Looking beyond MHD: Implications for the plasma Universe

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Overview

Introduction

- The statement that 99% of the visible Universe is comprised of plasma is well-known.
- Astrophysical plasmas range from the tenuous intergalactic medium and interstellar medium to the interiors of stars.

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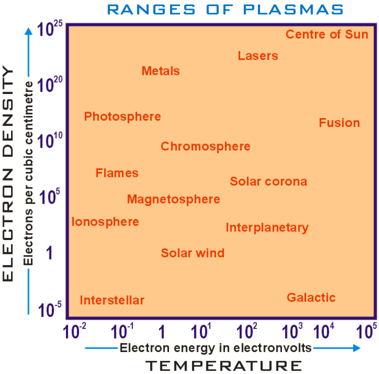


Motivation

- From the previous figure, it is apparent that there exists considerable variation in plasma parameters (e.g. number density, temperature) because of the diverse array of astrophysical environments under consideration.
- Hence, using a one-model-fits-all approach without due caution can be dangerous since several astrophysical systems can lie outside any particular model's domain of validity.
- In this talk, I will delineate a couple of simple plasma fluid models that possess a broader domain of validity compared to magnetohydrodynamics.
- The ensuing ramifications of using these models in the context of astrophysical mechanisms/systems will then be outlined.

Magnetohydrodynamics

- The most widely used plasma model in astrophysics is ideal magnetohydrodynamics (MHD).
- MHD is a fluid model with the following dynamical equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) = 0, \tag{3}$$

where p is the total pressure and $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$. Here, ρ, $\mathbf{V}$ and $\mathbf{B}$ are the mass density, velocity and magnetic field respectively.

Looking beyond MHD

Looking beyond MHD

- Plasma physics is replete with fluid models that have a broader scope than MHD.
- In this talk, I will focus on a class of models that include ideal MHD as a limiting case.
- One of the chief disadvantages of ideal MHD is that it becomes inaccurate when the characteristic length scale L becomes comparable to (or smaller than) the ion and electron skin depths, denoted by $d_i = c/\omega_{pi}$ and $d_e = c/\omega_{pe}$ respectively; here, ω_{ps} is the plasma frequency of species s.
- The "beyond MHD" models presented here include either or both of these length scales and are therefore valid over a broader domain compared to MHD.

Hall MHD

- Hall MHD is the simplest of the "beyond MHD" models and has been invoked in a wide range of astrophysical environments ranging from protoplanetary discs and the solar wind to neutron star crusts and planetary magnetospheres.
- Hall MHD has the same governing equations as ideal MHD, except for a different Ohm's law. The barotropic Hall MHD Ohm's law is

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times \left[\left(\mathbf{V} - \frac{\mathbf{J}}{ne}\right) \times \mathbf{B}\right] = 0. \tag{4}$$

- The Hall drift, i.e. the $\mathbf{J} \times \mathbf{B}$ term in (4), encapsulates the fact that the electron and ion fluid velocities are not necessarily comparable in Hall MHD.

Extended MHD

- In Hall MHD, the electrons are effectively assumed to be massless. The inclusion of finite electron inertia leads to extended MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{5}$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} - \frac{m_e}{e^2} \mathbf{J} \cdot \nabla \left(\frac{\mathbf{J}}{n} \right), \tag{6}$$

$$\begin{aligned} \mathbf{E} + \mathbf{V} \times \mathbf{B} &= \frac{\mathbf{J} \times \mathbf{B} - \nabla p_e}{en} \\ &+ \frac{m_e}{ne^2} \left[\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left(\mathbf{VJ} + \mathbf{JV} - \frac{1}{en} \mathbf{JJ} \right) \right], \end{aligned} \tag{7}$$

where n is the number density, and m_e (m_i) is the electron (ion) mass.

Extended MHD

- We can also rewrite (6) and (7):

$$\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + (\nabla \times \mathbf{V}) \times \mathbf{V} &= -\nabla \left(h + \frac{V^2}{2} \right) + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}^\star}{\rho} \\ &- d_e^2 \nabla \left[\frac{(\nabla \times \mathbf{B})^2}{2\rho^2} \right], \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial \mathbf{B}^\star}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}^\star) - d_i \nabla \times \left(\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}^\star}{\rho} \right) \\ &+ d_e^2 \nabla \times \left[\frac{(\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{V})}{\rho} \right], \end{aligned} \tag{9}$$

for the barotropic case in Alfvénic units, with d_i and d_e denoting the normalized skin depths. The dynamical variable $\mathbf{B}^\star$ is defined as

$$\mathbf{B}^\star = \mathbf{B} + d_e^2 \, \nabla \times \left[\frac{\nabla \times \mathbf{B}}{\rho} \right], \tag{10}$$

Magnetic fields and the dynamo mechanism [*Astrophys. J.*, **829**, 51 (2016)]

- Magnetic fields are ubiquitous in astrophysics.
- They have been documented in stars, accretion discs, compact objects, galaxies and clusters of galaxies.
- The dynamo mechanism is widely invoked to explain the observed magnetic fields.
- Although dynamo theory has made considerable advances, it has typically relied upon resistive MHD (ideal MHD + resistivity) as the base physical model.
- Since the Hall effect has been argued to be important in several astrophysical settings, it is instructive to ask how it can affect the generation of these magnetic fields.

Incompressible Hall MHD [*Astrophys. J.*, **829**, 51 (2016)]

- Let us recall the equations of resistive incompressible Hall MHD.
- The dynamical equation for the velocity is

$$\left[\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right]\mathbf{V} = (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla\left(\frac{p}{\rho}\right), \qquad (11)$$

  which is the same as that of ideal MHD.
- The Ohm's law for the model is

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}_E\times\mathbf{B}) - \eta\nabla\times\mathbf{J}, \qquad (12)$$

  where $\mathbf{J} := \nabla\times\mathbf{B}$, $\mathbf{V}_E = \mathbf{V} - d_i\mathbf{J}$, η is the resistivity, and d_i is the ion skin depth.
- The $\mathbf{J}\times\mathbf{B}$ term in the Ohm's law of Hall MHD.

Mean-field dynamo theory [*Astrophys. J.*, **829**, 51 (2016)]

- Each field is decomposed into a mean-field (large-scale) component and a fluctuating (turbulent) component, i.e. $\mathbf{X} = \mathbf{X}_0 + \mathbf{x}$ for a given field $\mathbf{x} \in (\mathbf{v}, \mathbf{b})$.
- In our simple model, large-scale velocity is taken to be negligible.
- The large-scale magnetic field evolves via

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times \boldsymbol{\mathcal{E}} - d_i \nabla \times (\mathbf{J}_0 \times \mathbf{B}_0) - \eta \nabla \times \mathbf{J}_0. \quad (13)$$

- The electromotive force $\boldsymbol{\mathcal{E}}$ is

$$\boldsymbol{\mathcal{E}} = \langle (\mathbf{v} - d_i \nabla \times \mathbf{b}) \times \mathbf{b} \rangle, \quad (14)$$

and must be specified in terms of $\mathbf{B}_0$.

Mean-field dynamo theory [*Astrophys. J.*, **829**, 51 (2016)]

- As per the above assumptions, the electromotive force is expanded as

$$\boldsymbol{\mathcal{E}} = \alpha \mathbf{B}_0 - \beta \nabla \times \mathbf{B}_0 + \dots, \tag{15}$$

and the goal of the classical dynamo theory is to compute the coefficients α and β.

- From (14), the EMF can be approximated as

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \tau_c \Big[\langle \partial_t \mathbf{v} \times \mathbf{b} \rangle + \langle \mathbf{v} \times \partial_t \mathbf{b} \rangle - d_i \langle (\nabla \times \mathbf{b}) \times \partial_t \mathbf{b} \rangle \\ & \quad - d_i \langle (\nabla \times \partial_t \mathbf{b}) \times \mathbf{b} \rangle \Big], \end{aligned} \tag{16}$$

where τ_c is the correlation time.

- Observe that the first term depends on the velocity evolution equation, whilst the last three do not.

The transport coefficients [*Astrophys. J.*, **829**, 51 (2016)]

- Computing the transport coefficients requires a closure - the standard First Order Smoothing Approximation (FOSA) is used. The results remain unchanged when other alternatives were employed.
- The coefficients are thus found to be

$$\alpha = -\frac{\tau_c}{3}\left\langle \mathbf{v}_E \cdot (\nabla \times \mathbf{v}_E) + d_i \mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{v}_E) \right\rangle + \frac{\tau_c}{3}\left\langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \right\rangle, \tag{17}$$

$$\beta = \frac{\tau_c}{3}\Big\langle \mathbf{v}_E^2 + d_i \left(\mathbf{v}_E \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{v}_E\right) + d_i^2\, \mathbf{b} \cdot \nabla \times \nabla \times \mathbf{b} \Big\rangle, \tag{18}$$

and these terms reduce to the their MHD counterparts when $d_i \to 0$.

Comments on the alpha coefficient [*Astrophys. J.*, **829**, 51 (2016)]

- The α coefficient in MHD becomes zero for Alfvénic states, i.e. provided that $\mathbf{v} = \pm\mathbf{b}$.
- In Hall MHD, it is found that the α coefficient does not vanish for Alfvénic states. Hence, we can ask: what is the particular state for which it vanishes?
- It turns out that α becomes zero for specific cases of the famous double Beltrami states of Hall MHD (Mahajan & Yoshida 1998).
- These states (with $\alpha = 0$) are characterized by the absence of equipartition ($v^2 \neq b^2$), unlike the MHD case. This feature also arises in many other contexts of Hall MHD turbulence and dynamos.

Comments on the beta coefficient [*Astrophys. J.*, **829**, 51 (2016)]

- Let us recall that β is given by

$$\beta = \frac{\tau_c}{3}\Big\langle \mathbf{v}_E^2 + d_i\left(\mathbf{v}_E \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{v}_E\right) + d_i^2\, \mathbf{b} \cdot \nabla \times \nabla \times \mathbf{b}\Big\rangle. \qquad (19)$$

- After subsequent simplification, it can be shown that the turbulent resistivity β is not necessarily positive-definite.
- This result is consistent with the numerical transport coefficients computed in Mininni, Alexakis & Pouquet (2007).
- The above fact is very important from the standpoint of dynamo action.

Comments on the beta coefficient [*Astrophys. J.*, **829**, 51 (2016)]

- Excluding the α-effect, the growth rate is proportional to $-\beta k^2$.
- Hence, even when the α-effect is absent, the validity of the $\beta < 0$ condition under certain circumstances can enable the generation of large-scale fields via the dynamo mechanism.
- Most MHD treatments that invoke a non-positive β rely on it being a tensor (with negative off-diagonal components), and require the presence of additional physics (such as large-scale velocity shear).
- In contrast, the consideration of the Hall effect alone may suffice to produce a negative turbulent resistivity without the necessity of including further constraints.
- Based on simple dimensional considerations, β may become negative when the Hall parameter (d_i/L) becomes sufficiently large.

Extended MHD - nonlinear waves [*Astrophys. J.*, **829**, 87 (2016)]

- We assume that the model is incompressible, i.e. $\rho = 1$ in the normalized units, and the ion skin depth is chosen to be the normalizing length scale.
- The magnetic field and the flow are split into the ambient and wave components:

$$\mathbf{B} = \widehat{\mathbf{e}}_{B_0} + \mathbf{b}, \qquad \mathbf{V} = \mathbf{v} \tag{20}$$

- The wave solutions are given by

$$\mathbf{b}^* = \mathbf{b}_k^* \; \exp\left[i\mathbf{k}\cdot\mathbf{x} + i\mu\left(\widehat{\mathbf{e}}_{B_0}\cdot\mathbf{k}\right)t\right], \tag{21}$$

$$\mathbf{v} = \mathbf{v}_k \; \exp\left[i\mathbf{k}\cdot\mathbf{x} + i\mu\left(\widehat{\mathbf{e}}_{B_0}\cdot\mathbf{k}\right)t\right], \tag{22}$$

where $\mathbf{b}^* = \mathbf{b} + d_e^2\nabla\times\left(\nabla\times\mathbf{b}\right)$.

Extended MHD - nonlinear waves [*Astrophys. J.*, **829**, 87 (2016)]

- The fluctuating magnetic and velocity fields are related via

$$\mathbf{b}_k = \mu \mathbf{v}_k. \tag{23}$$

- The quantity μ, calculated by solving a quadratic equation, is

$$\mu_{\pm}(k) = \frac{1}{(1+d_e^2 k^2)} \left[-\frac{k}{2} \pm \sqrt{\frac{k^2}{4} + (1 + d_e^2 k^2)} \right] \tag{24}$$

- In the MHD regime, characterized by $k \ll 1$, we obtain $\mu \to \pm 1$, implying that there is an equipartition of magnetic and kinetic energies, i.e. $b_k^2 = v_k^2$. However, in all other cases, we have $b_k^2 \neq v_k^2$.

Extended MHD - Kolmogorov scalings [*Astrophys. J.*, **829**, 87 (2016)]

- It is assumed that the energy cascade rate ϵ_E, taken to be independent of the wavenumber, is the product of the eddy turnover time $\tau = (k|v_k|)^{-1}$ and the energy.

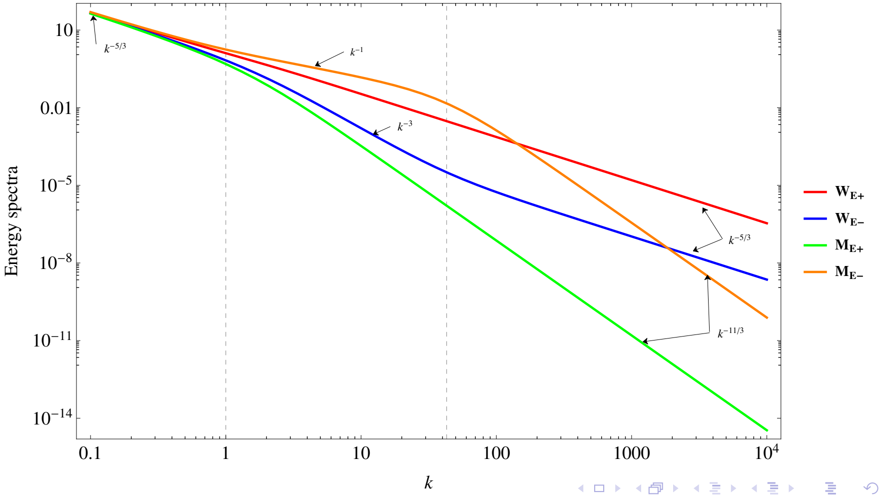
$$\epsilon_E = k|v_k|\left(1+\mu^2\left(1+d_e^2k^2\right)\right)\frac{|v_k^2|}{2}, \quad (25)$$

- The kinetic energy per wavenumber $W_E(k)$ is thus given by

$$W_E(k) = \frac{|v_k^2|}{k} = (2\epsilon_E)^{2/3}k^{-5/3}\left[1+\mu^2\left(1+d_e^2k^2\right)\right]^{-2/3}, \quad (26)$$

and the magnetic energy per wavenumber $M_E(k)$ follows from $M_E(k) = \mu^2 W_E(k)$.

Extended MHD - Kolmogorov scalings [*Astrophys. J.*, **829**, 87 (2016)]



Extended MHD - Kolmogorov scalings [*Astrophys. J.*, **829**, 87 (2016)]

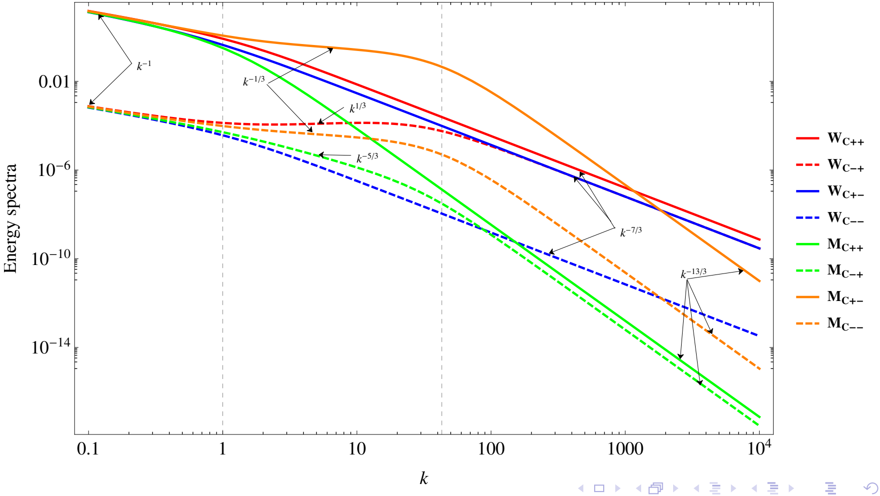
- Extended MHD has two generalized helicities:

$$C_\pm = \int_\Omega \mathbf{P}_\pm^* \cdot \left(\nabla \times \mathbf{P}_\pm^*\right)\, d^3x, \tag{27}$$

where $\mathbf{P}_\pm^* = \mathbf{V} + \theta_\pm \mathbf{A}^*$ and $\theta_\pm = \left(-1 \pm \sqrt{1+4d_e^2}\right)/\left(2d_e^2\right)$.

- The similarity between $C_\pm$ and the fluid/magnetic helicity will be discussed in more detail during Wednesday's talk.
- Kolmogorov-type arguments can be used to obtain W and M in the various ranges based on the assumption that the cascade rates of the two helicities are constant.

Extended MHD - Kolmogorov scalings [*Astrophys. J.*, **829**, 87 (2016)]



Extended MHD - Basic findings [*Astrophys. J.*, **829**, 87 (2016)]

- The magnetic energy spectrum is predicted to have a $-5/3$ slope in the MHD range, i.e. for $L > d_i$.
- It displays a steeper slope in the Hall range ($d_e < L < d_i$). At scales smaller than d_e, the magnetic spectrum becomes very steep and can attain the power-law exponent of $-13/3$.
- Thus, a total of two spectral breaks exist, which are approximately manifested at the ion and electron skin depths.
- The equipartition of kinetic and magnetic energies is not applicable in the Hall regime where $L \lesssim d_i$.
- Most of these features are consistent with observations of the solar wind.

Solar wind observations [*Astrophys. J.*, **829**, 87 (2016)]

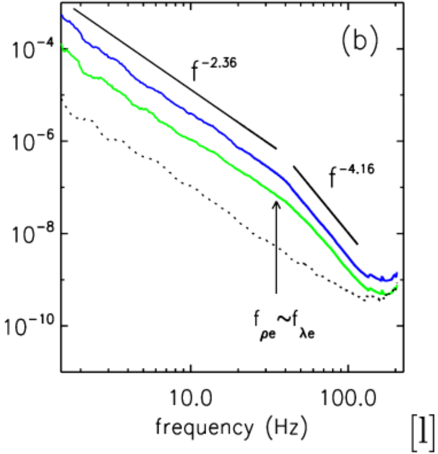


Figure: Magnetic energy spectra in the Hall and electron inertia regimes [Credit: Sahraoui et al., *Phys. Rev. Lett.*, **102**, 231102 (2009)]

Solar wind observations [*Astrophys. J.*, **829**, 87 (2016)]

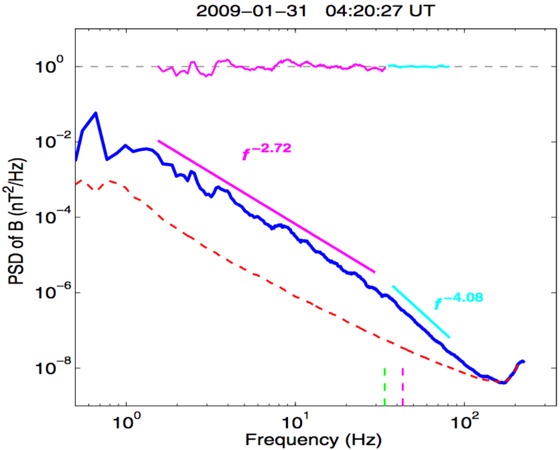


Figure: Magnetic energy spectra in the Hall and electron inertia regimes [Credit: Sahraoui et al., *Astrophys. J.*, **777**, 15 (2013)]

Seed magnetic fields in MHD [*Phys. Plasmas*, **23**, 112104 (2016)]

- In order to explain the origin of a seed (i.e. small, but finite) magnetic field, a number of exotic (and a few non-exotic) mechanisms have been proposed.
- For the ideal MHD Ohm's law, a non-zero magnetic field cannot be generated from an initially zero magnetic field.
- One of the most common explanations is via the baroclinic instability, i.e. the fluid is no longer barotropic implying that the term $\nabla p \times \nabla \rho \neq 0$. In this case, Ohm's law is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \dots \frac{\nabla p \times \nabla \rho}{\rho^2} \tag{28}$$

  and this mechanism is commonly called the Biermann battery in astrophysics.

Seed generation in "beyond MHD" models [*Phys. Plasmas*, **23**, 112104 (2016)]

- Consider a barotropic 2-fluid model, whose dynamics can be compactly expressed as

$$\frac{\partial \hat{\mathbf{B}}_s}{\partial t} - \nabla \times \left(\mathbf{V}_s \times \hat{\mathbf{B}}_s \right) = 0, \tag{29}$$

with $\mathbf{V}_s$ denoting the fluid velocity of species s and $\hat{\mathbf{B}}_s = \mathbf{B} + \frac{m_s}{q_s}\mathbf{V}_s$. One can therefore rewrite the above equation as

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times \left(\mathbf{V}_s \times \mathbf{B} \right) = \mathbf{S}, \tag{30}$$

where $\mathbf{S}$ now serves as a source term for the generation of the magnetic field, and is given by

$$\mathbf{S} = -\frac{m}{q}\left[\frac{\partial \mathbf{V}_s}{\partial t} - \nabla \times \left(\mathbf{V}_s \times [\nabla \times \mathbf{V}_s] \right)\right] \tag{31}$$

Seed generation in "beyond MHD" models [*Phys. Plasmas*, **23**, 112104 (2016)]

- Although our discussion was for a non-relativistic model, the basic principle is still valid when relativistic (or even quantum-mechanical) effects are included.
- In all these cases, one can construct an appropriate generalized vorticity $\mathbf{\Omega}$ that obeys $\partial_t \mathbf{\Omega} - \nabla \times (\mathbf{V} \times \mathbf{\Omega}) = 0$. Thus, if $\mathbf{\Omega}$ was initially zero, a finite value cannot be subsequently generated.
- However, since there is no equivalent constraint on $\mathbf{B}$, it is still possible to generate a finite value even if the magnetic field was initially absent.
- An explicit example of a relativistic non-baroclinic drive can be found in Mahajan & Yoshida (2011).
- The form of $\mathbf{\Omega}$ has been outlined in *Phys. Plasmas*, **23**, 112104 (2016) for a hot, relativistic fluid and explicit analytical solutions are also provided wherein a seed magnetic field can be generated.

Conclusions

- As astrophysical (and even fusion) plasmas are very diverse, it is important to make use of an appropriate physical model when studying associated phenomena.
- Although MHD is a powerful theory, it should not be regarded as being universally applicable.
- Two notable "beyond MHD" effects become increasingly prominent as one moves to smaller scales - the Hall current and electron inertia; of course, there are many other models not considered herein.
- We have shown that these extended MHD effects have a wide array of consequences for astrophysical mechanisms such as dynamos, turbulence and magnetogenesis.

Conclusions

- The Hall effect modifies the α-coefficient such that it becomes non-zero for Alfvénic states.
- The Hall current may lead to the turbulent resistivity becoming negative and thereby serve as an anti-diffusion term enabling the growth of large-scale magnetic fields via the dynamo mechanism.
- The inclusion of the Hall drift and electron inertia (via extended MHD) can result in the steepening of the turbulence spectra, with two spectral breaks, in a manner that is consistent with solar wind observations.
- These "beyond MHD" terms also provide alternatives to the Biermann battery mechanism for generating seed magnetic fields.

Conclusions

- In this talk, I have only barely scratched the surface of the many implications arising from the inclusion of extended MHD effects.
- I have not covered the role of extended MHD in magnetic reconnection despite its undoubted prominence.
- Magnetic reconnection entails the conversion of magnetic energy into kinetic and thermal energy, as well as particle acceleration.
- It has been argued to drive many "explosive" astrophysical phenomena (e.g. flares, GRBs) in the Universe, but classical resistive MHD models are too slow (by several orders).
- In contrast, the use of extended MHD models has been shown to yield timescales commensurate with observations.
- Thus, there are plenty of vistas waiting to be explored wherein "beyond MHD" effects will undoubtedly play a significant role.