(r,t) = \dot{q}(a,t)$; RHS is evaluated at $a = q^{-1}(r,t)$.
- Relations between attributes and their corresponding observables determined via imposition of physical laws. Locally, mass conservation dictates

$$\rho_0(a) d^3 a = \rho(r,t) d^3 r \tag{1}$$

and this leads to $\rho = \rho_0/\mathcal{J}$, where $\mathcal{J}$ is the Jacobian.

- Upon using the fact that $D\mathcal{J}/Dt = (\nabla \cdot v)\,\mathcal{J}$, and simplifying further, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2}$$

can be obtained.

# Lagrange - Euler maps (contd)

- Specific entropy (entropy per unit mass) follows via $s(r,t) = s_0$, i.e. advection along streamlines. This leads to

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0, \tag{3}$$

- In the case of ideal MHD, the magnetic flux is assumed to be 'frozen-in':

$$B_0(a) \cdot d^2 a = B \cdot d^2 r \tag{4}$$

which leads to $B^i = \partial q^i / \partial a^j B_0^j / \mathcal{J}$ and this leads to the ideal MHD induction equation

$$\frac{\partial B}{\partial t} - \nabla \times (v \times B) = 0, \tag{5}$$

after using the $\nabla \cdot B = 0$ condition.

# Formulating the ideal MHD action

- Pick the domain and the choice of observables/attributes at first.
- Build each term in the action by appealing to physical reasoning.
- All terms must satisfy the *Eulerian Closure Principle* (ECP) - the action should be entirely expressible in Eulerian variables after using the Lagrange-Euler maps.
- The kinetic energy density has the property

$$T[q] = \frac{1}{2}\int_D d^3a\, \rho_0 |\dot{q}|^2 \leftrightarrow \frac{1}{2}\int_D d^3r\, \rho |v|^2 \qquad (6)$$

- Construct the action, vary with respect to $q$ and Eulerianize the equation(s) of motion.

# Formulating the ideal MHD action

- The potential energy term is given by

$$V[q] = \int_D d^3a\, \rho_0 U(\rho_0/\mathcal{J}, s_0) + \frac{q^i_{,j} q^i_{,k} B^j_0 B^k_0}{2\mathcal{J}} = \int_D d^3r\, \rho U(\rho, s) + \frac{|B|^2}{2} \quad (7)$$

- The action for ideal MHD is therefore given by

$$S = \int_{t_0}^{t_1} (T[q] - V[q])\, dt \quad (8)$$

- The dynamical equation(s) found from $\delta S = 0$.

# The ideal MHD equations

- We have seen earlier that the Eulerian equations for the density, entropy and the magnetic field follow automatically as a result of imposing local mass conservation, invariance and flux conservation respectively.
- Taking the variation of $S$ leads to the Euler-Lagrange equation:

$$\rho_0 \ddot{q}_i + A_i^j \frac{\partial}{\partial a^j} \left( \frac{\rho_0^2}{\mathcal{J}^2} \frac{\partial U}{\partial \rho} \right) + \cdots = 0, \tag{9}$$

and upon using the Euler-Lagrange maps and several identities (e.g. Morrison 2009), the MHD dynamical equation for the velocity is obtained:

$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + J \times B. \tag{10}$$

# A brief comment on gyroviscosity

- In addition to the term quadratic in $v$, one can also introduce a term linear in $v$ in the action, i.e. of the form

$$S = \int_D d^3 r \, v \cdot M^\star, \tag{11}$$

where $M^\star$ can be viewed as an intrinsic momentum density.

- In the specific scenario where $M^\star$ is expressible as $\nabla \times L^\star$, choosing a suitable ansatz for $L^\star$ leads to the inclusion of gyroviscosity - an important plasma term.
- In a simplified 2D limit, the gyroviscosity is given by

$$\begin{aligned} \pi_{ls} &= N_{sjlk} \beta \partial_k \left(\frac{M_j}{\rho}\right) \\ N_{sjlk} &= \frac{m}{2e} \left(\delta_{sk}\epsilon_{jl} - \delta_{jl}\epsilon_{sk}\right) . \end{aligned} \tag{12}$$

# Towards the Hamiltonian formulation

- Compute the canonical momentum $\Pi = \partial L / \partial \dot{q}$.
- Carry out a Legendre transform of the Lagrangian to obtain the Hamiltonian (in Lagrangian variables), and express it in terms of Eulerian variables.
- The Hamiltonian of ideal MHD is given by

$$H = \int_D d^3r \left[ \frac{\rho v^2}{2} + \rho U(\rho, s) + \frac{B^2}{2} \right] \tag{13}$$

- These terms represent the kinetic, internal and magnetic energy densities respectively.
- It is more advantageous sometimes to work with $\sigma = \rho s$ and $M = \rho v$.

# Towards the Hamiltonian formulation (contd)

- As noted earlier, physical functionals ought to be equally expressible in terms of Eulerian and Lagrangian variables:

$$\begin{aligned} \delta \bar{F} &\equiv \int_D d^3a \, \frac{\delta \bar{F}}{\delta \Pi} \cdot \delta \Pi + \frac{\delta \bar{F}}{\delta q} \cdot \delta q \\ &= \delta F \equiv \int_D d^3r \, \frac{\delta F}{\delta \rho} \delta \rho + \frac{\delta F}{\delta \sigma} \delta \sigma + \frac{\delta F}{\delta M} \cdot \delta M + \frac{\delta F}{\delta B} \cdot \delta B. \end{aligned} \tag{14}$$

- Compute Lagrangian functional derivatives in terms of Eulerian ones, and map to find the Eulerian Poisson bracket.
- For example, the Euler-Lagrange map for the density can be written in integral form:

$$\rho = \int_D d^3a \, \rho_0 \delta \left( r - q \right). \tag{15}$$

# Towards the Hamiltonian formulation (contd)

- The variation in the density is therefore given by

$$\delta\rho = -\int_D d^3a\, \rho_0 \nabla\delta\left(r-q\right)\delta q \tag{16}$$

$$\int_D d^3a\, \frac{\delta \bar{F}}{\delta q}\cdot\delta q + \cdots = -\int_D d^3r\, \frac{\delta F}{\delta\rho}\delta\rho \int_D d^3a\, \rho_0 \nabla\delta\left(r-q\right)\delta q + \ldots \tag{17}$$

- Interchanging the order of integration and equating the coefficients of $\delta q$ and $\delta\Pi$ enables us to determine $\delta F/\delta q$ and $\delta F/\delta\Pi$ in terms of Eulerian functional derivatives such as $\delta F/\delta\rho$ ....
- We plug in $\delta F/\delta q$ and $\delta F/\delta\Pi$ into the canonical Lagrangian-variable Poisson bracket to obtain the Poisson bracket of ideal MHD in Eulerian variables.

# Ideal MHD Poisson bracket

$$\begin{aligned}\{F,G\}_{MHD} &= -\int_D d^3r \Bigg[M_i\left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta F}{\delta M_i}\right)\\
&\quad +\rho\left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta F}{\delta \rho}\right)\\
&\quad +\sigma\left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta F}{\delta \sigma}\right)\\
&\quad +B_i\left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta G}{\delta B_i} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta F}{\delta B_i}\right)\\
&\quad +B_i\left(\frac{\delta G}{\delta B_j}\frac{\partial}{\partial x^i}\frac{\delta F}{\delta M_j} - \frac{\delta F}{\delta B_j}\frac{\partial}{\partial x^i}\frac{\delta G}{\delta M_j}\right)\Bigg], \qquad (18)\end{aligned}$$

# Ideal MHD Poisson bracket

It is more common to express the MHD bracket in terms of $\rho$, $v$ and $B$ as they are the dynamical variables of interest.

$$\begin{aligned}\{F,G\}^{MHD} &= -\int_D d^3x \Big\{ [F_\rho \nabla\cdot G_v + F_v\cdot\nabla G_\rho] \\ &\quad -\frac{(\nabla\times v)}{\rho}\cdot(F_v\times G_v) \\ &\quad -\frac{B}{\rho}\cdot(F_v\times(\nabla\times G_v)) \\ &\quad +\frac{B}{\rho}\cdot(G_v\times(\nabla\times F_B))\Big\} \end{aligned} \tag{19}$$

Note that this is the barotropic MHD bracket, where the entropy is absent.

# Preliminaries and motivation

- MHD is a powerful theory, but it is not applicable in every domain. In some systems, 2-fluid effects must be considered.
- Such effects include the Hall current, electron inertia, etc. Must build MHD models with 2-fluid effects for such a purpose.
- Such models used widely in reconnection, as they do not conserve the magnetic flux. Also applicable in several astrophysical systems - protoplanetary discs, solar wind, etc.
- Unfortunately, many “beyond MHD” models in the plasma literature fail to even retain the basic feature of energy conservation.
- Approach: We start from the parent (i.e. 2-fluid) action, and will impose successive orderings within the action to obtain different extended MHD models.
- Strategy: adopt a mixed Eulerian-Lagrangian action.

# The two-fluid action

$$L = \frac{1}{8\pi}\int d^3r \left[\left|-\frac{1}{c}\frac{\partial A(r,t)}{\partial t} - \nabla\phi(r,t)\right|^2 - |\nabla\times A(r,t)|^2\right] \tag{20}$$

$$+\sum_s \int d^3a\, n_{s0}(a)\int d^3r\, \delta\left(r - q_s(a,t)\right)\times\left[\frac{e_s}{c}\,\dot{q}_s\cdot A(r,t) - e_s\phi(r,t)\right] \tag{21}$$

$$+\sum_s \int d^3a\, n_{s0}(a)\left[\frac{m_s}{2}\,|\dot{q}_s|^2 - m_s U_s\left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0}\right)\right]. \tag{22}$$

Electromagnetic potentials are Eulerian in nature, while fluid ‘particles’ are Lagrangian. The Lagrange-Euler maps for the species $s$ are defined via $n_s = n_{s0}/\mathcal{J}_s$ and $\dot{q}_s = v_s$.

# The two-fluid action (contd)

- The $\delta\phi$ variation leads to

$$\nabla \cdot E = 4\pi e \left(n_e - n_i\right) \tag{23}$$

- The $\delta A$ variation results in

$$\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \tag{24}$$

- The variation wrt $\delta q_s$ yields

$$m_s n_s \left(\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s\right) = e_s n_s \left(E + v_s \times B\right) - \nabla p_s \tag{25}$$

- Collectively, they represent the 2-fluid equations of motion.

# Towards one-fluid variables and the orderings

We begin by introducing the one-fluid variables

$$
\begin{aligned}
Q(a,t) &= \frac{1}{\rho_{m0}(a)}\left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right) \\
D(a,t) &= e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right) \\
\rho_{m0}(a) &= m_i n_{i0}(a) + m_e n_{e0}(a) \\
\rho_{q0}(a) &= e\left(n_{i0}(a) - n_{e0}(a)\right) .
\end{aligned}
\tag{26}
$$

and normalize the two-fluid action in Alfvenic units. The electric field is ordered out, as it is $\mathcal{O}\left(v_A^2/c^2\right)$. Statement of quasineutrality on the Lagrangian level necessitates $\mathcal{J}_i = \mathcal{J}_e$, and $n_{i0} = n_{e0}$. Terms that are first order in $\mu = m_e/m_i$ retained.

# Extended MHD action

$$S = -\frac{1}{8\pi}\int dt \int d^3r \; |\nabla \times A(r,t)|^2$$

$$+ \int dt \int d^3r \int d^3a \; n_0 \Big\{ \delta(r - q_i(Q,D))$$

$$\times \left[ \frac{e}{c}\dot{Q}(a,t) + \frac{\mu}{c n_0}\dot{D}(a,t)\cdot A(r,t) - e\phi(r,t) \right] \Big\}$$

$$+ \int dt \int d^3r \int d^3a \; n_0 \Big\{ \delta(r - q_e(Q,D))$$

$$\times \left[ -\frac{e}{c}\dot{Q}(a,t) + \frac{(1-\mu)}{c n_0}\dot{D}(a,t)\cdot A(r,t) + e\phi(r,t) \right] \Big\}$$

$$+ \frac{1}{2}\int dt \int d^3a \, n_0 m_i \left( (1+\mu)|\dot{Q}|^2(a,t) + \frac{\mu}{e^2 n_0^2}|\dot{D}|^2(a,t) \right)$$

# Extended MHD action

- In addition, there are two internal energy terms (ions and electrons).
- The variables $q_i$ and $q_e$ are short-hand notation for

$$q_i(Q, D) = Q(a, t) + \frac{\mu}{en_0} D(a, t)$$
$$q_e(Q, D) = Q(a, t) - \frac{1-\mu}{en_0} D(a, t) \qquad (28)$$

- Extended MHD has two-fluid effects, thus its Lagrange-Euler maps are very complex, comprising of contributions from both ions and electrons.
- The dynamical equations for the velocity $(\partial v/\partial t = \dots)$ and current $(\partial J/\partial t = \dots)$ are obtained by varying wrt $Q$ and $D$.

# Dynamical equations from the action

$$nm\left(\frac{\partial V}{\partial t}+(V\cdot\nabla)V\right)=-\nabla p+\frac{J\times B}{c}-\frac{m_e}{e^2}(J\cdot\nabla)\left(\frac{J}{n}\right). \qquad (29)$$

$$E+\frac{V\times B}{c}=\frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t}+\nabla\cdot(VJ+JV)\right) - \frac{m_e}{e^2 n}(J\cdot\nabla)\left(\frac{J}{n}\right)+\frac{(J\times B)}{enc}-\frac{\nabla p_e}{en}. \qquad (30)$$

- Last term on the RHS of (29) is necessary for energy conservation.
- Hall MHD is a subset of extended MHD, wherein the electrons are assumed to be massless. Thus, only terms that are zeroth order in $\mu$ are retained.

# The Hall MHD action

.

$$S = -\frac{1}{8\pi}\int dt\int d^3r\ |\nabla\times A(r,t)|^2$$

$$+\int dt\int d^3r\int d^3a\ n_0\left\{\delta(r-q_i(Q,D))\right.$$
$$\left.\times\left[\frac{e}{c}\dot{Q}(a,t)-e\phi(r,t)\right]\right\}$$

$$+\int dt\int d^3r\int d^3a\ n_0\left\{\delta(r-q_e(Q,D))\right.$$
$$\left.\times\left[-\frac{e}{c}\dot{Q}(a,t)+\frac{1}{cn_0}\dot{D}(a,t)\cdot A(r,t)+e\phi(r,t)\right]\right\}$$

$$+\frac{1}{2}\int dt\int d^3a\, n_0 m\,|\dot{Q}|^2(a,t)$$

# Hall MHD dynamical equations

- In Hall MHD, we define $q_i$ and $q_e$ as follows:

$$\begin{aligned} q_i(Q,D) &= Q(a,t) \\ q_e(Q,D) &= Q(a,t) - \frac{1}{en_0} D(a,t) \end{aligned} \tag{32}$$

- The Hall MHD equations follow upon varying wrt Q and D.

$$nm\left(\frac{\partial V}{\partial t} + (V\cdot\nabla)V\right) = -\nabla p + \frac{J\times B}{c}\,. \tag{33}$$

$$E + \frac{V\times B}{c} = \frac{(J\times B)}{enc} - \frac{\nabla p_e}{en}\,. \tag{34}$$

# Comments on other models and the energy

- Electron MHD obtained by demanding $\dot{q}_i = 0$ in the action since this model has stationary ions.
- The energy can be derived from the Lagrangian via the Legendre transformation

$$\mathcal{E} = \int d^3r \left[ \frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2} \right] \tag{35}$$

- Note that the last term on the RHS is present only when electron inertia is finite, i.e. absent in ideal/Hall MHD.
- Also verified the existence of momentum and angular momentum conservation for these models.

# On Extended MHD Hamiltonian formulations

- As we have seen, several models of extended MHD emerged via a common action principle.
- What about the Hamiltonian formulations of these models?
- The process of deriving the Eulerian Poisson brackets from their Lagrangian counterpart (along the lines of ideal MHD) is straightforward, but turns out to be rather lengthy.
- Alternatively, one can "guess" the Poisson bracket as done in Abdelhamid et al. (2015).
- The Hamiltonian formulation can be used to arrive at some interesting similarities between the different "beyond MHD" models which points to their common origin from the 2-fluid model.

# A snapshot of deriving the Hall MHD Poisson bracket

Here, the $\partial^2 q_f/\partial a\partial a$ term in the integration by parts vanishes because it is a symmetric object contracted with an antisymmetric one, and the second factor of $\partial q_f/\partial a$ appears because we want the delta-function derivative to give a derivative with respect to $q$ (and thus $x$). These factors may be eliminated in the following manner:

$$\begin{aligned}\epsilon^{jkl}\frac{\partial q_f^i}{\partial a^j}\frac{\partial q_f^m}{\partial a^k} &= \frac{1}{2}\epsilon^{jkl}\left(\frac{\partial q_f^i}{\partial a^j}\frac{\partial q_f^m}{\partial a^k}-\frac{\partial q_f^i}{\partial a^k}\frac{\partial q_f^m}{\partial a^j}\right)\\ &= \frac{1}{2}\epsilon^{jkl}\frac{\partial q_f^a}{\partial a^j}\frac{\partial q_f^b}{\partial a^k}\delta_{ab}^{im} = \frac{1}{2}\epsilon^{jkl}\frac{\partial q_f^a}{\partial a^j}\frac{\partial q_f^b}{\partial a^k}\epsilon^{nim}\epsilon^{nab} = \frac{1}{2}C^{ln}\epsilon^{nim}.\end{aligned}$$

Thus, using (30), that portion of the $\delta f$ variation becomes

$$\int\int \frac{c}{2n_0 e}\frac{\delta f}{\delta B^i}\mathcal{J}_f\delta\pi_d^j\epsilon^{jik}\delta_k'(x-q-q_d)d^3a\,d^3x.$$

Comparison of the expanded Eulerian $\delta f$ with the right side of (41) then gives expressions for the Lagrangian functional derivatives in terms of the Eulerian ones

$$\begin{aligned}
\frac{\delta f}{\delta\pi^i} &= \int\frac{\delta f}{\delta m^i}\delta(x-q(a,t))d^3x = \frac{\delta f}{\delta m^i}\bigg|_{x=q(a,t)}\\
\frac{\delta f}{\delta q^i} &= -\int\left(\frac{\delta f}{\delta\rho}\rho_0+\frac{\delta f}{\delta\sigma}\sigma_0+\frac{\delta f}{\delta m^j}\pi\right)\delta_i'(x-q)+\frac{\delta f}{\delta B^j}B_0^k\frac{\partial q_f^j}{\partial a^k}\delta_i'(x-q-q_d)-\frac{\delta f}{\delta B^i}B_0^k\frac{\partial q_f^j}{\partial a^k}\delta_j'(x-q-q_d)d^3x\\
&= \int\left[\rho_0\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta\rho}\right)+\sigma_0\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta\sigma}\right)+\pi^j\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta m^j}\right)\right]\delta(x-q)+\mathcal{J}_f\left[B^j\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right)-B^j\frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right)\right]\delta(x-q-q_d)d^3x\\
\frac{\delta f}{\delta q_d^i} &= -\int\frac{\delta f}{\delta B^j}B_0^k\frac{\partial q_f^j}{\partial a^k}\delta_i'(x-q-q_d)-\frac{\delta f}{\delta B^i}B_0^k\frac{\partial q_f^j}{\partial a^k}\delta_j'(x-q-q_d)d^3x\\
&= \int\mathcal{J}_f\left[B^j\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right)-B^j\frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right)\right]\delta(x-q-q_d)d^3x\\
\frac{\delta f}{\delta\pi_d^i} &= \int\frac{\delta f}{\delta B^j}\frac{c}{2n_0 e}\mathcal{J}_f\,\epsilon^{ijk}\delta_k'(x-q-q_d)\,d^3x = \frac{c}{2ne}\int\left(\nabla\times\frac{\delta f}{\delta\mathbf{B}}\right)_i\delta(x-q-q_d)d^3x = -\frac{c}{2ne}\left(\nabla\times\frac{\delta f}{\delta\mathbf{B}}\right)_i\bigg|_{x=q(a,t)+q_d(a,t)}.
\end{aligned}$$

# On Hamiltonian formulations of Hall MHD

- The Hall MHD Poisson bracket can be expressed as

$$\{F,G\}^{HMHD} = \{F,G\}^{MHD} + \{F,G\}^{Hall}, \tag{36}$$

where the first term is the ideal MHD Poisson bracket and

$$\{F,G\}^{Hall} \;=\; -d_i \int_D d^3x \, \frac{B}{\rho} \cdot \left[ (\nabla \times F_B) \times (\nabla \times G_B) \right], \tag{37}$$

where $d_i$ is the normalized skin depth.

- Magnetic helicity $M = \int_D d^3x \, A \cdot B$ is an invariant of Hall MHD (as well as for ideal MHD).
- In addition, the canonical helicity $C = \int_D d^3x \, (A + d_i V) \cdot (B + d_i \nabla \times V)$ is also conserved.

# Inertial MHD and its bracket

- Inertial MHD has finite electron inertia, but no Hall term. The Ohm's law given by

$$\frac{\partial B^\star}{\partial t} = \nabla \times (V \times B^\star) + d_e^2 \nabla \times \left[\frac{(\nabla \times B) \times (\nabla \times V)}{\rho}\right]. \quad (38)$$

$$B^\star = B + d_e^2 \nabla \times \left(\frac{\nabla \times B}{\rho}\right), \quad (39)$$

- Although inertial MHD lacks the Hall current and Hall MHD lacks electron inertia, their Poisson brackets are interchangeable:

$$\{F, G\}^{IMHD} \equiv \{F, G\}^{HMHD} [\mathcal{B}_\pm; \mp 2d_e], \quad (40)$$

where $\mathcal{B}_\pm = B^\star \pm d_e \nabla \times V$.

- There are two conserved helicities in this model:

$$\mathcal{C} = \int_D d^3x \, (A^\star \pm d_e V) \cdot (B^\star \pm d_e \nabla \times V), \quad (41)$$

# Extended MHD equations

Let us recall the equations of extended MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0, \tag{42}$$

$$\begin{aligned}\frac{\partial V}{\partial t} + (\nabla \times V) \times V &= -\nabla \left( h + \frac{V^2}{2} \right) + \frac{(\nabla \times B) \times B^*}{\rho} \\ &\quad - d_e^2 \nabla \left[ \frac{(\nabla \times B)^2}{2\rho^2} \right],\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial B^*}{\partial t} &= \nabla \times (V \times B^*) - d_i \nabla \times \left( \frac{(\nabla \times B) \times B^*}{\rho} \right) \\ &\quad + d_e^2 \nabla \times \left[ \frac{(\nabla \times B) \times (\nabla \times V)}{\rho} \right].\end{aligned} \tag{44}$$

# Extended MHD bracket and general properties

- Similar process of mapping the extended MHD Poisson bracket yields

$$\{F, G\}^{ExtMHD} \equiv \{F, G\}^{HMHD} [d_i - 2\kappa; \mathcal{B}_\kappa] , \qquad (45)$$

where $\mathcal{B}_\kappa := B^\star + \kappa\nabla \times V$ and $\kappa$ satisfies $\kappa^2 - d_i\kappa - d_e^2 = 0$.

- Two helicities exist for extended MHD:

$$\mathcal{C}_{I,II} = \int_D d^3x \, (A^\star + \kappa V) \cdot (B^\star + \kappa\nabla \times V) , \qquad (46)$$

- All of the "beyond MHD" models discussed here possess two helicities of the form $\int_D d^3r \, P \cdot (\nabla \times P)$ - akin to the fluid/magnetic helicity - and two frozen-in generalizations of the magnetic flux.

# Summary

- I have briefly sketched the derivation of the ideal MHD action principle (in Lagrangian variables).
- The corresponding Poisson bracket in Lagrangian variables can be mapped to obtain the Hamiltonian formulation of ideal MHD in Eulerian variables.
- Subsequently, the derivation of various "beyond MHD" models from the 2-fluid model action was outlined.
- Lastly, I discussed how certain "beyond MHD" models with mutually (or partially) exclusive effects possess a certain degree of commonality, as their Poisson brackets have the same underlying structure.
- The extended MHD bracket can also be used to determine the existence of two different helicities akin to the fluid/magnetic helicity.

# Whence next?

- One can include additional non-dissipative plasma effects arising from FLR contributions using the HAP approach. This has already been done for some simple models (e.g. the inclusion of gyroviscosity).
- The Hamiltonian formulation can be used to extract the equilibria and to study their stability for the beyond MHD models presented here (Kaltsas et al. 2018).
- Action principles for relativistic MHD and extended MHD have also been formulated recently (Kawazura et al. 2017).
- Ongoing work to construct variational integrators for these models, and apply them to study astrophysical and fusion phenomena of interest (e.g. magnetic reconnection).

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