APPLIED MATH PERSPECTIVES ON STOCHASTIC CLIMATE MODELS

BY

ANDREW J. MAJDA

MORSE PROFESSOR OF ARTS & SCIENCES
DEPARTMENT OF MATHEMATICS
AND CENTER FOR ATMOSPHERE OCEAN SCIENCE
COURANT INSTITUTE, NYU
AND
CENTER FOR PROTOTYPE CLIMATE MODELING
NYU ABU DHABI

Main collaborators this talk

I. B. Khouider (University of Victoria), Y. Frenkel (CIMS

II. T. Sapsis (CIMS-MIT)

III. I. Grooms (CIMS)

Entering a New Era of Stochastic Climate Modelling

- Model crucial poorly represented processes in contemporary GCM’s such as intermittent organized tropical convection in atmosphere; mesoscale/submesoscale eddies in ocean

- Quantity Uncertainty in Intermediate and Long range forecasts where both Uncertainty in Initial Data and Forcing play a role.

- Represent unresolved stochastic backscatter from small scales to large scales in midlatitude dynamics and increase variability of GCM’S (Palmer, et al 1999, Frederiksen, et al 1997-2012, Berner at al 2010.)

Applied Mathematics Perspective

- Multi-scale Asymptotic Modelling
  - Majda and Klein, JAS 2003, Tropics
  - Klein. Ann Rev Fluids 2010
  - Khouider, Majda, Stechmann, Nonlinearity 2012 (Tropics, PDE)
- Stochastic Models for Low Frequency Dynamics
  - Majda, Timofeyev, Vanden-Eijnden, PNAS, 1999
  - Del Sole, Surveys in Geophysics, 2004
  - Palmer and Williams, Phil. Trans. Roy. Soc. (Special Issue), 2008
- Superparameterization
  - Grabowski and Smolarkiewicz, 1999
  - Grabowski, JAS, 2001
  - Majda, JAS, 2007 (Multi-scale Algo)
  - Xing, Majda, Grabowski, MWR 2010 (Sparse space-time)
  - Slawinska, Pauluis, Majda, Grabowski, 2012

Lecture Plan-Three Topics

I. Improving Tropical Convection Par Through Stochastic Multi-cloud Model

Majda and Khouider, PNAS 2002, SLC

Khouider and Majda, JAS, 2006, MCM

Khouider, Biello, Majda, Comm. Math. Sci.,2010

Frenkel, Majda, Khouider, JAS 2011, Climate Dynamics, 2012 (SMCM)

II. Uncertainty Quantification

Majda and Branicki, DCDS, 2012

Sapsis and Majda, Phys. D., 2012

Branicki and Majda, Nonlinearity, 2012

III. Stochastic Superparameterization in a Model for Wave Turbulence

Majda and Grote, PNAS, 2010 (Test Models)

Majda, CPAM, 2012 (Syst. Stoch. Model)

Grooms and Majda, SIAM J. MMS, 2012

Available on the Majda faculty website

Using the Stochastic Multicloud Model to Improve Cumulus Parameterization

Collaboration with

Boualem Khouider (University of Victoria)

Yevgeniy Frenkel (Courant Institute)

Outline

- A stochastic multicloud model for organized convection
- Deterministic simulation with clear deficiencies
- Improvement of convectively coupled waves and variability by stochastic model
- Papers:
  - ✓ K., Biello, and Majda, 2010: A Stochastic Multicloud model for tropical convection. Comm. Math. Sci.
  - ✓ Frenkel, Majda, and K., 2012: Using the stochastic multi-cloud model to improve tropical convective parameterization: A paradigm example. J. Atmos. Sci.

Why a stochastic model for convection?

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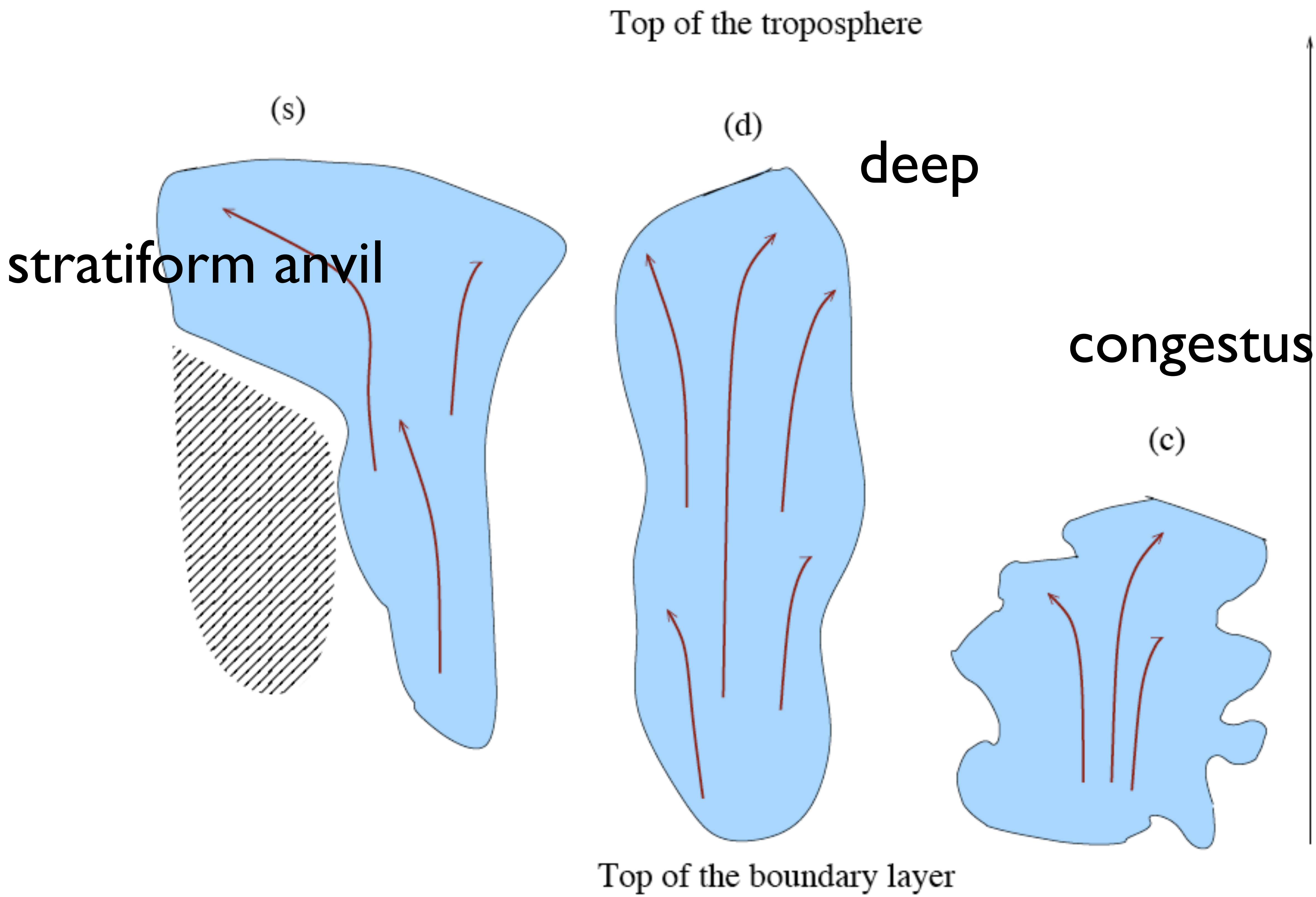
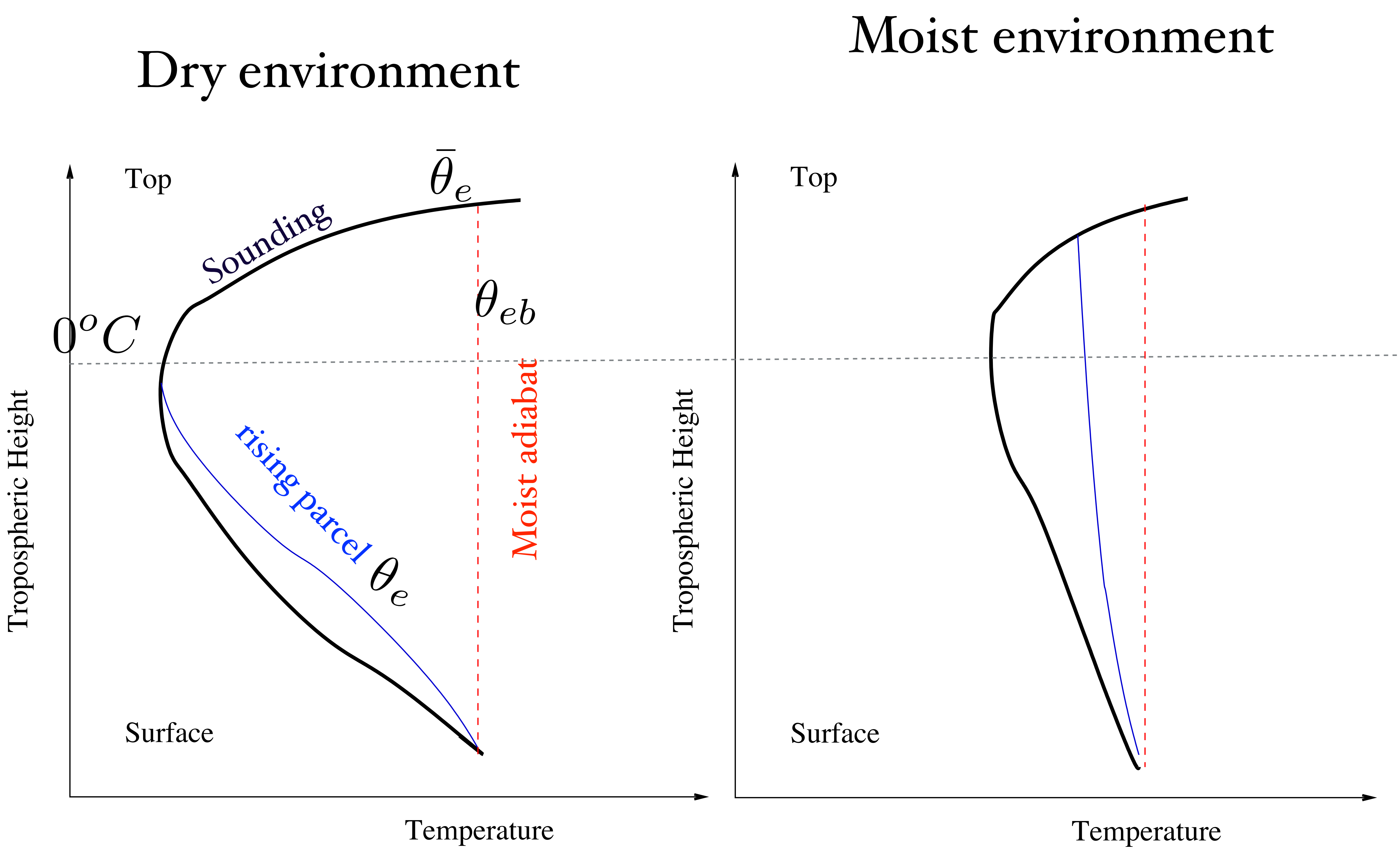
- Statistical-self similarity across-scales of tropical convective systems
- How cloud systems interact with each other and with the environment?
- Adequate representation of sub-grid dynamics
- Capture deviations from quasi-equilibrium paradigm
- Improve tropical variability in climate models ---> reduce model error
- We propose a stochastic model for area-fractions of various cloud types...

Schematic of three characteristic cloud types



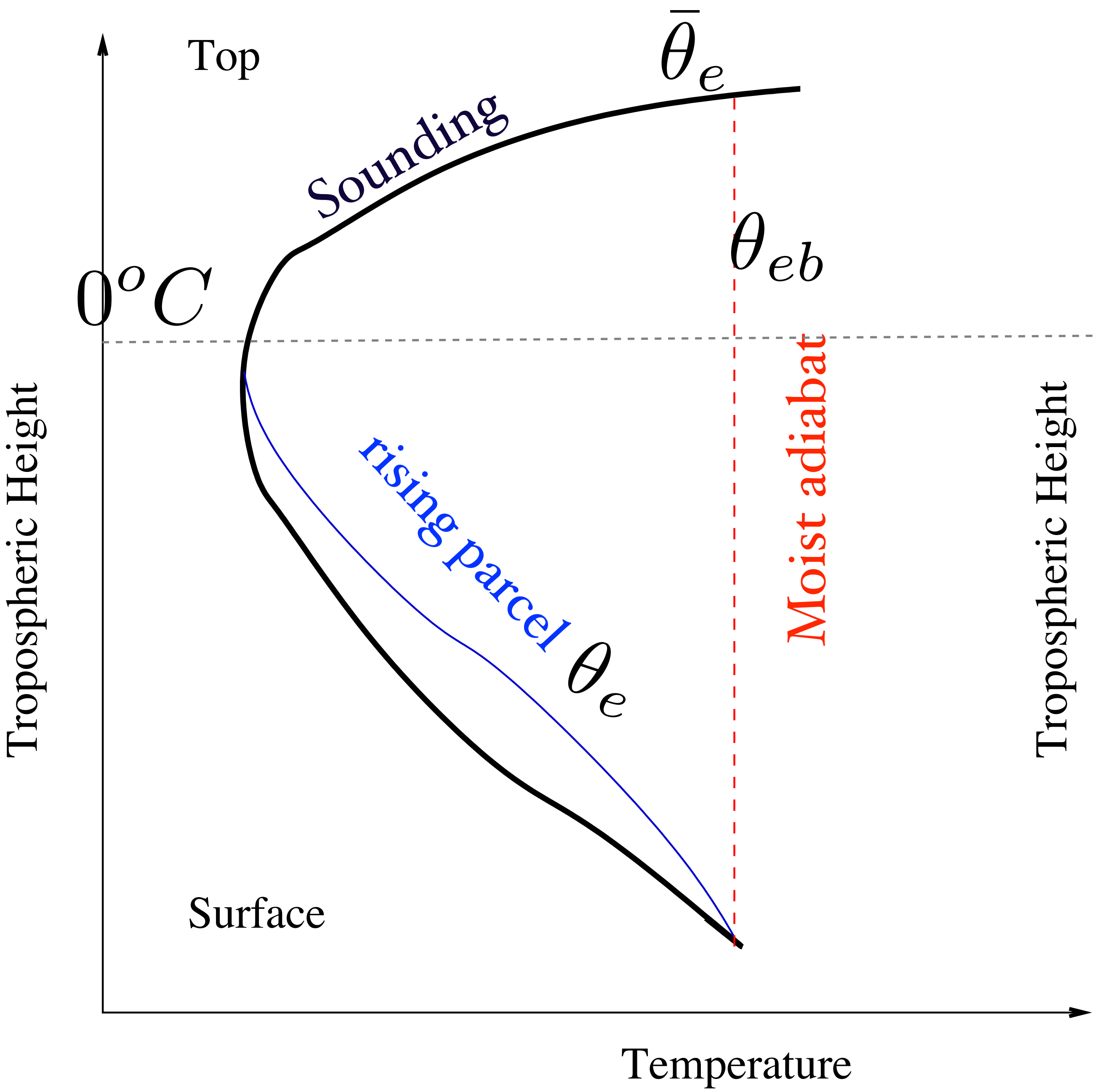
Dilute parcel lifting

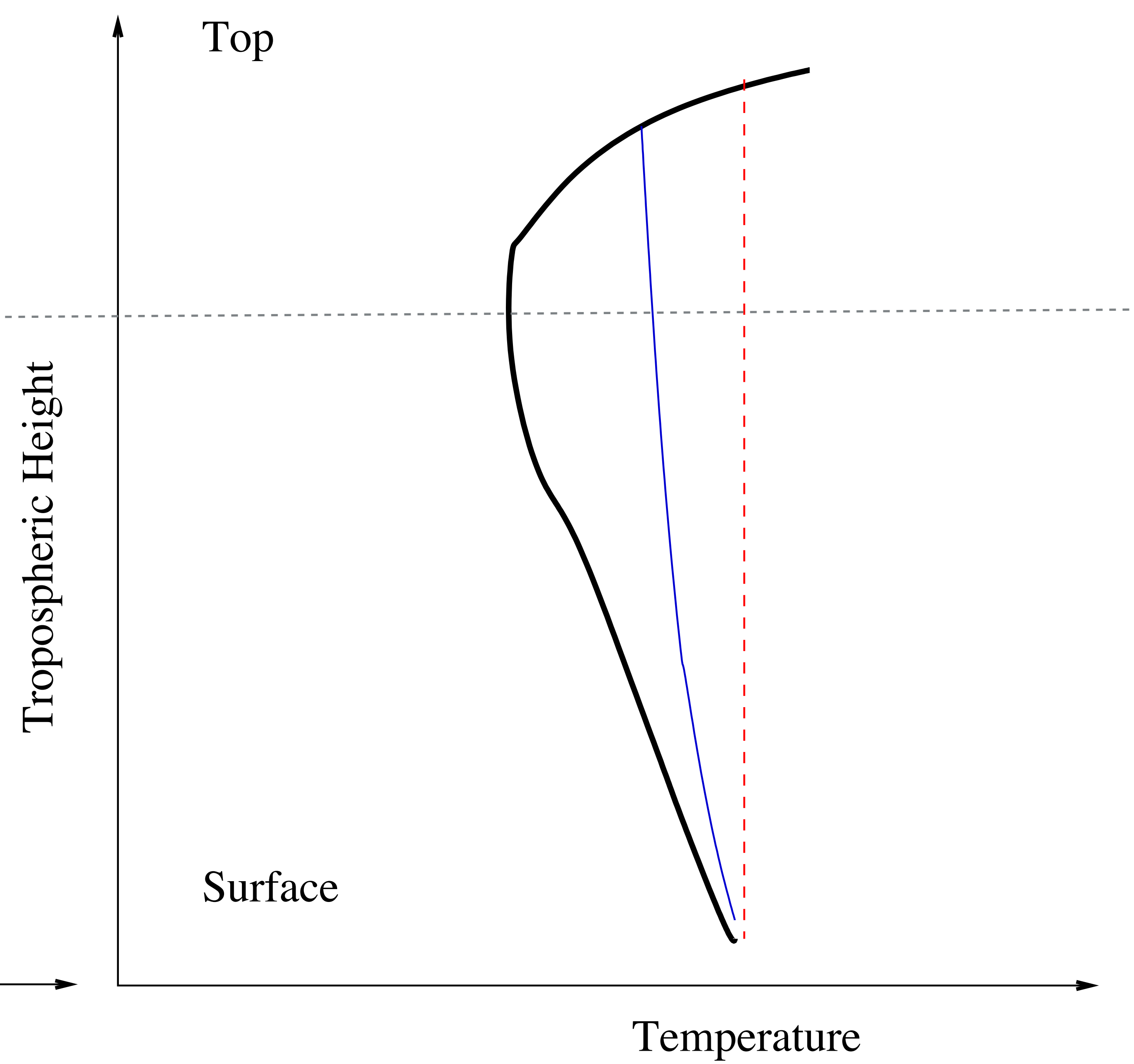


Dilute parcel lifting

Dry troposphere with positive CAPE favors congestus clouds

Moist environment

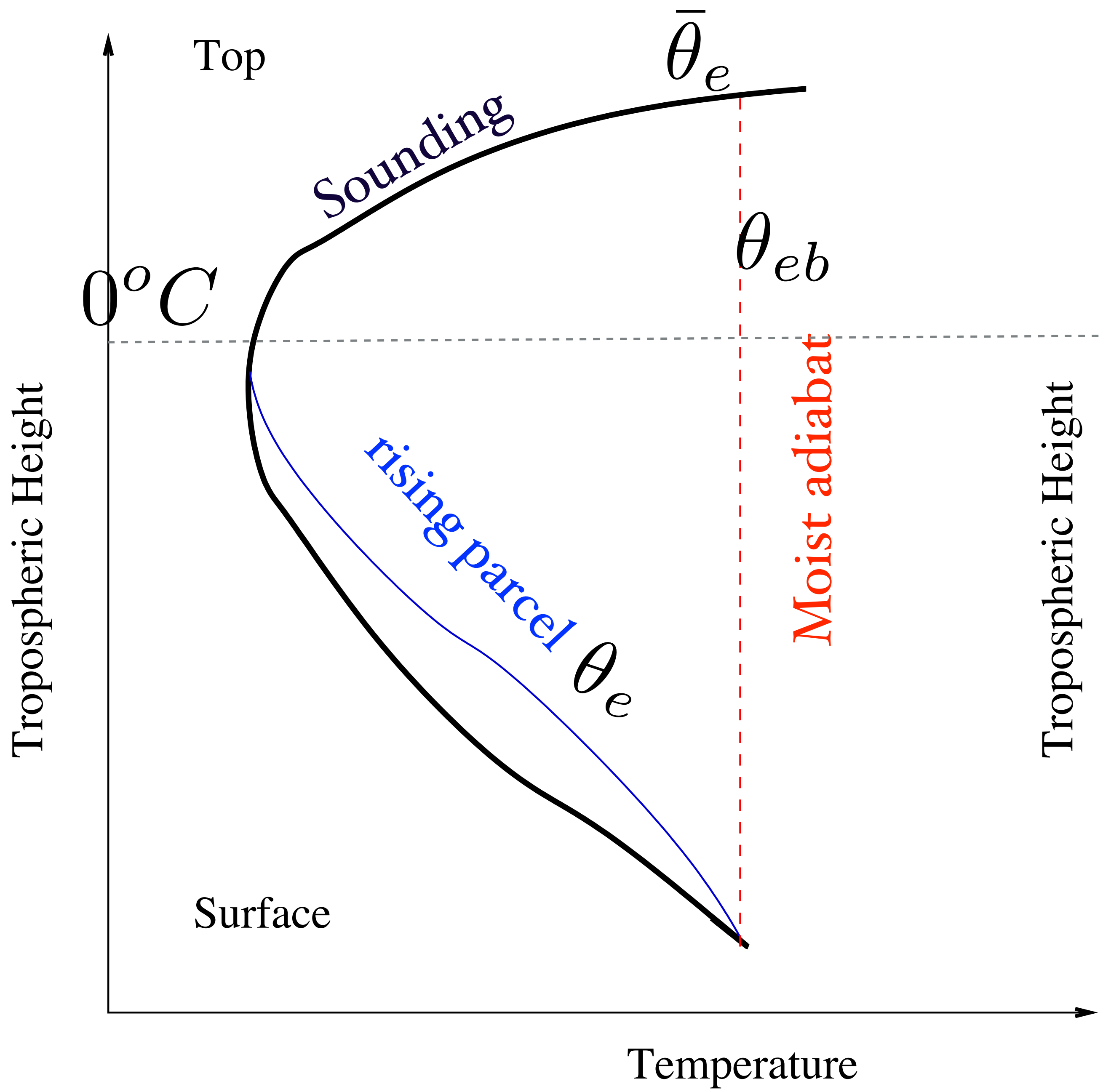


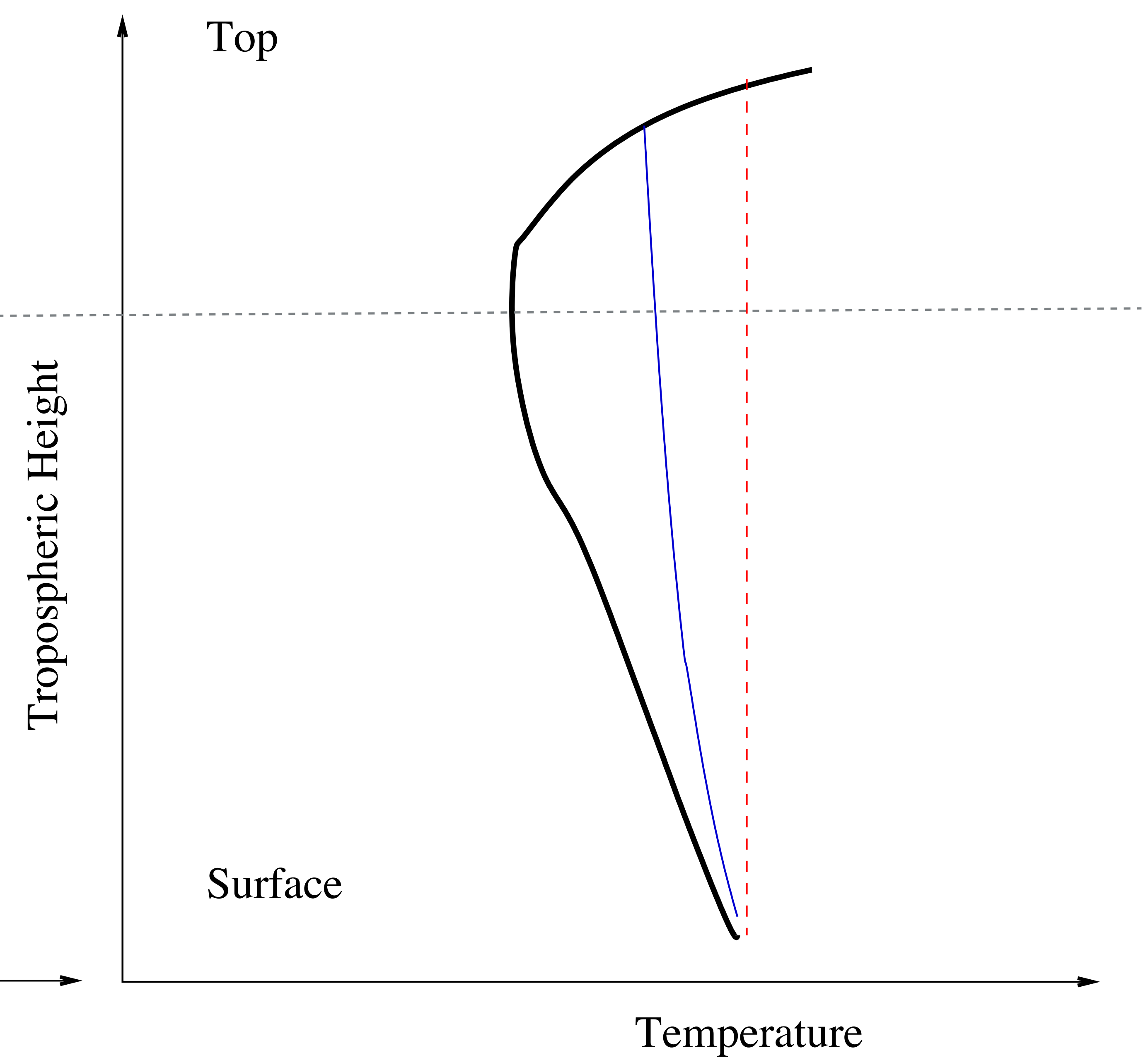


Dilute parcel lifting

Dry troposphere with positive CAPE favors congestus clouds

Deep convection is allowed (beyond freezing level) when troposphere is moist





Stochastic Multicloud Model

- Lattice points take values 0,1,2, or 3
- Three order parameters c,d,s taking values 1 or 0, at a given lattice point, depending on whether we have a congestus, a deep, or a stratiform cloud, or none, respectively.
- A sequence of four states, multivariable Markov chain.

C		S
	D	
C		
	C	
S		D

Intuitive transition rules

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- A clear sky site turns into a congestus site with high probability if CAPE>0 and middle troposphere is dry.
- A congestus or clear sky site turns into a deep site with high probability if CAPE>0 and middle troposphere is moist.
- A deep site turns into a stratiform site with high probability.
- All three cloud types decay naturally according to prescribed decay rates.

Transition probabilities

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- Four state Markov chain at given site

$$\boxed{X_t = \begin{cases} 0 & \text{at clear sky site} \\ 1 & \text{at congestus site} \\ 2 & \text{at deep site} \\ 3 & \text{at statiform site} \end{cases}}$$

- Prob{ $X_{t+\Delta}t = k/X_t = l$} = $R_{lk}\Delta t + O(\Delta t^2), l \neq k$

- Transition probability matrix, with $P_{lk} = R_{lk}\Delta t$

$$M = \begin{bmatrix} 1 - P_{01} - P_{02} & P_{01} & P_{02} & 0 \\ P_{10} & 1 - P_{10} - P_{12} & P_{12} & 0 \\ P_{20} & 0 & 1 - P_{20} - P_{23} & P_{23} \\ P_{30} & 0 & 0 & 1 - P_{30} \end{bmatrix}$$

Transition rates

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- C = normalized CAPE, D = mid-trop. dryness
- tau_kl transition time scales
- Local interactions ignored for simplicity. Transition rates depend only on external parameters (potential)

$$R_{01} = \frac{1}{\tau_{01}}\Gamma(C)\Gamma(D) \qquad R_{02} = \frac{1}{\tau_{02}}\Gamma(C)(1-\Gamma(D))$$

$$R_{10} = \frac{1}{\tau_{10}}\Gamma(D) \qquad R_{12} = \frac{1}{\tau_{12}}\Gamma(C)(1-\Gamma(D))$$

$$R_{20} = \frac{1}{\tau_{20}}[1-\Gamma(C)] \qquad R_{23} = \frac{1}{\tau_{23}} \text{ or } \frac{\Gamma(C)}{\tau_{23}}$$

$$\Gamma(x) = 1 - e^{-x} \text{ if } x > 0$$

$$R_{30} = 1/\tau_{30} \qquad \Gamma(x) = 0 \text{ if } x \leq 0$$

Transition rates and Effective dynamics

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- Depend on large-scale (thermodynamics and fluid mechanics) and intrinsic microscopic dynamics
- The microscopic dynamics are represented through time scales tau_{kl}, which can be estimated through in situ or simulation data (de la Chevrotiere's poster, P3.7)
- Here, we use intuition to assign values for these parameters:
- As a design principle Equilibrium Distribution will favour congestus, deep, or stratiform clouds according to (external) large-scale state.

Coarse graining and mean-field Eqns

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- Nx = number of sites filled with cloud type x
- Local interactions between sites ignored for simplicity
- Multi-D Birth-death process:

$$Prob\{N_c^{t+\Delta t} = k+1/N_c^t = k\} = N_{cs}R_{01}\Delta t + o(\Delta t)$$
$$Prob\{N_c^{t+\Delta t} = k-1/N_c^t = k\} = N_c(R_{10}+R_{12})\Delta t + o(\Delta t)$$
$$Prob\{N_d^{t+\Delta t} = k+1/N_d^t = k\} = (N_{cs}R_{01}+N_cR_{12})\Delta t + o(\Delta t)$$

$\dots$

- Mean-Field equations

$$\dot{\sigma}_c = (1-\sigma_c\sigma_d-\sigma_s)R_{01} - \sigma_c(R_{10}+R_{12})$$
$$\dot{\sigma}_d = (1-\sigma_c\sigma_d-\sigma_s)R_{02} + \sigma_cR_{12} - \sigma_d(R_{20}+R_{23})$$
$$\dot{\sigma}_s = \sigma_dR_{23} - \sigma_sR_{30}$$

Time evolution and statistics of filling fraction

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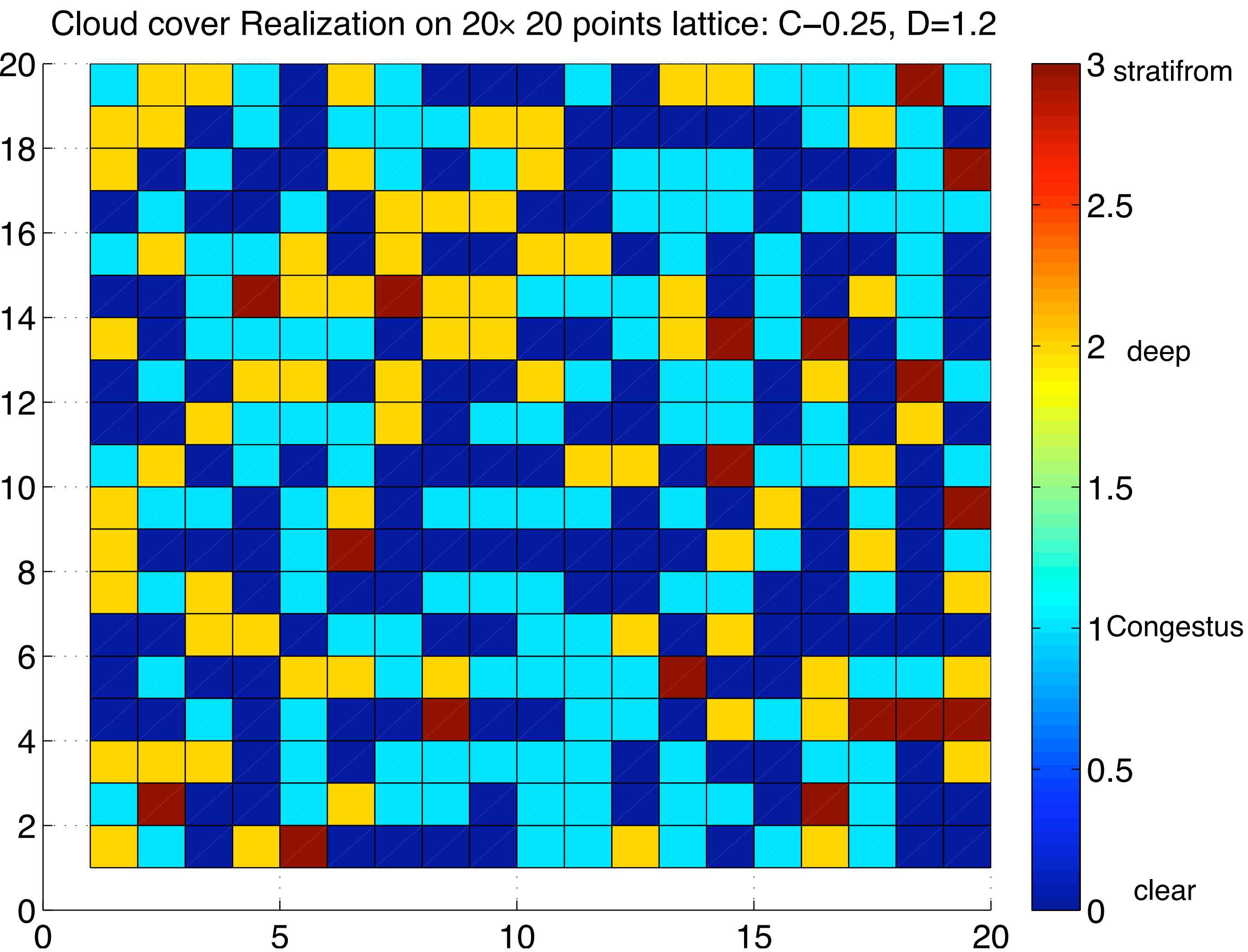

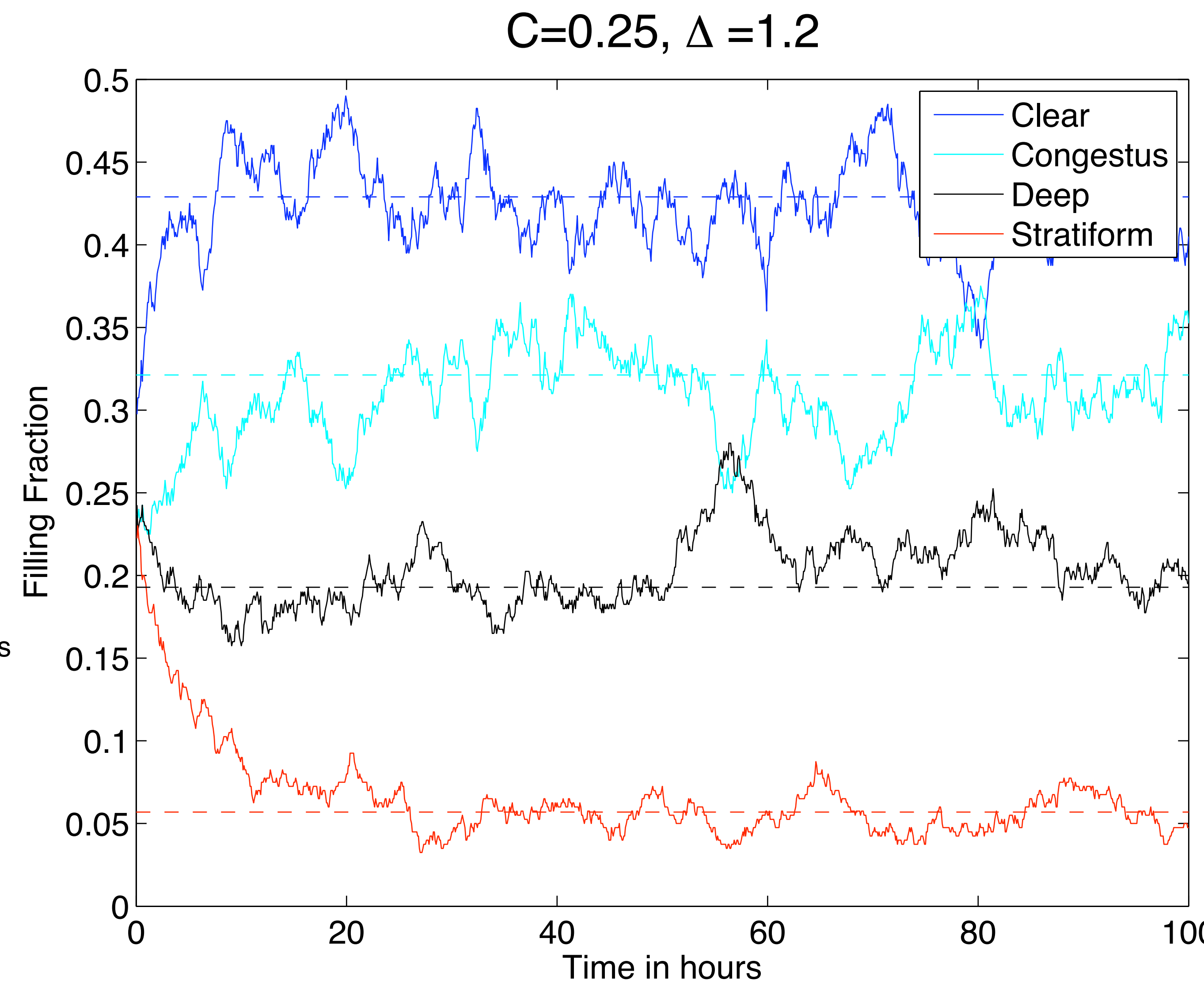

Deterministic simulation with clear deficiencies

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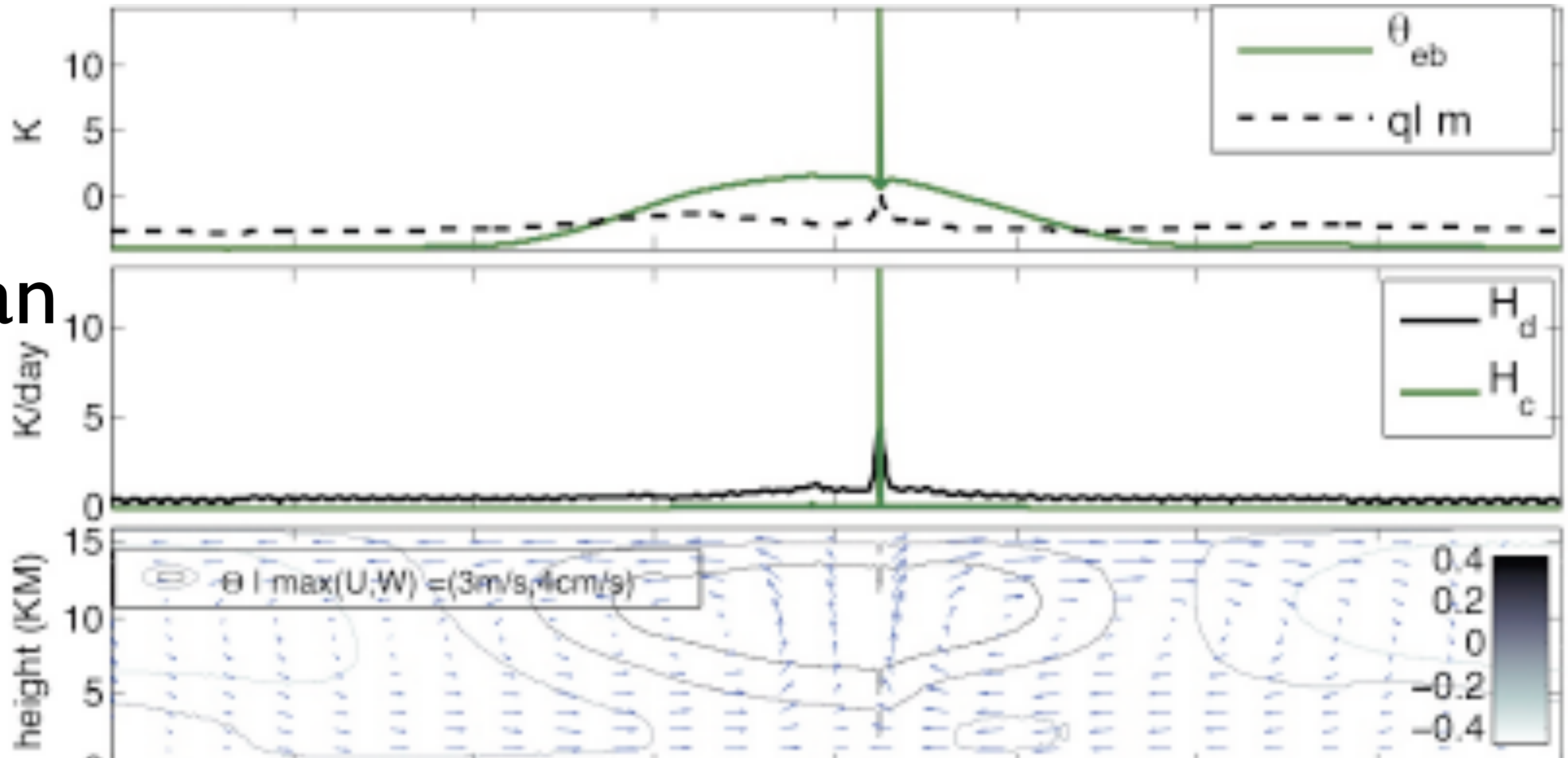
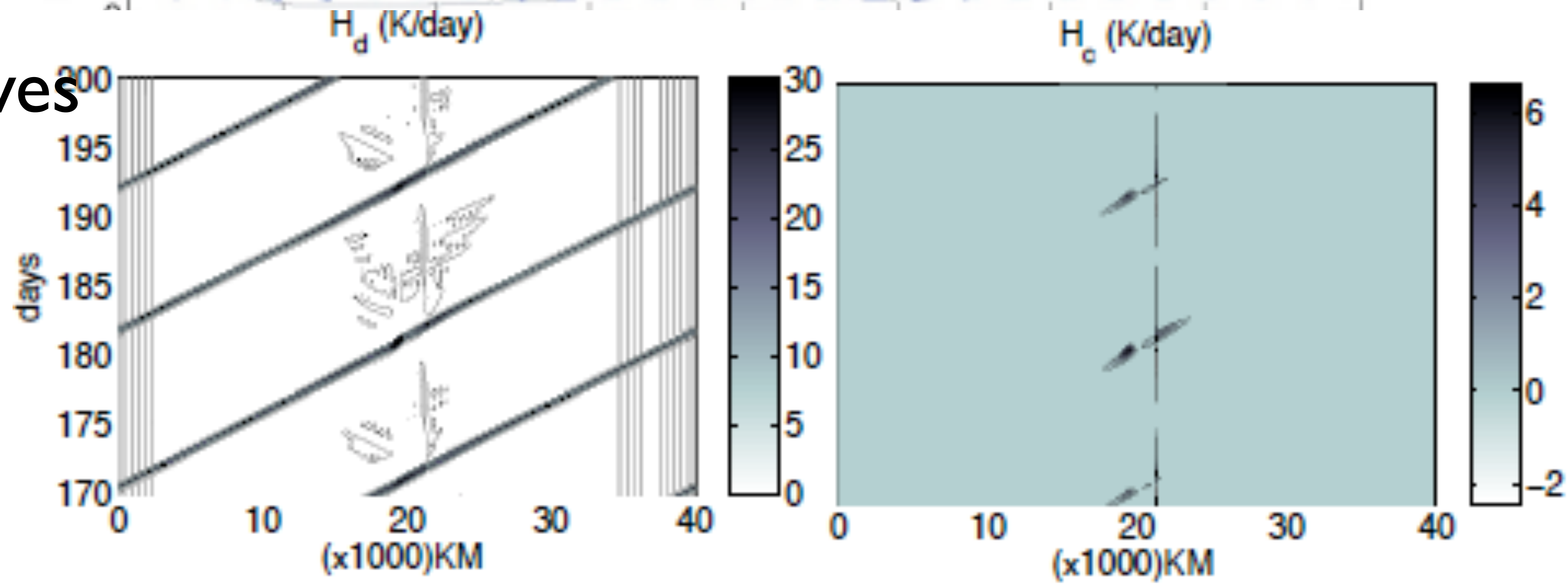
- Multicloud model of K. and Majda (JAS 2006,2008):
  - ✓ Two baroclinic-mode model, cumulus parametrization
  - ✓ based on three cloud types, mid-tropospheric moisture closures.
  - ✓ Captures well physics and morphology of CCWs.
- Parameter regime with large stratiform rain fraction that reduces the variability of the model by reducing downdrafts from evaporation of stratiform rain
- Warm pool-like SST profile for a 2d flow over the equator.
- Leads to unrealistic results .....

Mean



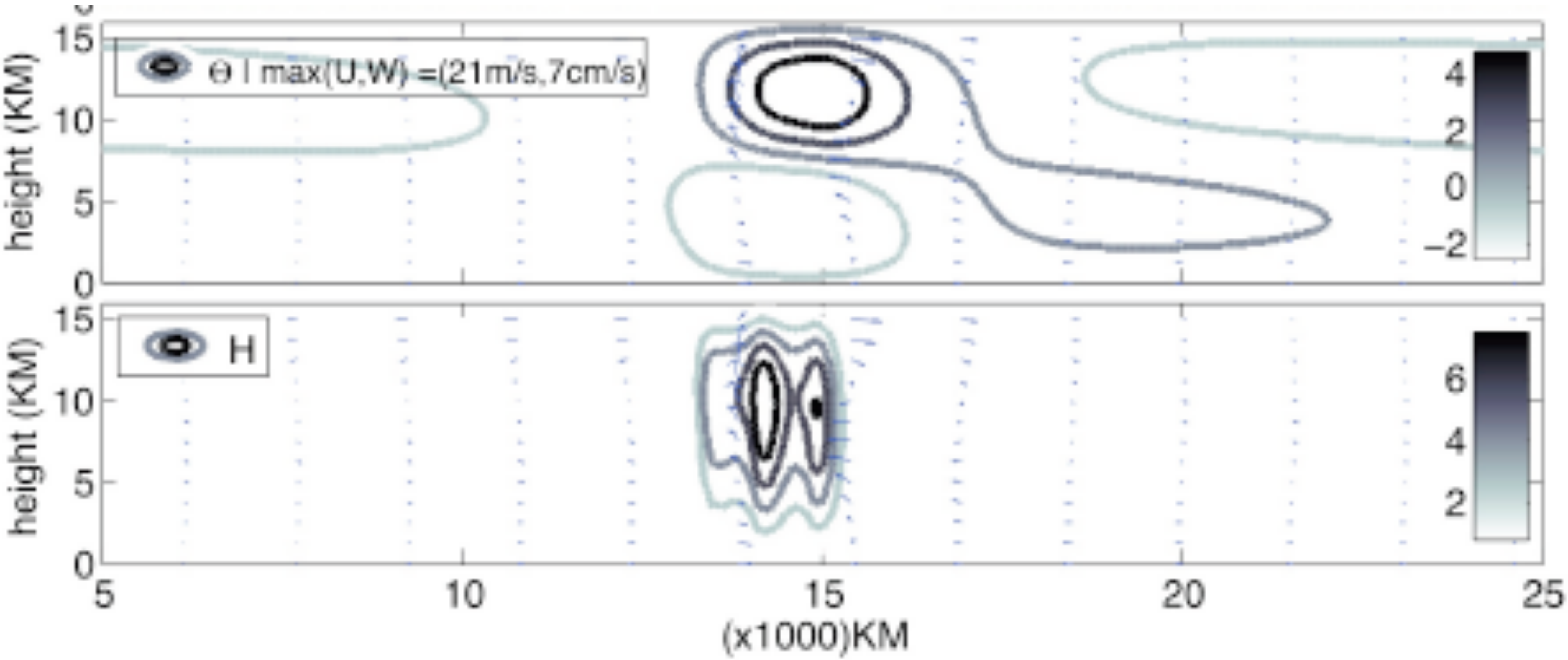
Grid scale Congestus peak

Waves



Waves are too regular, No (or weak) congestus, too much deep convection as in (some) GCMs

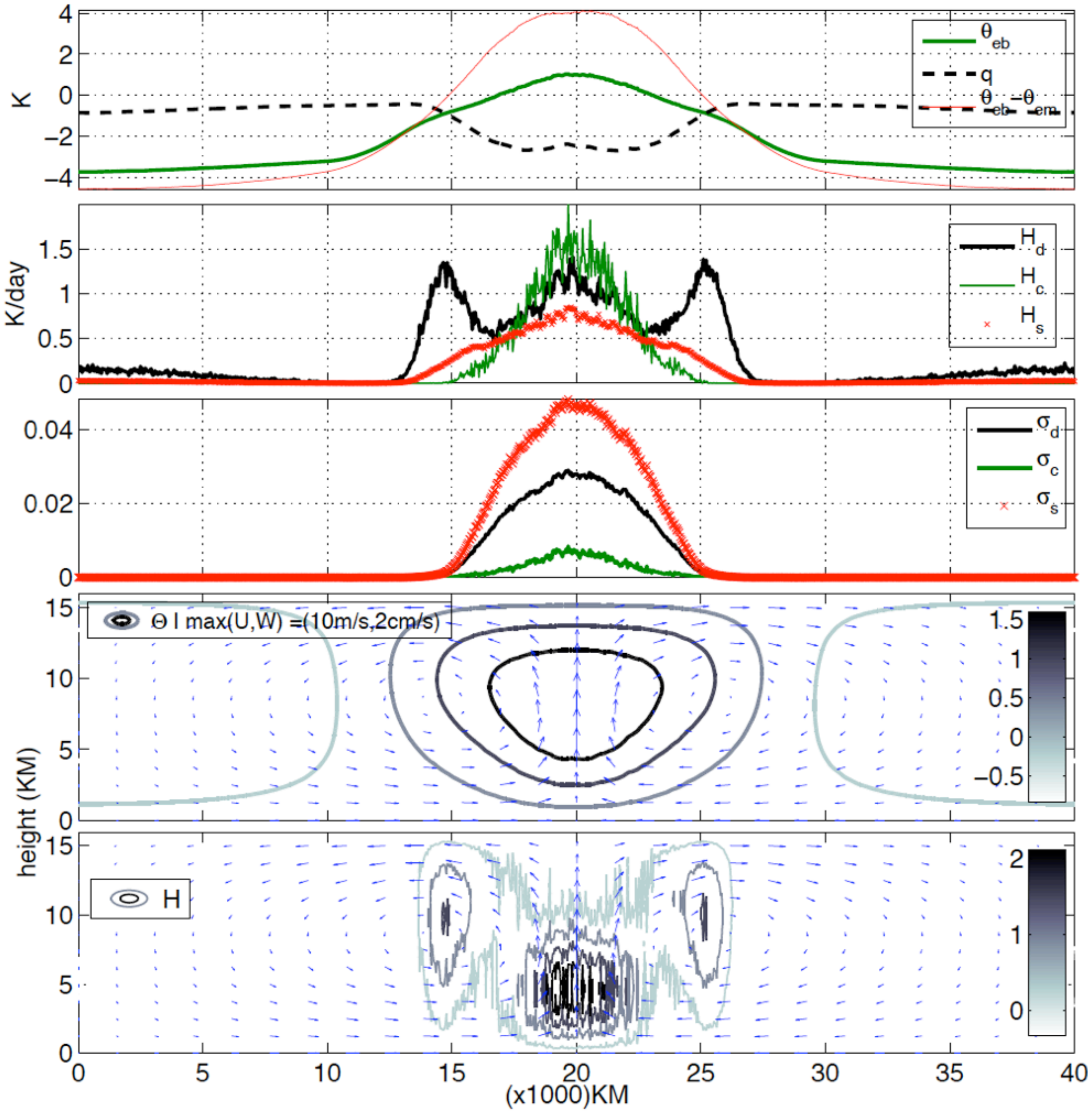
Wave Structure



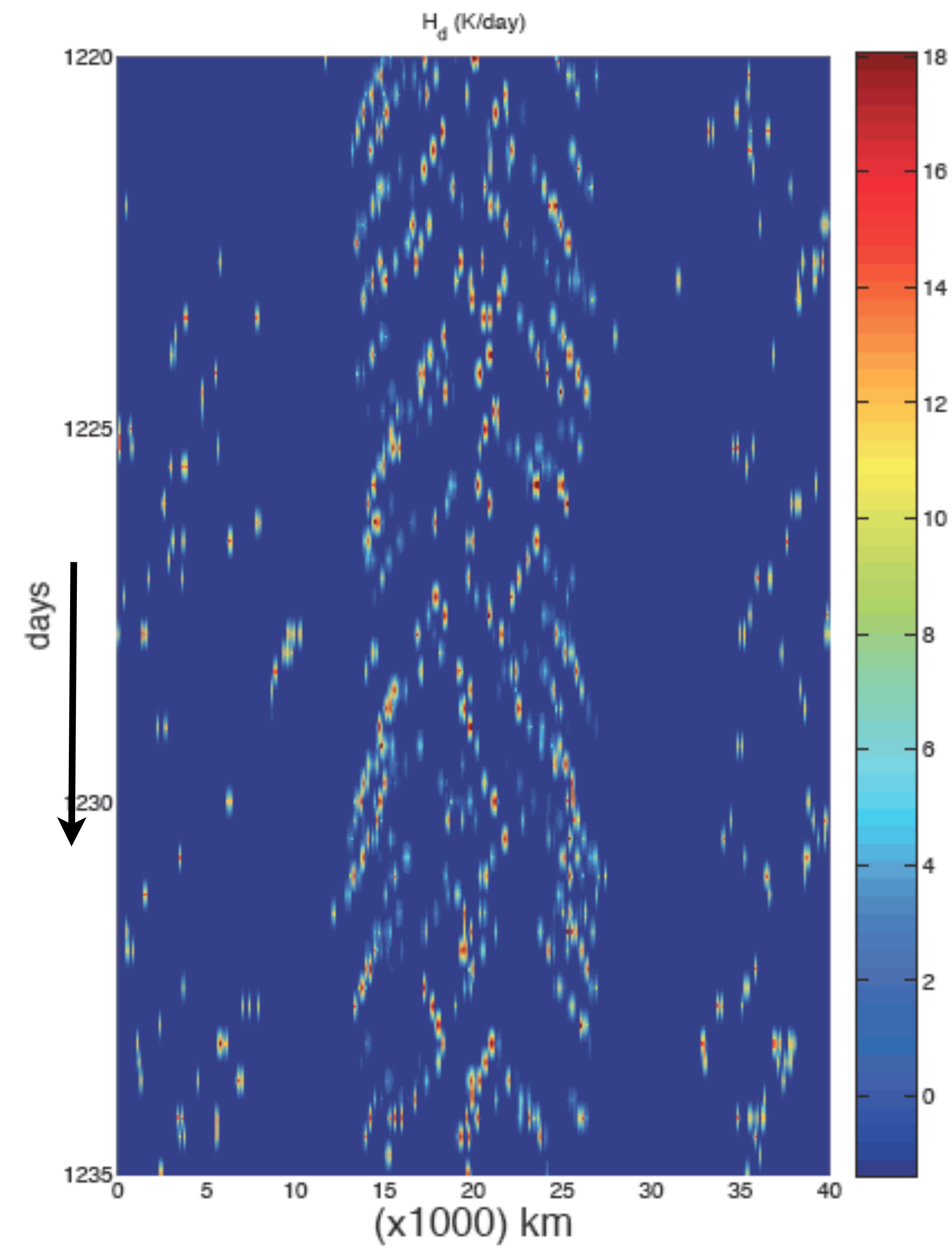
Up right convection ... no tilt in heating field

Stochastic simulations

Mean

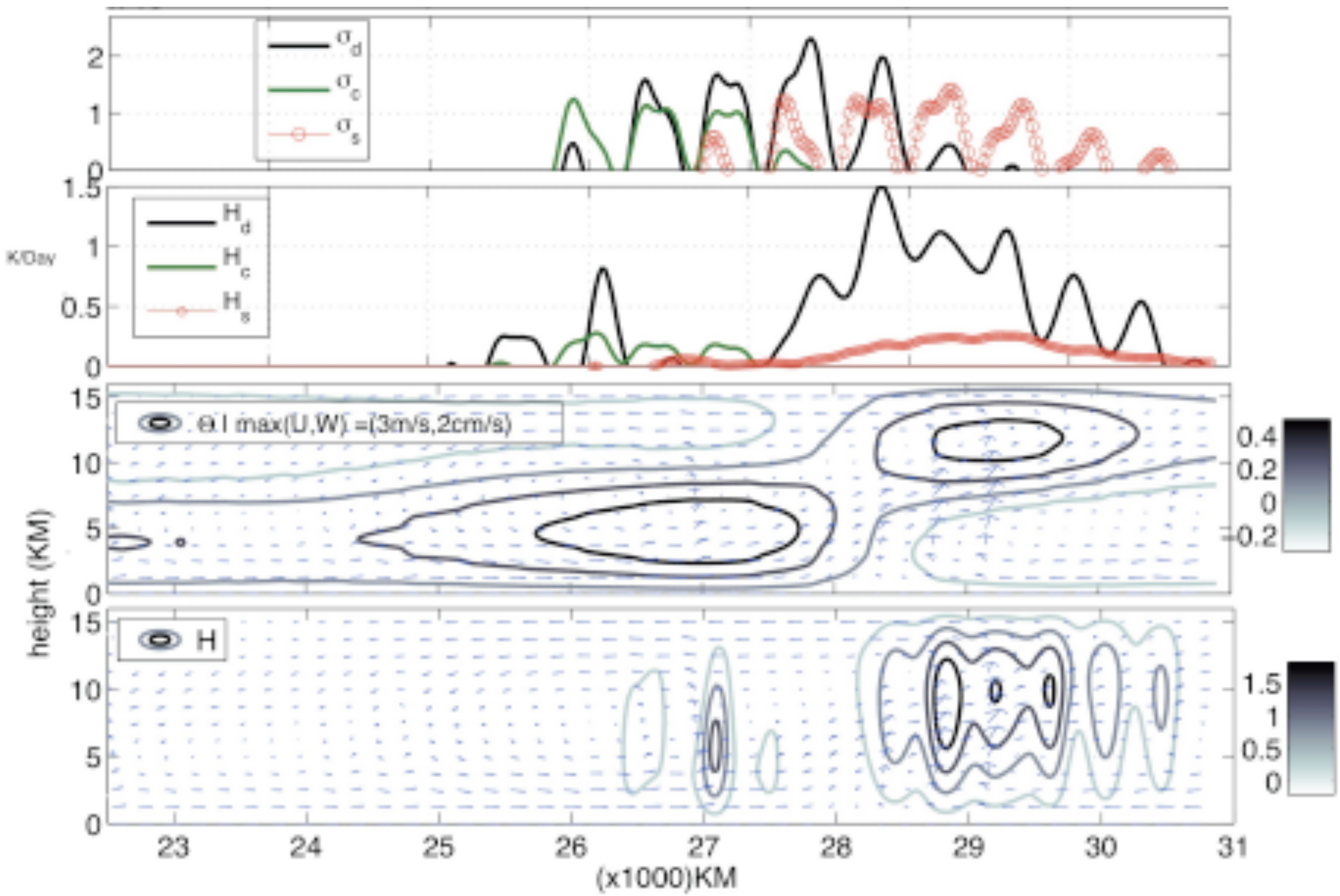


Waves



Some similarity to CRM simulations of Grabowski et al. (2000)

Wave structure



Notice strong stratiform fluctuations, unlike the mean... stratiform heating is carried by the waves!

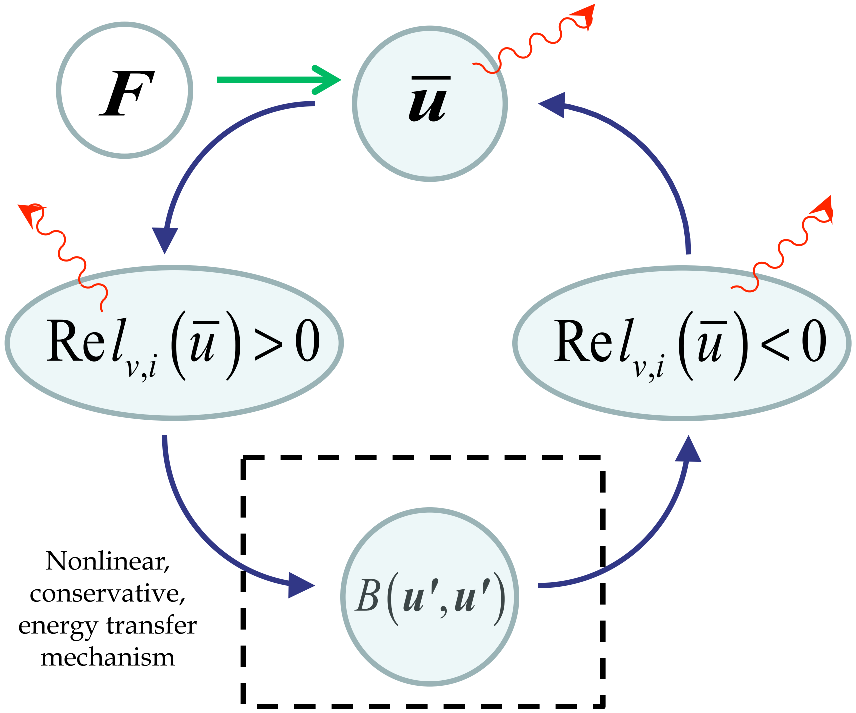
Conclusion

- Presented a strategy for stochastic modelling framework to improve cumulus parametrization
- Focus on organized tropical convection--characterized by three cloud types (congestus, deep, and stratiform)
- Model for stochastic dynamics of area fractions conditional on background instability and thermodynamics
- Tested in context of a simple slab model of equatorial circulation
- Both variability and wave structure were drastically improved

Uncertainty quantification for Turbulent Dynamical Systems

Collaboration with Themis Sapsis (CIMS-MIT)

UQ in systems with internal instabilities



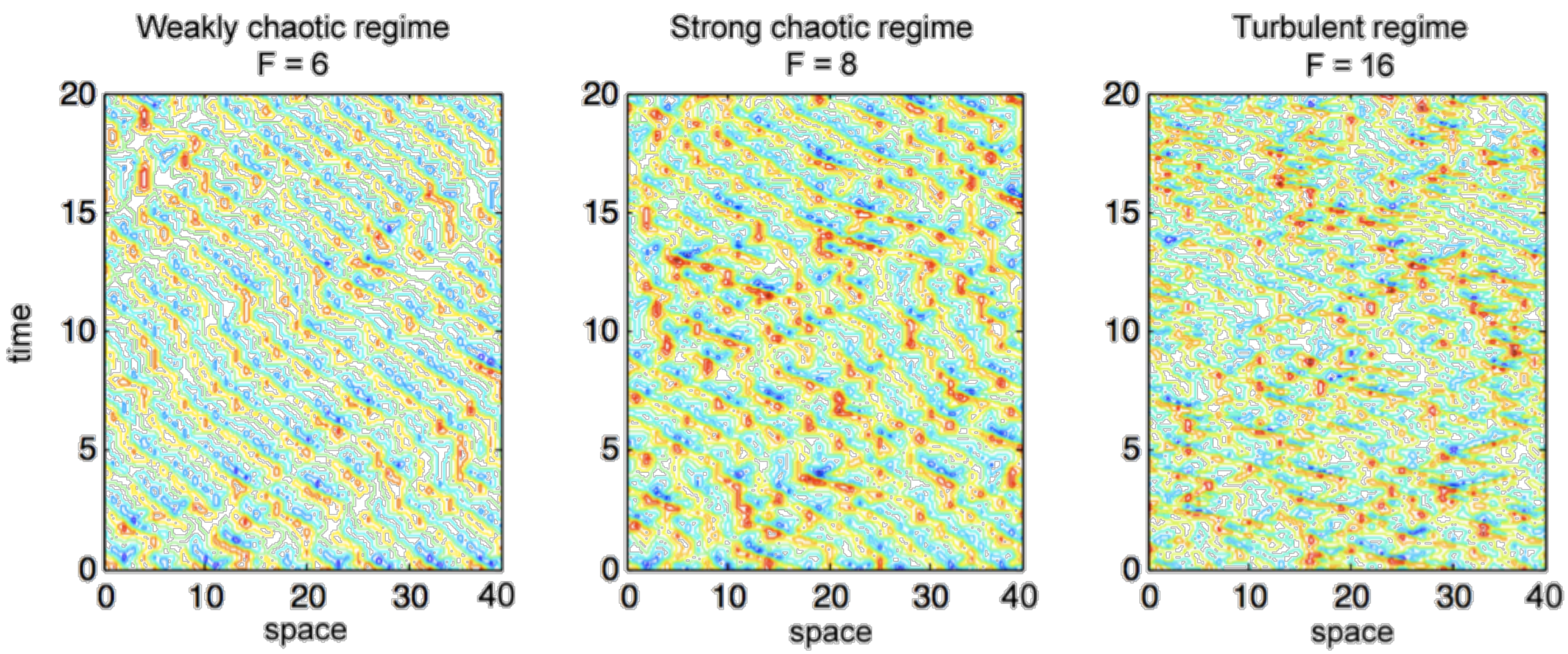
- Some systems with very high complexity may be characterized by stochastic attractors with <u>too many dimensions</u> to resolve.
- This is equivalent with having a system with a <u>large number of positive Lyapunov exponents</u>, i.e. instabilities.
- Reduction, no matter how good it is, will <u>neglect some natural frequencies</u> and <u>non-normal dynamics</u>.
- Still however, these may play an important role.

An overview of Lorenz 96

$$\dot{u}_i = u_{i-1}\left(u_{i+1} - u_{i-2}\right) - u_i + F, \quad i = 1, \ldots, 40 \qquad \text{Periodicity}$$

L-96 is designed to mimic baroclinic turbulence in the midlatitude atmosphere.

Effects of energy conserving nonlinear advection and dissipation are included.



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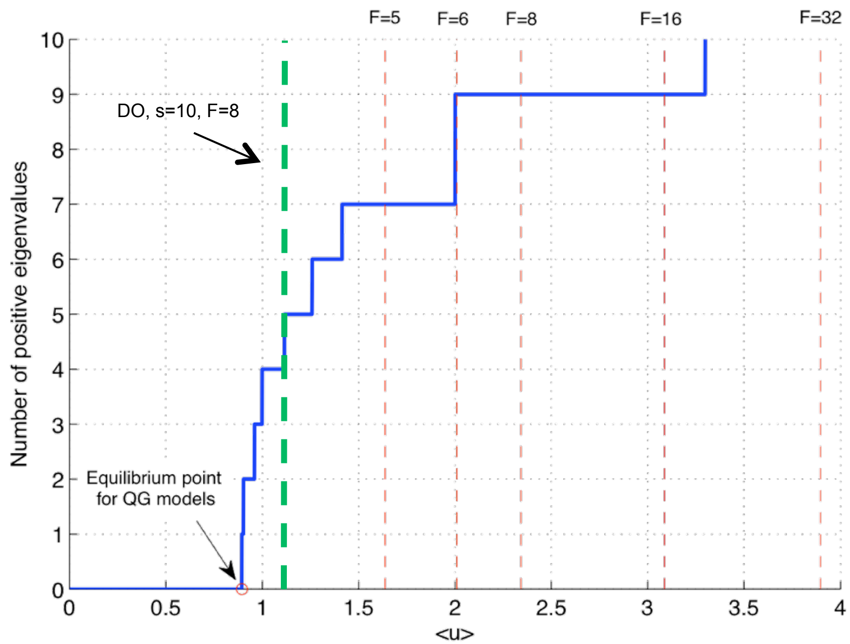
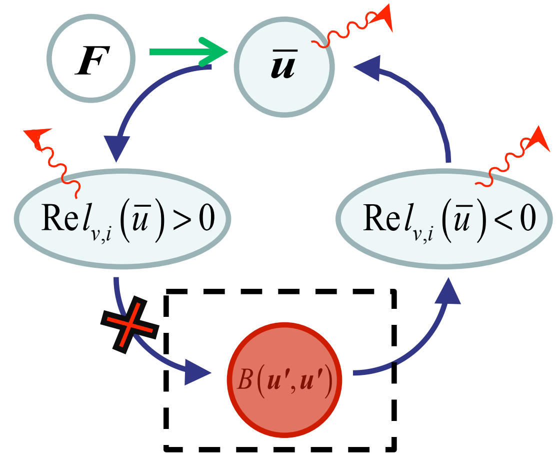
A critical overview of Lorenz 96



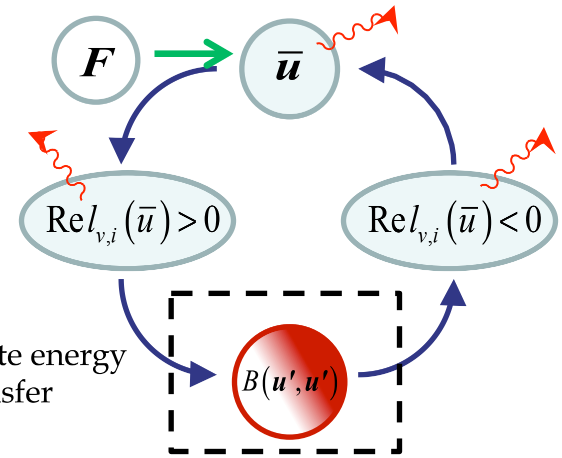
Ignoring nonlinear energy transfers…

- Will drive the stable models to zero energy.
- Will cause the mean flow to reduce its energy so that there are no positive Lyapunov exponents
- Will result in wrong energy distribution among modes and of course wrong mean flow energy
- Reduction methods model partially these energy transfers; therefore the same problems occur.

Gaussian or linearized models



In the absence of nonlinear terms the model has two choices: either to blow-up or reduce its mean flow energy to a completely stable regime



Incomplete energy transfer

T. Sapsis, A. Majda, A statistically accurate modified quasilinear Gaussian closure for UQ in turbulent dynamical systems, *Submitted, Physica D (2012)*.

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Modified Quasi-Gaussian closure method

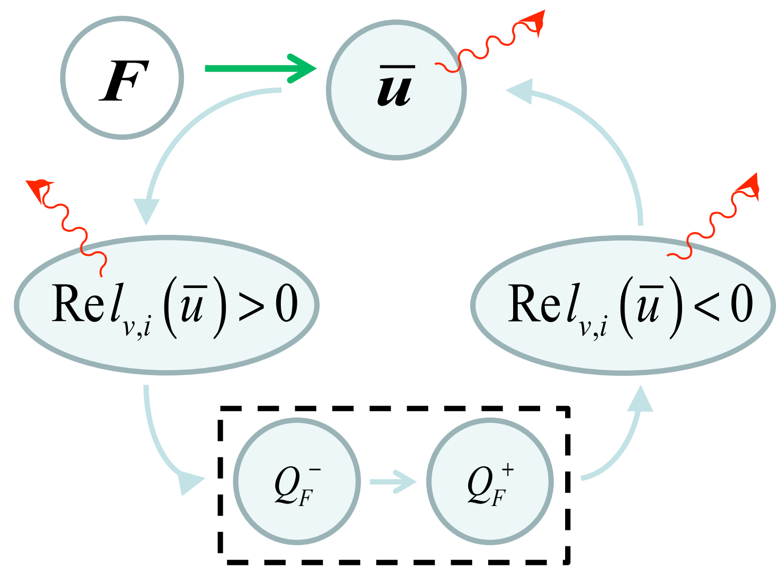
Modified Quasi-Gaussian closure

Evolution of 'full' covariance

$$\frac{dR}{dt} = L_v R + R L_v^* + Q + \sigma\sigma^*$$

L_v : Linearized dynamics around $\bar{u}$

$\{L_v\}_{ij} = v_i \cdot \left[[L+D] v_j + B(\bar{u}, v_j) + B(v_j, \bar{u}) \right]$



In steady state…

$$Q_\infty = -L_{v,\infty} R_\infty - R_\infty L_{v,\infty}^* - \sigma\sigma^*$$

Split into positive and negative definite fluxes

Q_F^- Must be represented as a direct effect on the eigenvalues of the linearized dynamics. In the terminology of dynamics it should be an effective damping.

Q_F^+ Is represented as an additive white noise contribution.

$$Q_F^- = N_\infty R + R N_\infty^* \qquad N_\infty = \frac{1}{2} Q_{F,\infty}^- R_\infty^{-1}$$

Consistency with correct steady state

$$Q_F^+ = \frac{Tr\left[Q_F^-\right]}{Tr\left[Q_F^+\right]} Q_F^+$$

Guarantees zero trace property

Modified Quasi-Gaussian closure method

Modified Quasi-Gaussian closure: Two important remarks

1) The way we setup the nonlinear fluxes we make the system marginally stable

We add some small additional dissipation and noise to make the attractor stable

$$N_\infty = \frac{1}{2}\left(Q_{F\infty}^{-} - qI\right)R_\infty^{-1} \qquad \text{and} \qquad Q_F^{+} = -\frac{Tr\left[Q_F^{-}\right]}{Tr\left[Q_{F\infty}^{+}\right]}\left(Q_{F\infty}^{+} + qI\right)$$

$$q = d_s \lambda_{\max}\left[Q_{F\infty}\right]$$

2) We obtain a much better UQ performance by scaling the fluxes with functionals of the covariance

$$N = \frac{f(R)}{f(R_\infty)} N_\infty$$

$$f(R) = \left[Tr(R)\right]^{\frac{1}{2}}, \qquad \text{Energy functional}$$

$$f(R, L_v) = \left[\sum_{\lambda_i[L_v]>0} \sigma_i^2\right]^{\frac{1}{2}}, \qquad \text{Energy of unstable modes}$$

$$f(R) = \sum_{i=1}^{N} \sigma_i, \qquad \text{Sum of typical deviations}$$

$$f(R, L_v) = \sum_{\lambda_i[L_v]>0} \lambda_i[L_v]\,\sigma_i, \qquad \text{Sum of positive nonlinear fluxes}$$

Summary of Modified Quasi-Gaussian closure method

Modified Quasi-Gaussian closure

$$\frac{d\bar{\mathbf{u}}}{dt} = [L+D]\,\bar{\mathbf{u}} + B(\bar{\mathbf{u}},\bar{\mathbf{u}}) + R_{ij}B(\mathbf{v}_i,\mathbf{v}_j) + \mathbf{F}$$

$$\frac{dR}{dt} = \left(L_v + N\right)R + R\left(L_v^* + N^*\right) + Q_F^+ + \sigma\sigma^*$$

L_v : Linearized dynamics around $\bar{u}$
$\{L_v\}_{ij} = v_i \cdot \left[[L+D]v_j + B(\bar{u},v_j) + B(v_j,\bar{u})\right]$

$$N = \frac{f(R)}{f(R_\infty)}N_\infty \quad \text{with} \quad N_\infty = \frac{1}{2}\left(Q_{F\infty}^- - qI\right)R_\infty^{-1} \quad \text{and} \quad f(R) = \sum_{i=1}^{N}\sigma_i,$$

$$Q_F^+ = -\frac{Tr\left[Q_F^-\right]}{Tr\left[Q_{F\infty}^+\right]}\left(Q_{F\infty}^+ + qI\right) \quad \text{with} \quad Q_F^- = NR + RN^*,$$

$$q = d_s \lambda_{\max}\left[Q_{F\infty}\right] \quad \text{with} \quad d_s \ll 1.$$

- Guaranteed convergence to the correct steady state
- We need only covariance and mean in steady state to learn nonlinear fluxes
- Non-local, nonlinear equation for the covariance

A new kind of non-local SODEs for UQ

$$\frac{d\bar{\mathbf{u}}}{dt} = [L+D]\,\bar{\mathbf{u}} + B(\bar{\mathbf{u}},\bar{\mathbf{u}}) + \overline{B(\mathbf{u}',\mathbf{u}')} + \mathbf{F}$$

$$\frac{d\mathbf{u}'}{dt} = ([L+D]\,\mathbf{u}' + B(\bar{\mathbf{u}},\mathbf{u}') + B(\mathbf{u}',\bar{\mathbf{u}})) + N\left(\overline{\mathbf{u}'\mathbf{u}'^*},\bar{\mathbf{u}}\right)\mathbf{u}' + \left[Q_F^+\left(\overline{\mathbf{u}'\mathbf{u}'^*},\bar{\mathbf{u}}\right)\right]^{\frac{1}{2}}\dot{\mathbf{W}}_1 + \sigma\dot{\mathbf{W}}_2$$

$$N = \frac{f(R)}{f(R_\infty)}N_\infty \quad \text{with} \quad N_\infty = \frac{1}{2}\left(Q_{F\infty}^- - qI\right)R_\infty^{-1} \quad \text{and} \quad f(R) = \sum_{i=1}^{N}\sigma_i,$$

$$Q_F^+ = -\frac{Tr\left[Q_F^-\right]}{Tr\left[Q_{F\infty}^+\right]}\left(Q_{F\infty}^+ + qI\right) \quad \text{with} \quad Q_F^- = NR + RN^*,$$

$$q = d_s\lambda_{\max}\left[Q_{F\infty}\right] \quad \text{with} \quad d_s \ll 1.$$

Quadratic term of the exact equation has been replaced by a pair of damping and noise terms which depend linear on the state of the perturbation and non-linearly, non-locally to the second order statistics of the systems.

New class of stochastic differential equations where the evolution of each stochastic realization depends on the global statistics – these novel SODEs merit further mathematical study.

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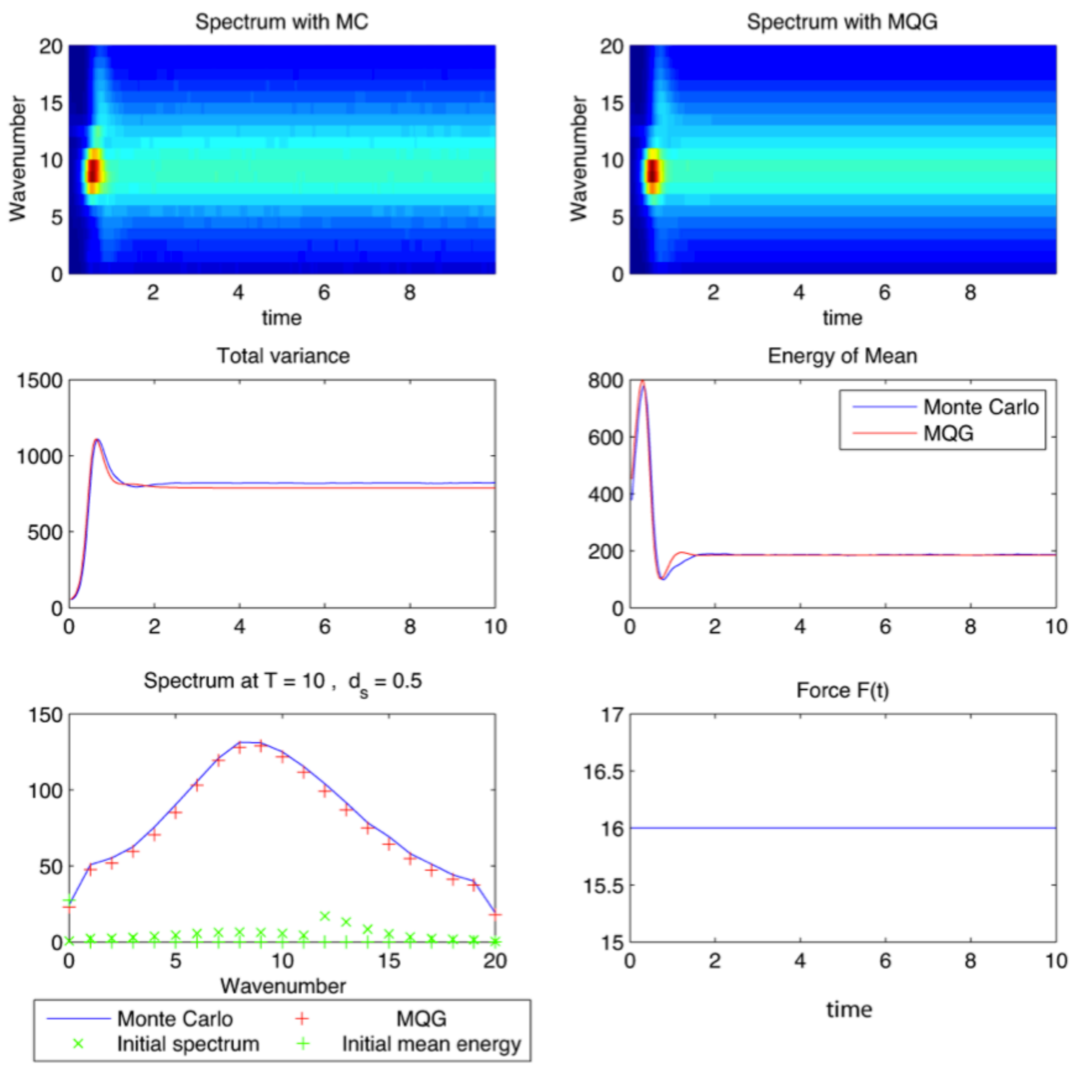
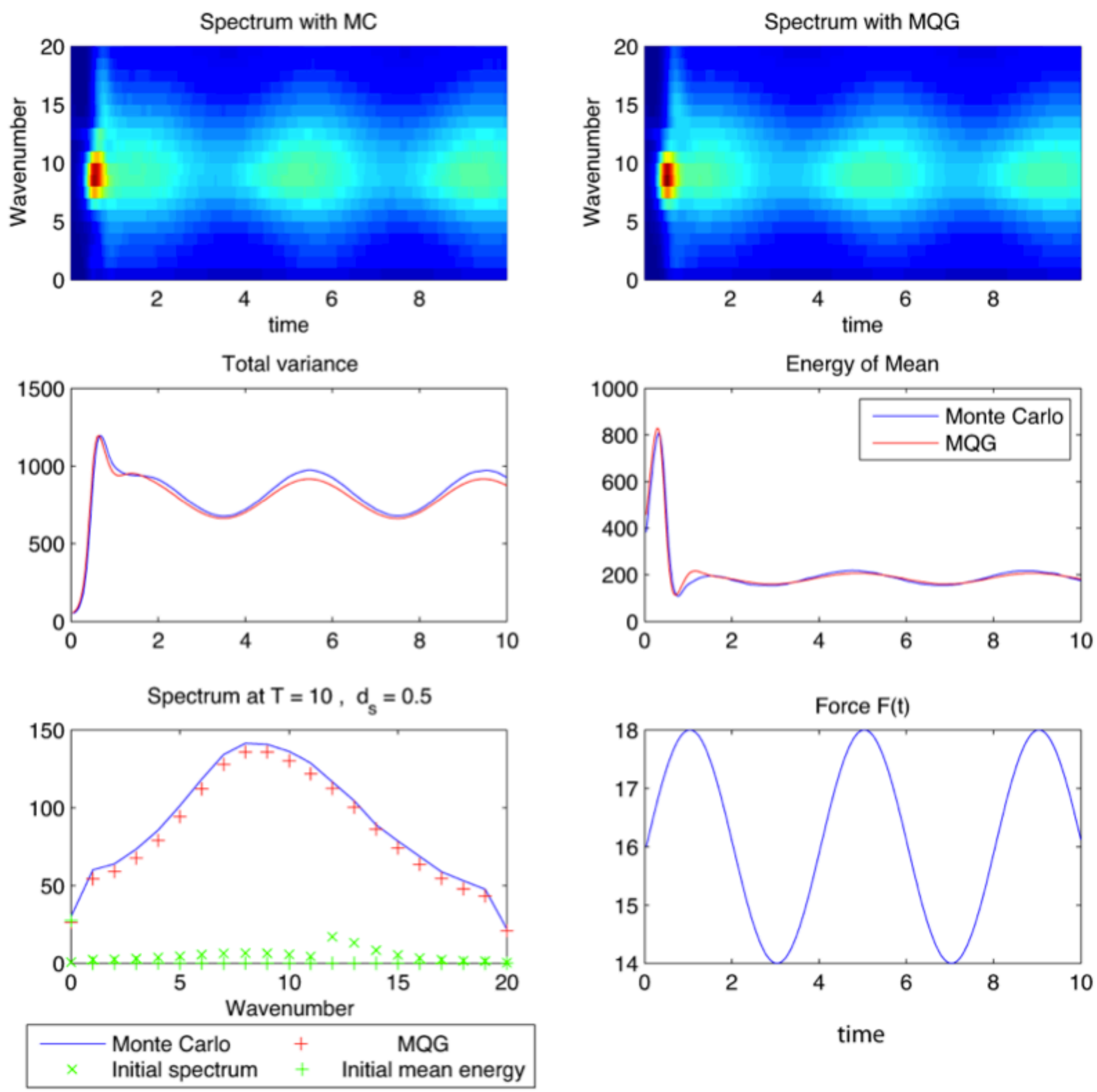
Modified Quasi-Gaussian closure method: Lorenz 96 (F=16)



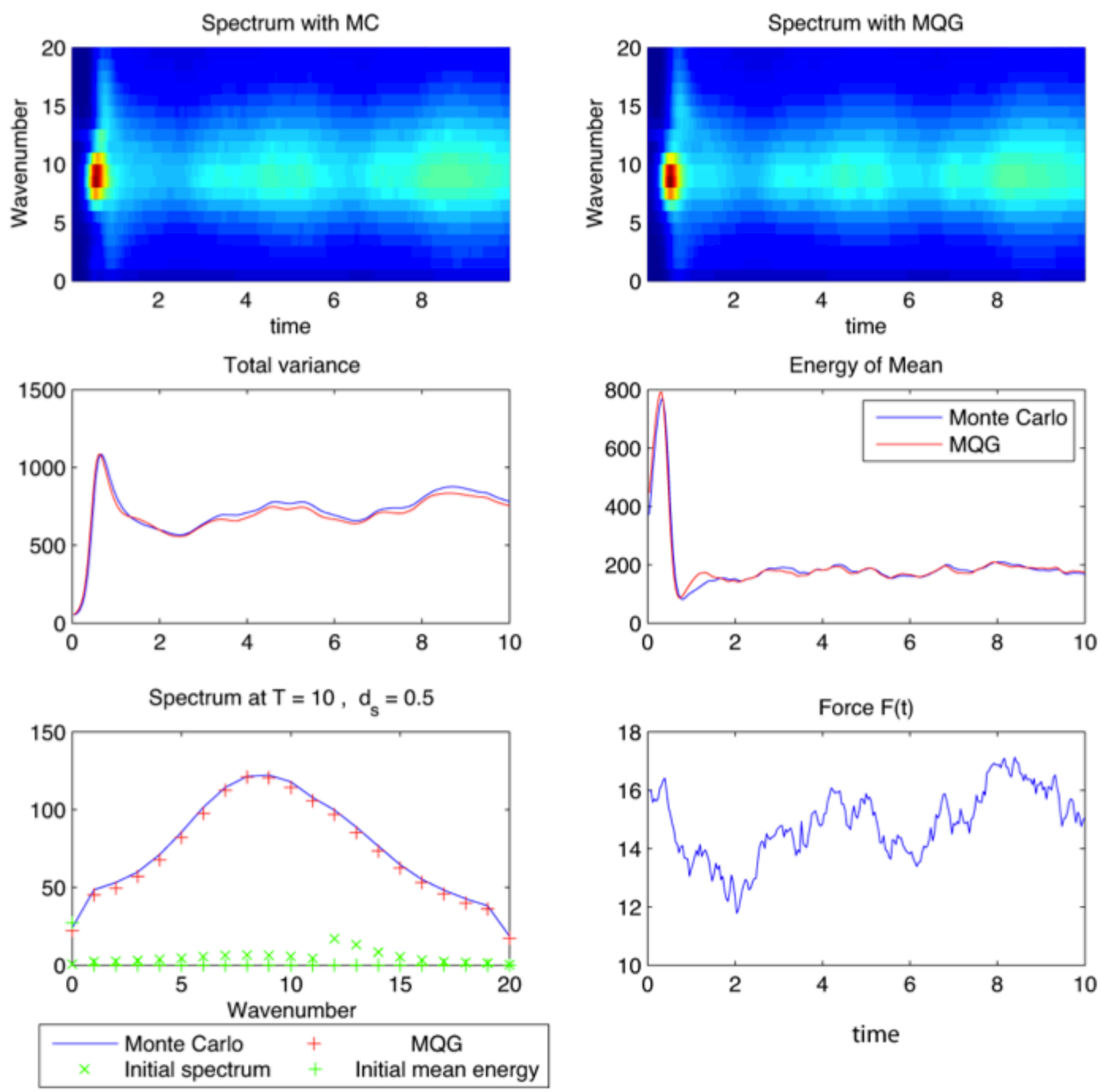
Modified Quasi-Gaussian closure method: Lorenz 96 (F=16)



Time periodic forcing F

Fluxes have been equilibrated using steady state for constant F=16

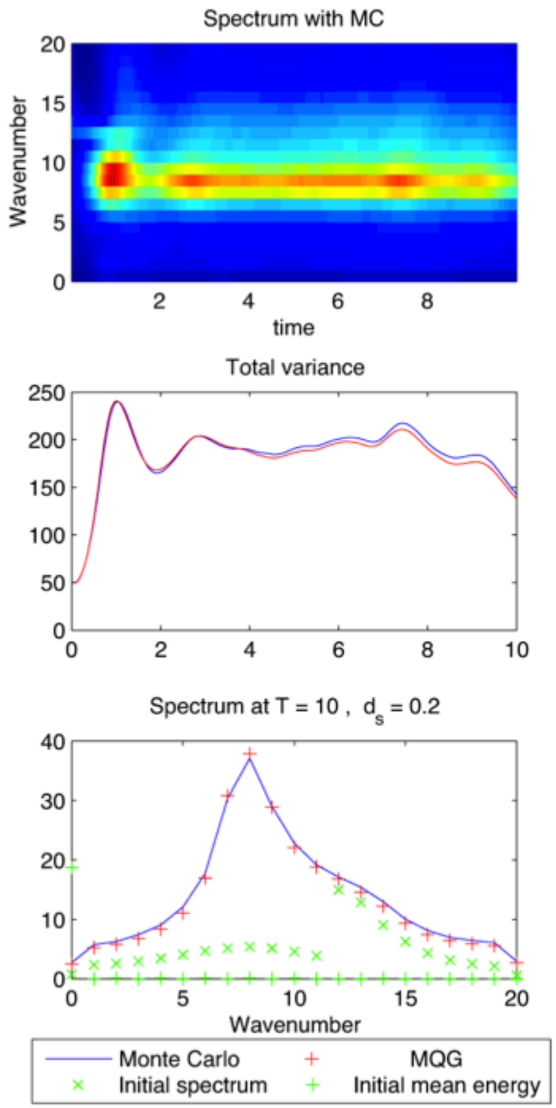
Modified Quasi-Gaussian closure method: Lorenz 96 (F=16)



Time dependent forcing F from an OU process

Fluxes have been equilibrated using steady state for constant F=16

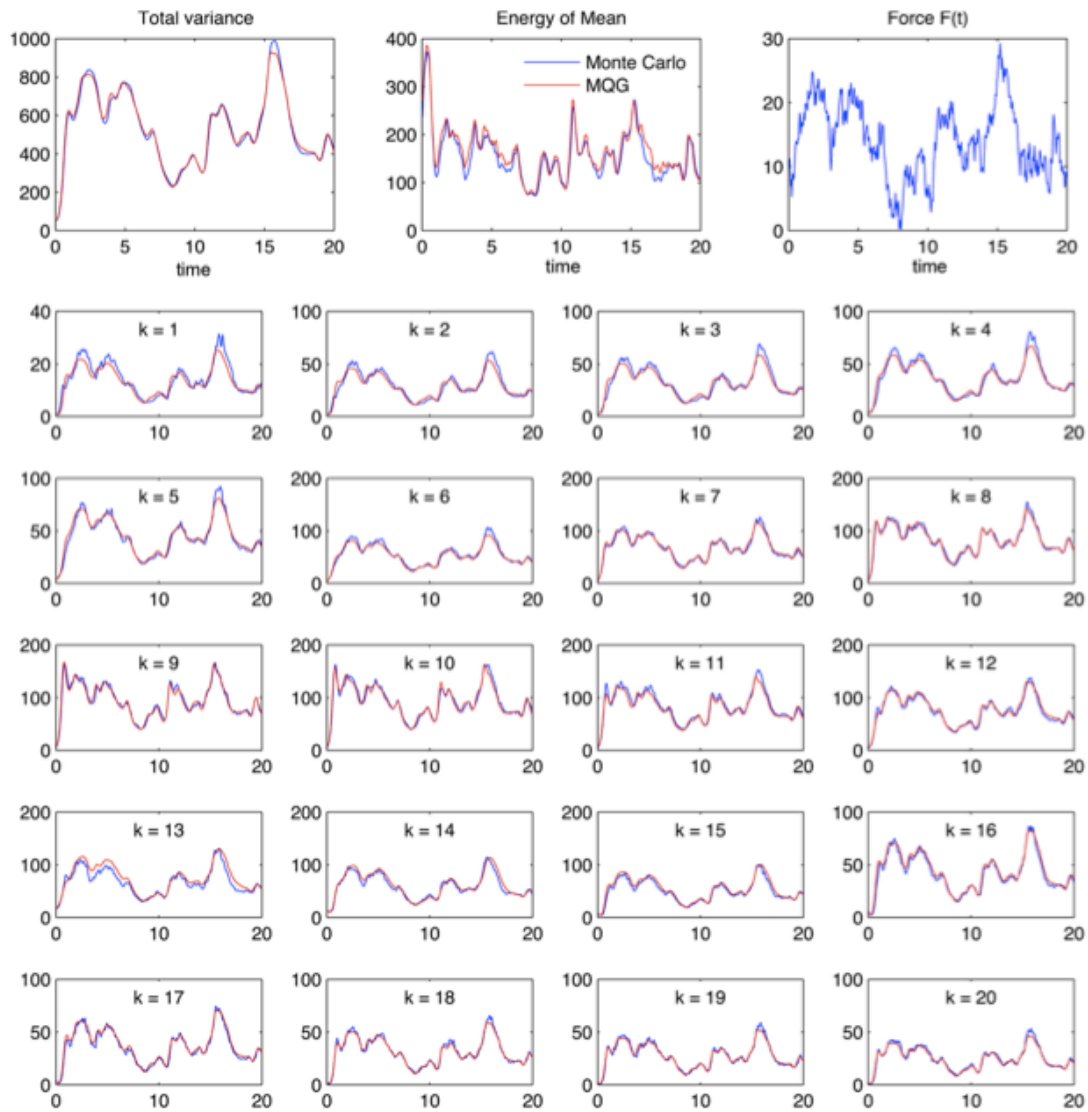
Modified Quasi-Gaussian closure method: Lorenz 96 (F=6)



Time dependent forcing F from an OU process

Fluxes have been equilibrated using steady state for constant F=6

An extremely time-dependent case



Time dependent forcing F from an OU process

Forcing fluctuates between 0 and 30

Fluxes have been equilibrated using steady state for constant F=10

Conclusions – Future directions

- We have developed an inexpensive second-order method for the UQ of complex systems with linear instabilities in the stochastic attractor. This method (MQG) follows from Gaussian closure with suitable modeling of the nonlinear fluxes.
- We have illustrated this approach through some very extreme setups involving Lorenz 96.
- Current work involves the blending of this inexpensive UQ method (designed to give second order statistics) with Reduced Order approaches towards the modeling of non-Gaussian characteristics for specific subspaces.

T. Sapsis, A. Majda, A statistically accurate modified quasilinear Gaussian closure for UQ in turbulent dynamical systems, *Submitted, Physica D (2012).*

- Blended algorithms for UQ of stable systems

T. Sapsis, A. Majda, Blended reduced subspace algorithms for UQ of quadratic systems with a stable mean state, *In Preparation, (2012).*

- Blending MQG and Reduced Subspace algorithms for UQ of turbulent systems

T. Sapsis, A. Majda, Blending modified Gaussian closure and non-Gaussian reduced subspace methods for turbulent dynamical systems, *In Preparation, (2012).*

Stochastic Superparameterization in a Model for Wave Turbulence

Collaboration with Ian Grooms (CIMS)

General Derivation of Gaussian Closure Stochastic Superparameterization

Consider a schematic system of PDEs of the form

$$\partial_t \mathbf{u} = L\mathbf{u} + B(\mathbf{u},\mathbf{u}) + S(\mathbf{u},\mathbf{u},\mathbf{u}) + F, \ \ \mathbf{u} \in \mathbb{R}^N$$

where B is quadratic (e.g. advection) and S is cubic (e.g. source terms due to moisture).
Decompose $\mathbf{u}$ into a mean $\overline{\mathbf{u}}$ and fluctuations (or 'eddies') $\mathbf{u}'$ using a statistical average

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$$

General Derivation of Gaussian Closure Stochastic Superparameterization

The mean equation is

$$\partial_t \overline{\mathbf{u}} = L\overline{\mathbf{u}} + B(\overline{\mathbf{u}}, \overline{\mathbf{u}}) + S(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \overline{\mathbf{u}}) + \overline{F} + \overline{B(\mathbf{u}', \mathbf{u}')} + \overline{[S_2(\overline{\mathbf{u}})](\mathbf{u}', \mathbf{u}')} + \overline{S(\mathbf{u}', \mathbf{u}'\mathbf{u}')}$$

where $[S_2(\overline{\mathbf{u}})](\mathbf{u}', \mathbf{u}') = S(\overline{\mathbf{u}}, \mathbf{u}', \mathbf{u}') + S(\mathbf{u}', \overline{\mathbf{u}}, \mathbf{u}') + S(\mathbf{u}', \mathbf{u}', \overline{\mathbf{u}})$.

The eddy equation is

$$\partial_t \mathbf{u}' = [\mathcal{L}(\overline{\mathbf{u}})]\mathbf{u}' + F' + \left[B(\mathbf{u}', \mathbf{u}') + [S_2(\overline{\mathbf{u}})](\mathbf{u}', \mathbf{u}') + S(\mathbf{u}', \mathbf{u}', \mathbf{u}')\right]'$$

where
$[\mathcal{L}(\overline{\mathbf{u}})]\mathbf{u}' = L\mathbf{u}' + B(\overline{\mathbf{u}}, \mathbf{u}') + B(\mathbf{u}', \overline{\mathbf{u}}) + S(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \mathbf{u}') + S(\overline{\mathbf{u}}, \mathbf{u}', \overline{\mathbf{u}}) + S(\mathbf{u}', \overline{\mathbf{u}}, \overline{\mathbf{u}})$.

General Derivation of Gaussian Closure Stochastic Superparameterization

Make two approximations in either order to arrive at Gaussian closure stochastic superparameterization equations:

1. The 'Gaussian Closure' approximation, wherein the terms in the eddy equation that are nonlinear in $\mathbf{u}'$ are replaced by stochastic forcing and damping, i.e. $\partial_t \mathbf{u}' = [\mathcal{L}(\overline{\mathbf{u}})]\mathbf{u}' + \sigma \dot{W} - \Gamma \mathbf{u}'$. This is a quasi-linear approximation which guarantees Gaussianity of the eddy statistics.
2. The 'point approximation' wherein mean variables and their spatial derivatives are both taken to be constant (spatially invariant) in the eddy equations, which are assumed to apply on a family of local domains embedded at each point in the original domain. This approximation imposes an *artificial* scale separation between the mean and the eddies.

The eddy equations can be further approximated by making the mean variables time-independent in the eddy equations; this imposes an artificial time scale separation.

General Derivation of Gaussian Closure Stochastic Superparameterization

The mean equation with approximate eddy statistics is

$$\partial_t \overline{\mathbf{u}} = L\overline{\mathbf{u}} + B(\overline{\mathbf{u}}, \overline{\mathbf{u}}) + S(\overline{\mathbf{u}}, \overline{\mathbf{u}}, \overline{\mathbf{u}}) + \overline{F} + \overline{B(\mathbf{u}', \mathbf{u}')} + \overline{[S_2(\overline{\mathbf{u}})](\mathbf{u}', \mathbf{u}')}.$$

The eddy equation on each embedded domain is

$$\partial_t \mathbf{u}' = [\mathcal{L}(\overline{\mathbf{u}})]\mathbf{u}' + \sigma \dot{W} - \Gamma \mathbf{u}'.$$

The overbar $\overline{(\cdot)}$ is interpreted as a statistical average and an average over the embedded domains.

General Derivation of Gaussian Closure Stochastic Superparameterization

The Fourier transform of the eddy equation is

$$\mathrm{d}\hat{\mathbf{u}}'_k = \left([\mathcal{L}_k(\overline{\mathbf{u}})] - \gamma_k\right)\hat{\mathbf{u}}'_k \mathrm{d}t + \sigma_k \mathrm{d}W_k.$$

Dependence on any spatial coordinates shared by the mean and eddies can be assumed to be discretized, so this is a system of stochastic ordinary differential equations.

Itō's lemma allows the derivation of a linear deterministic ordinary differential equation for the covariance of $\hat{\mathbf{u}}'_k$

$$\frac{\mathrm{d}}{\mathrm{d}t} C_k = \left([\mathcal{L}_k(\overline{\mathbf{u}})] - \gamma_k\right) C_k + C_k \left([\mathcal{L}_k(\overline{\mathbf{u}})] - \gamma_k\right)^* + \sigma_k \sigma_k^*$$

where

$$C_k = \mathbb{E}\left[\hat{\mathbf{u}}'_k (\hat{\mathbf{u}}'_k)^*\right].$$

General Derivation of Gaussian Closure Stochastic Superparameterization

The quadratic eddy terms in the mean equation are derivable from the eddy covariance, which can be recovered from the eddy Fourier covariance, e.g. via

$$\overline{\mathbf{u}'(\mathbf{u}')^T} = \int_{-\infty}^{\infty} C_k \mathrm{d}k.$$

Stochastic superparameterization solves the linear, deterministic, autonomous eddy Fourier covariance equation at each grid point of the large scale domain and at each time step of the mean equation to generate an approximation to the eddy terms in the mean equation.

Stochastic Superparameterization in MMT

The test model:

$$i\partial_t\psi = |\partial_x|^{1/2}\psi - |\psi|^2\psi + i(F_0\sin(4\pi x/L) - D\psi)$$

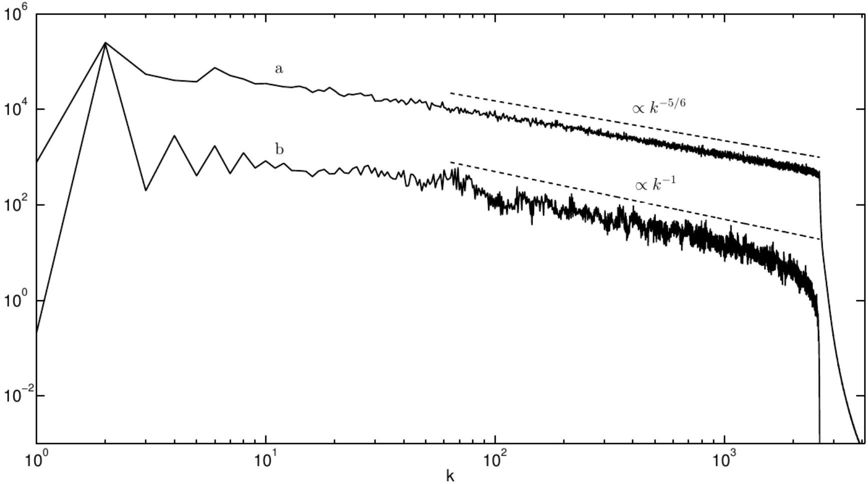
F_0 is a real constant, $L = 400$ and $D\psi$ denotes damping at high and low wavenumbers.

$|\partial_x|^{1/2}\psi$ is defined by $\widehat{|\partial_x|^{1/2}\psi} = |k|^{1/2}\hat{\psi}$.

Includes a turbulent inverse energy cascade, unstable solitons, and dispersive waves.

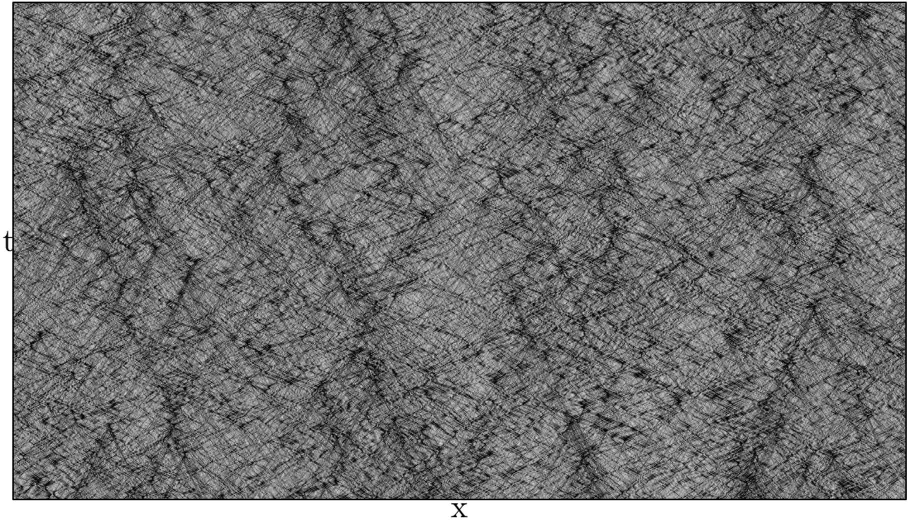
Stochastic Superparameterization in MMT

High-resolution reference simulations



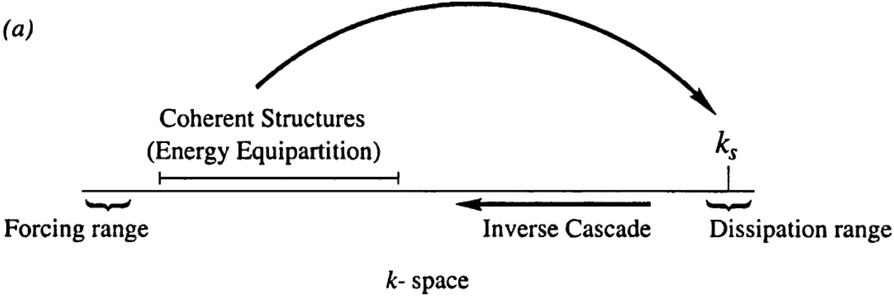
Simulation (a) uses $F_0 = 0.0163$; (b) uses $F_0 = 0.01625$. Both simulations are damped only for $2600 < |k| < 4096$ and $|k| = 1$.

Stochastic Superparameterization in MMT



Visualization of $|\psi(x,t)|$ from simulation with $F_0 = 0.0163$; darker colors indicate higher amplitudes.

Stochastic Superparameterization in MMT



Cai et al. Physica D (2001)

Stochastic Superparameterization in MMT

Model Equations

Mean

$$i\partial_t\overline{\psi} = |\partial_x|^{1/2}\overline{\psi} - |\overline{\psi}|^2\overline{\psi} - 2\overline{|\psi'|^2}\,\overline{\psi} - \overline{(\psi')^2}\,\overline{\psi}^* + i(F_0\sin(4\pi x/L) - D\overline{\psi})$$

Eddy Fourier

$$\mathrm{d}\hat{\mathbf{u}}_k = (\mathcal{L}_k - (\gamma_k + d_k)\mathbf{I})\hat{\mathbf{u}}_k\mathrm{d}t + \sigma_k\mathrm{d}W_k$$

where $\psi' = \psi'_r + i\psi'_i$, $\mathbf{u} = (\psi'_r, \psi'_i)^T$, $\mathbf{u} = \int \hat{\mathbf{u}}_k e^{ikx}\mathrm{d}W_k$ and

$$\mathcal{L}_k = \begin{bmatrix} -\mathcal{I}\{\overline{\psi}^2\} & |k|^{1/2} - 2|\overline{\psi}|^2 + \mathcal{R}\{\overline{\psi}^2\} \\ -|k|^{1/2} + 2|\overline{\psi}|^2 + \mathcal{R}\{\overline{\psi}^2\} & \mathcal{I}\{\overline{\psi}^2\} \end{bmatrix}$$

Stochastic Superparameterization in MMT

Implementation Details

The linear, autonomous, deterministic eddy Fourier covariance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}C_k = \left(\mathcal{L}_k - (\gamma_k + d_k)\mathbf{I}\right)C_k + C_k\left(\mathcal{L}_k - (\gamma_k + d_k)\mathbf{I}\right)^* + \sigma_k\sigma_k^*$$

- is non-normal but diagonalizable at almost all k
- allows exponential growth, depending on the damping $\gamma_k + d_k$, for $|k| \in \left(|\overline{\psi}|^4, 9|\overline{\psi}|^4\right)$

Damping is set to $\gamma_k + d_k = 10^{-5}$ for $|k| < 2600$ and to $10^{-5} + (|k| - 2600)^2$ for $|k| \geq 2600$.

Forcing $\sigma_k\sigma_k^*$ is chosen such that the eddy covariance relaxes towards $\mathbb{E}\left[|\widehat{\psi_r'}|^2\right] = \mathbb{E}\left[|\widehat{\psi_i'}|^2\right] \propto 1/(k^{5/6} + \exp\{|k| - 2600\})$ and $\mathbb{E}\left[\widehat{\psi_r'}^*\widehat{\psi_i'}\right] = 0$ when $\overline{\psi} = 0$.

Stochastic Superparameterization in MMT

Implementation Details

The initial condition for the eddy covariance is uniform on the spatiotemporal domain of the mean equations: the eddy state is reset after each time step of the large scale equations. The initial condition is

$$\mathbb{E}\left[|\widehat{\psi'_r}|^2\right] = \mathbb{E}\left[|\widehat{\psi'_i}|^2\right] \propto 1/(k^{5/6} + \exp\{|k| - 2600\}) \text{ and } \mathbb{E}\left[\widehat{\psi'_r}^{*}\widehat{\psi'_i}\right] = 0.$$

This allows the eddy terms to be pre-computed as functions of $\overline{\psi}$, resulting in significant computational savings.

The eddy terms are averaged over a time interval $\epsilon^{-1} = 1/2, 1/10, 1/50$ comparable to the time step size of the mean equations. Results are largely insensitive to ϵ.

Stochastic Superparameterization in MMT

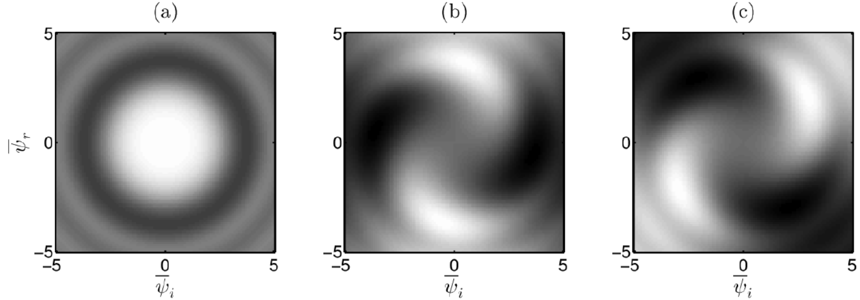
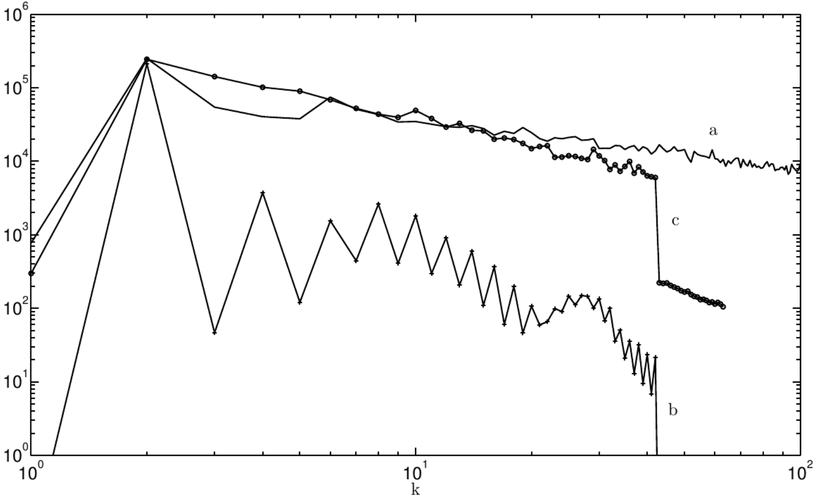


FIG. 4.1. **Precomputed eddy terms as functions of the mean** *The eddy terms in the mean equation (3.4) are precomputed as functions of* $\overline{\psi}$ *using a time interval* $\epsilon^{-1} = 1/10$ *and a large-scale cutoff* $k_0 = 8\pi/25$. *(a) The value of* $\overline{|\psi'|^2}$, *(b) the real part of* $\overline{(\psi')^2}$, *and (c) the imaginary part of* $\overline{(\psi')^2}$. *Higher values are darker, and the exact magnitudes are dependent on the eddy amplitude A.*

Stochastic Superparameterization in MMT

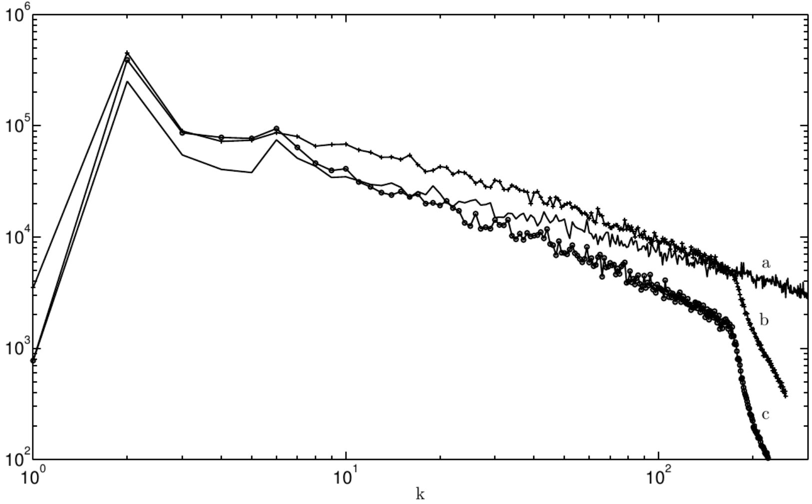
Results



Spectra from simulations with $1/64$ as many points as the reference simulation (a), with no eddy terms (b) and with eddy terms (c).

Stochastic Superparameterization in MMT

Results



Spectra from simulations with 1/16 as many points as the reference simulation (a), with no eddy terms (b) and with eddy terms (c).

Stochastic Superparameterization in MMT

Summary

- A general framework for stochastic subgridscale modelling with no scale separation and no small-scale equilibration based on the Gaussian closure approximation and the point approximation
- Success in a difficult test problem with no scale separation ($k^{-5/6}$ spectra), coherent structures, dispersive waves, and an inverse cascade from unresolved scales into the large scales
- The cost of pre-computing the eddy terms is small and the parameterization incurs negligible run-time cost

References

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- A.J. Majda, *Challenges in Climate Science and Contemporary Applied Mathematics*, Commun. Pur. Appl. Math., **65** (2012)
- A.J. Majda and M.J. Grote, *Mathematical Test Models for Superparameterization in Anisotropic Turbulence*, Proc. Natl. Acad. Sci. U.S.A., **106** (2009)