Log algebraic surfaces

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1 Log algebraic surfaces

The ground field is an algebraically closed field of characteristic zero. Let $\overline{V}$ be the spectrum of the local ring of a normal algebraic surface at a point P. We say that the pair $(\overline{V}, P)$ is a *germ* of a normal algebraic surface. Let $f : V \to \overline{V}$ be the minimal resolution of singularity. Let $\{E_j\}_{1 \le j \le n}$ be the set of irreducible exceptional curves of f. We say that the germ $(\overline{V}, P)$ has *log terminal singularity* if

(i) the canonical divisor $K_{\overline{V}}$ is a $\mathbf{Q}$-Cartier divisor, i.e., an integral multiple of $K_{\overline{V}}$ is a Cartier divisor, and

(ii) $K_V = f^*(K_{\overline{V}}) + \sum_{j=1}^{n} a_j E_j$, where $a_j \in \mathbf{Q}$ and $-1 < a_j \le 0$, where the equality holds in Pic $(V) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Lemma 1.1 *With the above notations, $(\overline{V}, P)$ has a log terminal singularity if and only if $(\overline{V}, P)$ has a quotient singularity.*

Let $\overline{V}$ be a normal projective surface and $\overline{\Delta}$ a reduced effective (Weil) divisor on $\overline{V}$. We can generalize the above definition of log terminal singularity for a pair $(\overline{V}, \overline{\Delta})$. We say that the pair $(\overline{V}, \overline{\Delta})$ has log terminal singularities if the following conditions are satisfied:

(1) $K_{\overline{V}} + \overline{\Delta}$ is a $\mathbf{Q}$-Cartier divisor.

(2) If $f : V \to \overline{V}$ is the minimal resolution of singularities, then the proper transform Δ of $\overline{\Delta}$ is a diviosr with simple normal crossings and

$$K_V + \Delta = f^*(K_{\overline{V}} + \overline{\Delta}) + \sum_{j=1}^{n} a_j E_j$$

with $a_j \in \mathbf{Q}$ and $0 \geq a_j > -1$, where $\{E_j\}_{1 \leq j \leq n}$ is the set of irreducible exceptional curves of f.

Lemma 1.2 *With the above notations, let $\Delta = \sum_{i=1}^{r} C_i$ be the irreducible decomposition of Δ and let $D = \sum_{i=1}^{r} C_i + \sum_{j=1}^{n} E_j$. Then $(\overline{V}, \overline{\Delta})$ has log terminal singularities if and only if the following conditions are satisfied:*

(1) *D is a divisor with simple normal crossings.*

(2) *$\overline{V}$ has only quotient singularities.*

(3) *If $\overline{V}$ has a singular point on $\overline{\Delta}$, say P, then the dual graph of exceptional curves of a minimal resolution of $(\overline{V}, P)$ is a linear chain such that Δ meets only one of the end components of the linear chain, the intersection being at a single point and transverse.*

With the above notations, we call the pair $(\overline{V}, \overline{\Delta})$ a *log projective surface* and $f : (V, \Delta) \to (\overline{V}, \overline{\Delta})$ the *minimal resolution* of $(\overline{V}, \overline{\Delta})$.

An irreducible curve $\overline{C}$ on a log projective surface $(\overline{V}, \overline{\Delta})$ is called a *log exceptional curve of the first kind* if $(K_{\overline{V}} + \overline{\Delta} \cdot \overline{C}) < 0$ and $(\overline{C}^2) < 0$. This is an analogy of an exceptional curve of the first kind, i.e., (-1) curve, on a nonsingular projective surface. A log projective surface $(\overline{V}, \overline{\Delta})$ is called *relatively minimal* if there are no log exceptional curves of the first kind on $\overline{V}$. Then $(\overline{V}, \overline{\Delta})$ is relatively minimal if and only if (V, D) is almost minimal.

Lemma 1.3 *Let $(\overline{V}, \overline{\Delta})$ be a log projective surface. Then there exists a birational morphism $\overline{\mu} : \overline{V} \to \overline{W}$ onto a normal projective surface $\overline{W}$ such that*

(1) *$(\overline{W}, \overline{\Gamma})$ is a log projective surface, where $\overline{\Gamma} = \overline{\mu}_*(\overline{\Delta})$;*

(2) *$(\overline{W}, \overline{\Gamma})$ is relatively minimal.*

The construction of relatively minimal models is done by the theory of peeling which will be explained in the next section.

2 Theory of peelings and almost minimal models

Let (V, D) be a pair of a smooth projective surface and a reduced effective divisor D with simple normal crossings. Let $X := V - D$. The Kodaira dimension $\overline{\kappa}(X)$ is defined as

$$\overline{\kappa}(X) = \begin{cases} -\infty & \text{if } N(D+K_V) = \emptyset \\ \sup\limits_{m \in N(D+K_V)} \dim \Phi_{m(D+K_V)}(V) & \text{if } N(D+K_V) \neq \emptyset \end{cases}$$

where

$$N(D+K_V) = \{m \in \mathbf{N} \mid |m(D+K_V)| \neq \emptyset\}.$$

In fact, $\overline{\kappa}(X)$ is independent of the choice of a pair (V, D) such that $X = V - D$. If $\overline{\kappa}(X) \geq 0$, the divisor $D + K_V$ has the Zariski-Fujita decomposition,

$$D + K_V = (D+K_V)^+ + (D+K_V)^-,$$

where $(D+K_V)^+$ is the nef component and $(D+K_V)^-$ is the negative component. Then we can find by Kawamata-Fujita-Tsunoda that there exist a nonsingular projective surface $\widetilde{V}$, a birational morphism $f : V \to \widetilde{V}$ and an effective $\mathbf{Q}$-divisor $\widetilde{D}^{\#}$ on $\widetilde{V}$ such that Supp $\widetilde{D}^{\#} \subseteq$ Supp $f_*(D)$ and $(D+K_V)^+ \equiv f^*(\widetilde{D}^{\#} + K_{\widetilde{V}})$. A pair $(\widetilde{V}, \widetilde{D})$ is thought of a *relatively minimal model* of the pair (V, D). The theory of peeling provides a concrete way of constructing the relatively minimal model $(\widetilde{V}, \widetilde{D})$.

Let $T = \{D_1, \ldots, D_r\}$ be a subset of irreducible components of D. We call L an *admissible maximal rational twig* if

(1) Each D_i is a smooth rational curve.
(2) $(D_i \cdot D_j) = 1$ if $j = i+1$ $(1 \leq i \leq r-1)$ and $(D_i \cdot D_j) = 0$ otherwise.
(3) T does not meet any component of $D - T$ except for D_r meeting $D - T$ transversally in one point.
(4) The intersection matrix of T is negative definite.

A connected component R (resp. F) is called an *admissible rational rod* (resp. *fork*) if

(1) Every irreducible component of R (resp. T) is a smooth rational curve.

(2) R is a linear chain (resp. F has the same dual graph as the exceptional graph of a quotient singularity).
(3) The intersection matrix of R (resp. F) is negative definite.

Let $\{T_\lambda\}_{\lambda\in\Lambda}$ (resp. $\{R_\mu\}_{\mu\in M}$, $\{F_\nu\}_{\nu\in N}$) be the set of all admissible maximal rational twigs (resp. rods, forks). Then there exists a unique effective $\mathbf{Q}$-divisor $\mathrm{Bk}\,(D)$ such that

(1) $\mathrm{Bk}\,(D)$ is a disjoint sum of $\mathrm{Bk}\,(T_\lambda), \mathrm{Bk}\,(R_\mu)$ and $\mathrm{Bk}\,(F_\nu)$, i.e.,

$$\mathrm{Bk}\,(D) = \sum_\lambda \mathrm{Bk}\,(T_\lambda) + \sum_\mu \mathrm{Bk}\,(R_\mu) + \sum_\nu \mathrm{Bk}\,(F_\nu).$$

(2)

$$\mathrm{Supp}\,\mathrm{Bk}\,(D) = \bigcup_\lambda T_\lambda \cup \bigcup_\mu R_\mu \cup \bigcup_\nu F_\nu.$$

(3) $(D^\# + K_V \cdot D_i) = 0$ for every irreducible component D_i of $\mathrm{SuppBk}(D)$, where $D^\# = D - \mathrm{Bk}\,(D)$.
(4) Let $D^\# = \sum_{i=1}^n a_i D_i$ be the irreducible decomposition of $D^\#$. Then $0 \le a_i \le 1$, where $a_i = 1$ if and only if D_i is not a component of $\mathrm{Bk}\,(D)$.
(5) $\mathrm{Supp}\,(D)$ and $\mathrm{Supp}\,(D - \mathrm{Bk}\,D)$ differs only by the exceptional loci of rational double points.

We call $\mathrm{Bk}\,(D)$ the *bark* of D and say that $D^\#$ is obtained by *peeling* the bark of D.

The following results are the crucial results in the theory of peeling.

Lemma 2.1 *Let V, D and $D^\#$ be as above. Suppose that there exists an irreducible component D_0 of D such that*

$$(D^* + K_V \cdot D_0) < 0 \quad \textit{and} \quad (D_0^2) \le -1\ .$$

Then the following assertions hold.

(1) *D_0 is a (-1) curve.*
(2) *D_0 is one of the following components of D:*
  (i) *With an admissible rational twig T which might be empty, D_0 forms a rational twig; we denote it by $T + D_0$;*

(ii) *With admissible rational twigs* T, T' *which might be empty,* D_0 *forms a rational rod,* $T + D_0 + T'$*;*

(iii) D_0 *is a component of the connected component* F *whose weighted dual graph looks like the one for an admissible rational fork except for that the component* D_0 *which has the branching number* 3 *has self-intersection* -1.

The component D_0 is called the *irrelevant component* of the non-admissible rational twig $T - D_0$, the non-admissible rational rod $T - D_0 - T'$ or the non-admissible rational fork F.

Lemma 2.2 $(D^{\#} + K_V \cdot Y) \geq 0$ *for every irreducible component* Y *of* D *except for the irrelevant components* D_0 *of the non-admissible rational twigs, the non-admissible rational rods and the non-admissible rational forks.*

Lemma 2.3 *Suppose there exists an irreducible curve* E *on* V *such that* $(D^{\#} + K_V \cdot E) < 0, E$ *is not an irreducible component of* D *and* $E + \mathrm{Bk}(D)$ *is negative definite. Then we have* $(E \cdot K_V) < 0$ *and* $(E^2) < 0$*, i.e.,* E *is a* (-1) *curve. Let* $\sigma : V \to V'$ *be the contraction of* E *and let* $D' = \sigma_*(D)$*. Then* D' *is a reduced effective divisor on* V' *with simple normal crossings.*

Lemma 2.4 *Let* E *now be an* exceptional component*, i.e.,* E *is an irreducible component of* D *which is a* (-1) *curve. Suppose that the braching number of* E *in* D *is less than or equal to* 2*,* $(D^{\#} + K_V \cdot E) < 0$ *and* $E + \mathrm{Bk}\,(D)$ *is negative definite. We call* E *a* superfluous exceptional component *of* D*. In view of Lemma 2.1, such a component* E *appears only in one of the following situations:*

(i) E *is an isolated component.*

(ii) E *is the irrelevant component of a non-admissible rational twig or a non-admissible rational rod.*

(iii) Let $f : V \to \overline{V}$ *be the contraction of* E *and consecutively contractible components of* $\mathrm{Bk}(D)$*. Let* $\overline{D} = f_*(D)$*. Then* $\overline{D}$ *is a reduced effective divisor with simple normal crossings.*

Theorem 2.5 *Let* V *be a smooth projective surface defined over* k *and let* D *be a reduced effective divisor with simple normal crossings. Then there* *exists a birational morphism* $\mu : V \to \widetilde{V}$ *onto a smooth projective surface* $\widetilde{V}$ *such that, with* $\widetilde{D} = \mu_*(D)$*, the following conditions are satisfied:*

(1) $h^0(V, n(D + K_V)) = h^0(\widetilde{V}, n(\widetilde{D} + K_{\widetilde{V}}))$ *for every integer* $n \geq 0$.
(2) $\mu_* \mathrm{Bk}\,(D) \leq \mathrm{Bk}\,(\widetilde{D})$ *and* $\mu_*(D^\# + K_V) \geq \widetilde{D}^\# + K_{\widetilde{V}}$.
(3) *For every irreducible curve* C *on* $\widetilde{V}$*, we have either* $(\widetilde{D}^\# + K_{\widetilde{V}} \cdot C) \geq 0$ *or* $(\widetilde{D} + K_{\widetilde{V}} \cdot C) < 0$ *and* $C + \mathrm{Bk}\,(\widetilde{D})$ *is not negative definite.*

The birational morphism $\mu : V \to \widetilde{V}$ is obtained as a composite of the following operations:

(1) Contract all possible superfluous exceptional components of D.
(2) If there are no superfluous exceptional components in D, then peel the barks of all admissible rational maximal twigs, all admisssible rational rods and all admissible rational forks of D.
(3) Find a (-1) curve E such that $E \not\subset \mathrm{Supp}\,(D)$, $(D^\# + K_V \cdot E) < 0$ and $E + \mathrm{Bk}\,(D)$ is negative definite. If there is none then we are done. If there is one, consider the contraction $\sigma : V \to \overline{V}$ of E and $\overline{D} = \sigma_*(D)$.
(4) Now repeat the operations (1), (2) and (3) all over again.

A pair (V, D) is called *almost minimal* if, for every irreducible curve C on V, either $(D^\# + K_V \cdot C) \geq 0$ or $(D^\# + K_V \cdot C) < 0$ and $C + \mathrm{Bk}\,(D)$ is not negative definite. The pair $(\widetilde{V}, \widetilde{D})$ given in the above theorem is called an *almost minimal model* of (V, D).

Let $(\overline{V}, \overline{\Delta})$ be a log projective surface and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. A relatively minimal model of $(\overline{V}, \overline{\Delta})$ is obtained from $(\widetilde{V}, \widetilde{D})$ by contracting $\mathrm{Bk}\widetilde{D}$ to the quotient singular points. We note that (V, D) is almost minimal if and only if $(\overline{V}, \overline{\Delta})$ is relatively minimal and that $D^\# + K_V = f^*(\overline{\Delta} + K_{\overline{V}})$ if (V, D) is almost minimal.

In the affine case, the construction of a relatively minimal model is just the subtraction of a disjoint union of topologically contractible curves. This is seen by the following result:

Lemma 2.6 *Let* $(\overline{V}, \overline{\Delta})$ *be a log projective surface such that* $\overline{V} - \overline{\Delta}$ *is affine and let* $\overline{C}$ *be a log exceptional curve of the first kind such that* $\overline{C} \not\subset \mathrm{Supp}\,\overline{\Delta}$*. Set* $\overline{C}_0 = \overline{C} - \overline{C} \cap \overline{\Delta}$*. Then we have:*

(1) *The Euler number* $e(\overline{C}_0) = 0$ *or* 1 *and* $\overline{C}_0$ *passes through at most one singular point of* $\overline{V} - \overline{\Delta}$.

(2) *Suppose either the irregularity* q *of (a minimal resolution of)* $\overline{V}$ *is zero or* $\kappa(\overline{V}, \overline{\Delta}) = -\infty$. *Then* $e(\overline{C}_0) = 1$.

(3) *In addition to the hypothesis of* (2) *above assume that* $\overline{V} - \overline{\Delta}$ *is nonsingular. Then* $\overline{C}_0$ *is isomorphic to the affine line* $\mathbf{A}^1$.

Corollary 2.7 *Let* X *be a normal affine surface with at worst quotient singularities. Suppose either* q *(= the irregularity of a nonsingular projective model of* X*) is zero or* $\overline{\kappa}(S - \mathrm{Sing}\, X) = -\infty$. *Then we have*

(1) *There exists an affine open set* U *of* X *such that*

(i) *either* $U = X$ *or* $X - U$ *is a disjoint union of contractible curves.*

(ii) U *is relatively minimal.*

(iii) *If* X *is nonsingular then* $X - U$ *is a disjoint union of the curves isomorphic to* $\mathbf{A}^1$.

(2) X *is relatively minimal provided* X *contains no contractible curves.*

3 Log algebraic surfaces, II

Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface and $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. An irreducible curve $\overline{E}$ on $\overline{V}$ is called a *log exceptional curve of the second kind* if $(\overline{\Delta} + K_{\overline{V}} \cdot \overline{E}) = 0$ and $(\overline{E}^2) < 0$, or equivalently, if $(K_V + D^\# \cdot E) = 0$ and $E + \mathrm{Bk}\,(D)$ is negative definite, where E is the proper transform of $\overline{E}$ on V.

Lemma 3.1 *Let* $\overline{E}$ *be an exceptional curve of the second kind such that* $\overline{E} \not\subset \mathrm{Supp}\, \overline{\Delta}$. *Then the following assertions hold.*

(1) E *is a* (-2) *curve if* $(E \cdot D^\#) = 0$ *and a* (-1) *curve if* $(E \cdot D^\#) > 0$. *If* E *is a* (-1) *curve then* $(E \cdot D^\#) = 1$.

(2) *Suppose* E *is a* (-1) *curve. Then* E *meets at most two connected components of* $\mathrm{Supp\,Bk}\,(D)$. *If* E *meets a connected component of* $\mathrm{Supp\,Bk}\,(D)$ *then* E *meets it in a single point transversally. Hence* $\overline{E}$ *passes through at most two singular points of* $\overline{V}$.

(3) *Suppose* E *meets* $\mathrm{Supp}\, f^*(\overline{\Delta})$. *Let* $\overline{E}_0 = \overline{E} - \overline{E} \cap \overline{\Delta}$. *Then* $e(\overline{E}_0) = 1$.

(4) *Let* $\sigma : V \to W$ *be a composite of the contraction of* E *and the contractions of all subsequently contractible components of* $\mathrm{Bk}\,(D)$*. Let* $B = \sigma_*(D)$*. Then* B *is a reduced effective divisor with simple normal crossings, and each connected component of* $\sigma(\mathrm{Supp}\mathrm{Bk}(D))$ *is an admissible rational twig, an admissible rational rod or an admissible rational fork of* B*.*

(5) *Let* $g : W \to \overline{W}$ *be the contraction of all the connected components of* $\mathrm{Supp}\ \mathrm{Bk}\,(B)$ *to the quotient singular points and let* $\overline{\Gamma} = g_*(B)$*. Then* $(\overline{W}, \overline{\Gamma})$ *is a log projective surface, and there exists a birational morphism* $\overline{\sigma} : \overline{V} \to \overline{W}$ *such that* $\overline{\sigma} \cdot f = g \cdot \sigma$ *and* $\overline{\Gamma} = \overline{\sigma}_*(\overline{\Delta})$*.*

(6) *Suppose* $\overline{\kappa}(V - D) \geq 0$*. Then* $D^{\#} + K_V = \sigma^*(B^{\#} + K_W)$ *and* $\overline{\Delta} + K_{\overline{V}} = \overline{\sigma}^*(\overline{\Gamma} + K_{\overline{W}})$*.*

(7) *Suppose* $\overline{\kappa}(V - D) \geq 0$*. Then* $\overline{\kappa}(V - D) = \overline{\kappa}(W - B)$*.*

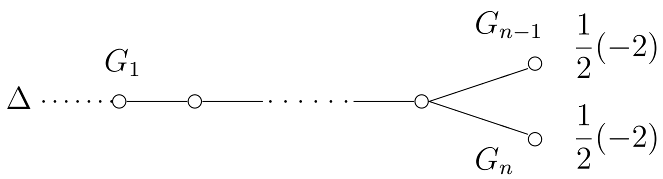
Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. *We assume, hereafter, that* $\overline{\kappa}(V - D) \geq 0$ *and that there are no log exceptional curves of the second kind whose proper transforms on* V *are* (-1) *curves lying outside* $\mathrm{Supp}\,\Delta$. So, if there exists a log exceptional curve of the second kind it is either contained in $\mathrm{Supp}\ \overline{\Delta}$ or disjoint from $\mathrm{Supp}\ \overline{\Delta}$.

Lemma 3.2 *Let* $G = G_1 + \cdots + G_n$ *be a connected reduced effective divisor on* V *with the irreducible components* G_i *such that* $\mathrm{Supp}\, G \bigcap \mathrm{Supp}\, D = \emptyset$*,* $(D^{\#} + K_V \cdot G) = 0$ *and* G *is negative definite. Then* G *is the exceptional locus of the minimal resolution of a rational double point.*

Lemma 3.3 *Let* $G = G_1 + \cdots + G_n$ *be a connected reduced effective divisor on* V *with the irreducible components* G_i *such that* $\mathrm{Supp}\ G \subset \mathrm{Supp}\ D, \mathrm{Supp}\ G \not\subset \mathrm{Supp}\ \mathrm{Bk}\,(D), (D^{\#} + K_V \cdot G) = 0$ *and* $G + \mathrm{Bk}\,(D)$ *is negative definite. Suppose that* G *is maximal among the divisors satisfying the above properties and that* G *is not a connected component of* D*. Then the weighted dual graph of* G *is one of the following:*

(1) *A linear chain of nonsingular rational curves, where only the end components* G_1 *and* G_n *meet* $D - G$*, actually each meeting* Δ *in a single point.*

(2) *A graph of Dynkin type D consisting of nonsingular rational curves (cf. Lemma 3.4), where the component* G_1 *meets* Δ *and other components have no intersection with* $D - G$*, where* G_{n-1} *and* G_n *have coefficients* $1/2$ *in* $D^{\#}$ *and self-intersection numbers* -2.



Lemma 3.4 *Let* $G = G_1 + \cdots + G_n$ *be a connected reduced effective divisor on* V *with the irreducible components* G_i *such that* $\mathrm{Supp}\, G \subset \mathrm{Supp}\, D, \mathrm{Supp}\, G \not\subset \mathrm{Supp}\, \mathrm{Bk}\,(D), (D^{\#} + K_V \cdot G) = 0$ *and* $G + \mathrm{Bk}\,(D)$ *is negative definite. Suppose that* G *is a connected component of* D*. Then one of the following cases takes place:*

(1) G *consists of a nonsingular elliptic curve, i.e.,* G *is the exceptional curve of the minimal resolution of an elliptic singular point.*

(2) $G = G_1 + \ldots + G_n$ *is a cycle of nonsingular rational curves with* $n \geq 2$*, i.e.,* $(G_i \cdot G_j) = 0$ *unless* $(G_i \cdot G_{i+1}) = 1$ *for* $1 \leq i \leq n-1$ *and* $(G_n \cdot G_1) = 1$*. Hence* G *is the exceptional curve of the minimal resolution of a cuspidal singular point.*

(3) G *consists of nonsingular rational curves, and the dual graph of* G *is the graph of the exceptional graph of minimal resolution of a quasi-elliptic or quasi-cuspidal singular point, where a quasi-elliptic (resp. quasi-cuspidal) singular point is the quotient of an elliptic (resp. cuspidal) singular point by a finite group action.*

Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface. We say that $(\overline{V}, \overline{\Delta})$ is *strongly minimal* if there are no log exceptional curves of the second kind on $\overline{V}$.

Let $\overline{V}$ be a normal projective surface and let $\overline{\Delta}$ be a reduced effective Weil divisor on $\overline{V}$. We say that the pair $(\overline{V}, \overline{\Delta})$ has *log canonical* singularities if the following conditions are satisfied:

(1) $K_{\overline{V}} + \overline{\Delta}$ is a **Q**-Cartier divisor;

(2) If $f : V \to \overline{V}$ is the minimal resolution of singularities then the proper transform Δ of $\overline{\Delta}$ plus the set of irreducible exceptional curves $\{E_j\}_{1 \le j \le n}$ of f is a divisor with simple normal crossings and

$$K_V + \Delta = f^*(K_{\overline{\Delta}} + \overline{\Delta}) + \sum_{j=1}^{n} a_j E_j \qquad (\dagger)$$

with $a_j \in \mathbf{Q}$ and $-1 \le a_j \le 0$.

Let $D = \Delta + \sum_{j=1}^{n} E_j$. Then the equality ($\dagger$) is written as

$$K_V + D^{\#} = f^*(K_{\overline{V}} + \overline{\Delta}) \ .$$

Let $\Xi = \sum_{j=1}^{n} E_j, \Xi_1 = \sum_{a_j=-1} E_j$ and $\Xi_2 = \sum_{a_j \neq -1} E_j$. Let $\sigma : V \to \widetilde{V}$ be the contraction of $\Xi_2, \widetilde{\Delta} = \sigma_* \Delta$ and $\widetilde{\Xi} = \sigma_*(\Xi) = \sigma_*(\Xi_1)$. Then $(\widetilde{V}, \widetilde{\Delta} + \widetilde{\Xi})$ has log terminal singularities and $(V, D) \to (\widetilde{V}, \widetilde{\Delta} + \widetilde{\Xi})$ is the minimal resolution of singularities. Furthermore, $\widetilde{\Xi}$ is a set of exceptional curves of the second kind on $\widetilde{V}$. Here we can define an irreducible exceptional curve $\widetilde{C}$ of the second kind on $(\widetilde{V}, \widetilde{\Delta} + \widetilde{\Xi})$ by the condition $(K_{\widetilde{V}} + \widetilde{\Delta} + \widetilde{\Xi} \cdot \widetilde{C}) = 0$, though $(\widetilde{V}, \widetilde{\Delta} + \widetilde{\Xi})$ is not necessarily relatively minimal. Hence log canonical singularities are elliptic singularities, cusp singularities, quasi-elliptic singularities, quasi-cuspidal singularities and quotient singularities.

Theorem 3.5 *Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface. Suppose that $\overline{\kappa}(V - D) = 2$. Let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution and let G be a connected component of D as in Lemma 3.4. Then there exists a birational morphism $\overline{\rho} : \overline{V} \to \overline{W}$ onto a normal projective surface $\overline{W}$ satisfying the following conditions:*

(1) *$P := \overline{\rho} \cdot f(G)$ is a point on $\overline{W}$, and $\overline{V} - f(G)$ is isomorphic to $\overline{W} - \{P\}$.*

(2) *$K_{\overline{W}} + \overline{\Gamma}$ is a **Q**-Cartier divisor, where $\overline{\Gamma} = \overline{\rho}_*(\overline{\Delta})$, and $K_{\overline{V}} + \overline{\Delta} = \overline{\rho}^*(K_{\overline{W}} + \overline{\Gamma})$.*

Hence $(\overline{W}, \overline{\Gamma})$ has log canonical singularities.

We have the following result of Kawamata.

Theorem 3.6 *Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. Assume that $\overline{\kappa}(V - D) = 2$. Let n_0 be the smallest positive integer such that $n_0(D^\# + K_V)$ is an integral divisor. Then the following assertions hold:*

(1) *The linear system $|nn_0(D^\# + K_V)|$ has no base points if n is sufficiently large.*
(2) *The graded ring $R^* := \bigoplus_{n \geq 0} H^0(V, nn_0(D^\# + K_V))$ is finitely generated over k, and the graded ring $\bigoplus_{n \geq 0} H^0(V, [n(D^\# + K_V)])$ is a finite R^*-module.*
(3) *The graded ring $R := \bigoplus_{n \geq 0} H^0(V, n(D + K_V))$ is finitely generated over k.*
(4) *Let $\overline{V}_c := \mathrm{Proj}\,(R)$. Then $\overline{V}_c$ is isomorphic to the image of V by $\Phi_{|nn_0(D^\# + K_V)|}$ for $n \gg 0$. Hence $(\overline{V}_c, \overline{\Gamma}_c)$ has only log canonical singularities, where $\overline{\Gamma}_c$ is the direct image of D by $\Phi_{|nn_0(D^\# + K_V)|}$.*

In the affine case, the construction of the strongly minimal model is also the subtraction of a disjoint union of topologically contractible curves.

Lemma 3.7 *Let X be a normal affine surface with at worst quotient singularities, let $(\overline{V}, \overline{\Delta})$ be a normal projective completion of X and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be a minimal resolution. Let $\overline{C}$ be a log exceptional curve of the second kind such that $\overline{C} \not\subset \mathrm{Supp}\,\overline{\Delta}$. Then C is a (-1) curve meeting $\mathrm{Supp}\, f^*(\overline{\Delta})$ transversally in a single smooth point. Furthermore, $\overline{C}$ passes through at most one singular point on X and $e(\overline{C}_0) = 1$, where $\overline{C}_0 = \overline{C} - \overline{C} \cap \overline{\Delta}$. If X is smooth then $\overline{C}_0 \cong \mathbf{A}^1$.*

Lemma 3.8 *Let X be a normal affine surface with at worst quotient singularities. Suppose that the following conditions are satisfied:*

(1) *Either the irregularity of a smooth projective model of X is zero or $\overline{\kappa}(X - \mathrm{Sing}\, X) = -\infty$.*
(2) *Let $(\overline{V}, \overline{\Delta})$ be a normal projective completion of X and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be a minimal resolution. Then the dual graph of D is a tree.*

Then there exists an affine open set U_0 of X such that U_0 is a strongly minimal model of X and that either $X = U_0$ or $X - U_0$ is a disjoint union *of contractible curves, which are isomorphic to* $\mathbf{A}^1$ *provided* X *is smooth. Furthermore,* $\overline{\kappa}(U_0 - \mathrm{Sing}\, U_0) = \overline{\kappa}(X - \mathrm{Sing}\, X)$.

4 Structure theorems

We have the following result.

Lemma 4.1 *Let* $(\overline{V}, \overline{\Delta})$ *be a log projective surface and let* $f : (V, D) \to (\overline{V}, \overline{\Delta})$ *be the minimal resolution. Suppose that the pair* $(\overline{V}, \overline{\Delta})$ *is relatively minimal. Then* $\overline{\kappa}(V - D) \geq 0$ *if and only if* $D^{\#} + K_V$ *is nef. Suppose* $D^{\#} + K_V$ *is not nef. Then we have either (i) or (ii) below:*

(i) $V - D$ *is affine-ruled,*
(ii) $\rho(\overline{V}) = 1$ *and* $-(\overline{D} + K_{\overline{V}})$ *is ample, where* $\overline{D} = \overline{\Delta}$.

We say that $(\overline{V}, \overline{D})$ is a *log del Pezzo surface* if $-(\overline{D} + K_{\overline{V}})$ is ample. If $\rho(\overline{V}) = 1$ we say that $(\overline{V}, \overline{D})$ has rank one. Furthermore, a log del Pezzo surface is said to be *open* (resp. *complete*) if $\overline{D} \neq 0$ (resp. if $\overline{D} = 0$). A complete log del Pezzo surface is simply called a log del Pezzo surface.

Let X be a non-complete nonsingular algebraic surface. We say that X has an $\mathbf{A}^1_*$*-fiber space* structure if there exists a surjective morphism $\varphi : X \to C$ from X onto a nonsingular curve C such that general fibers of φ are isomorphic to $\mathbf{A}^1_*$, the affine line $\mathbf{A}^1_k$ minus one point. We say that the morphism φ is an $\mathbf{A}^1_*$*-fibration.* A fiber $\varphi^*(P)$ is *singular* if either $\varphi^{-1}(P)$ is reducible or $\varphi^*(P) = n_P C_P$, where $n_P \geq 2$ and C_P is irreducible. If $\varphi^*(P) = \sum_{i=1}^r n_i C_i$ is the irreducible decomposition and if $n_P := \gcd(n_1, \ldots, n_r) > 1$, we say that the fiber $\varphi^*(P)$ is a *multiple fiber with multiplicity* n_P. Given an $\mathbf{A}^1_*$-fiber space $\varphi : X \to C$ with a complete curve C, there exist a smooth completion (W, B, X) of X and a surjective $\mathbf{P}^1$-fibration $\pi : W \to C$ such that φ is the resteriction of π onto X. An $\mathbf{A}^1_*$-fiber space $\varphi : X \to C$ is called a *Platonic* $\mathbf{A}^1_*$*-fiber space* if the following conditions are satisfied:

(1) $C \cong \mathbf{P}^1$.
(2) φ has no singular fibers but three multiple fibers $\Gamma_i = \mu_i \Delta_i, 1 \leq i \leq 3$, such that Δ_i is isomorphic to $\mathbf{A}^1_*$ and that $\{\mu_1, \mu_2, \mu_3\}$ is, up to

a permutation, one of the triplets $\{2,2,n\}(n \geq 2), \{2,3,3\}, \{2,3,4\}$ and $\{2,3,5\}$.

(3) There exist a smooth completion (W, B, X) of X and a surjective morphism $\pi : W \to C$ as above such that:

  (i) B contains two irreducible components S_0 and S_1 which are cross-sections of π with $S_0 \cap S_1 = \emptyset$, and other irreducible components of B are contained in the fibers of π;

  (ii) every fiber of π has a linear chain as its weighted dual graph.

Theorem 4.2 *Let $(\overline{V}, \overline{D})$ be an open log del Pezzo surface of rank one and let $f : (V, D) \to (\overline{V}, \overline{D})$ be the minimal resolution. Let $X = V - D$. Then X is affine-ruled or X is a Platonic $\mathbf{A}^1_*$-fiber space.*

This result implies the following result.

Theorem 4.3 *Let X be a smooth open algebraic surface with $\overline{\kappa}(X) = -\infty$. Suppose that X is not affine-ruled. Suppose furthermore that there exists an open immersion of X into a smooth projective surface V such that*

(1) *$V - X$ is a reduced effective divisor with simple normal crossings.*

(2) *If we write $V - X = \bigcup_{i=1}^{r} C_i$ with irreducible components C_i, the intersection matrix $((C_i \cdot C_j))_{1 \leq i,j \leq r}$ is not negative definite.*

Then there exist a Zariski open set U of X and a proper birational morphism $\varphi : U \to T'$ onto a smooth algebraic surface T' defined over k such that

(i) *Either $U = X$ or $X - U$ has pure dimension one.*

(ii) *T' is an open set of a Platonic $\mathbf{A}^1_*$-fiber space T with* $\dim(T - T') \leq 0$.

Consider next the case of Kodaira dimension 0 or 1.

Lemma 4.4 *Let $(\overline{V}, \overline{D})$ be a relatively minimal log projective surface. If $\overline{\kappa}(V - D) = 0$ or 1 then $((D^\# + K_V)^2) = 0$.*

Lemma 4.5 *Let $(\overline{V}, \overline{D})$ be a relatively minimal log projective surface. If $D^\# + K_V$ is nef then $\overline{\kappa}(V - D) \geq 0$. This implies, in particular, that $n(D^\# + K_V) \sim 0$ for some positive integer n provided $D^\# + K_V \equiv 0$.*

The following result is a sort of the converse to the previous tw results.

Theorem 4.6 *Let* $(\overline{V}, \overline{D})$ *be a relatively minimal log projective surface such that* $\overline{\kappa}(V - D) \geq 0$, $D^{\#} + K_V \not\equiv 0$ *and* $(D^{\#} + K_V{}^2) = 0$. *Then* $\overline{\kappa}(V - D) = 1$.

In the case of Kodaira dimension 1, we have the following theorems due mainly to Kawamata.

Theorem 4.7 *Let* $(\overline{V}, \overline{D})$ *be a relatively minimal log projective surface with* $\overline{\kappa}(V - D) = 1$. *Then, for a positive integer* n *with* $n(D^{\#} + K_V)$ *a Cartier divisor, the linear system* $|n(D^{\#} + K_V)|$ *is composed of an irreducible pencil* Λ *without base points. Let* $\varphi : V \to B$ *be a fibration defined by* Λ. *Then one of the following assertions holds:*

(1) Λ *is a pencil of rational curves. Let* ℓ *be a general fiber of* φ. *Then* $(D^{\#} \cdot \ell) = ([D^{\#}] \cdot \ell) = 2$ *and every irreducible component of* Bk (D) *is a fiber component of* φ, *i.e., contained in a fiber of* φ.

(2) Λ *is a pencil of elliptic curves. Every irreducible component of* D *is a fiber component of* φ.

Applying Theorem 4.7 to affine surfaces, we obtain the following result.

Theorem 4.8 *Let* X *be a normal affine surface with at worst quotient singularities. Suppose that* $\overline{\kappa}(X - \mathrm{Sing}\, X) = 1$ *and that one of the following conditions are satisfied:*

(i) X *is relatively minimal, i.e., there exists a relatively minimal log projective surface* $(\overline{V}, \overline{\Delta})$ *such that* $X = \overline{V} - \overline{\Delta}$.

(ii) X *is smooth.*

Then there exists an $\mathbf{A}^1_*$*-fibration* $\rho : X \to B$.

As in the complete case, we have the following canonical divisor formula.

Theorem 4.9 *Let* $(\overline{V}, \overline{D})$ *be a relatively minimal log projective surface with* $\overline{\kappa}(V - D) = 1$. *Let* $h : V \to W$ *be the birational morphism* $h : (V, D) \to (W, C)$ *with a* $\mathbf{Q}$*-divisor* $C = h_*(D^{\#})$ *such that* $D^{\#} +$

$K_V = h^*(C + K_W)$ *and that there are no* (-1)*-curves* E *on* W *such that* $(C + K_W \cdot E) = 0$*. Suppose that* $|n(C + K)|$ *defines an elliptic fibration* $\pi : W \to B$ *for a sufficiently large integer* n *with* $n(C + K)$ *a Cartier divisor and* $\dim |n(C + K)| > 0$*. Then the following assertions hold.*

(1) *The fibration* $\pi : W \to B$ *is relatively minimal.*

(2) $C = \sum_i d_i F_i$*, where* $0 < d_i \leq 1$ *and* $m_i F_i$ *is a (distinct) fiber of* π *for some integer (multiplicity)* $m_i \geq 1$*.*

(3) *We have the following formula:*

$$C + K_W = \pi^*(K_B - \delta) + \sum_s (m_s - 1)E_s + \sum_i d_i F_i \ ,$$

where $m_s E_s$ *ranges over all multiple fibers of* π *with multiplicity* m_s *and the reduced form* E_s *and where* δ *is a divisor on* B *with* $t := -\deg \delta = \chi(\mathcal{O}_V)$*.*

(4) *For a sufficiently large integer* n *as above, we have*

$$h^0(V, n(D^\# + K_V)) = \\ n(2g - 2 + t) + \sum_s \left[n\left(1 - \frac{1}{m_s}\right)\right] + \sum_i \left[\frac{nd_i}{m_i}\right] + (1 - g),$$

where g *is the genus of* B*.*

(5) *Suppose* $\kappa(V) = -\infty$*. Then the reduced inverse image* $\pi^*(F_i)_{\mathrm{red}}$ *is contained in* $\mathrm{Supp}\, D$ *for some* i*.*

Theorem 4.10 *Let* $(\overline{V}, \overline{D})$ *be a relatively minimal log projective surface with* $\overline{\kappa}(V - D) = 1$*. Let* $h : V \to W$ *be the birational morphism as above such that* $D^\# + K_V = h^*(C + K_W)$ *with* $C = h_*(D^\#)$ *and that there are no* (-1)*-curves* E *on* W *such that* $(C + K_W \cdot E) = 0$*. Suppose that* $|n(C + K)|$ *defines a* $\mathbf{P}^1$*- fibration* $\pi : W \to B$ *for a sufficiently large integer* n *with* $n(C+K)$ *a Cartier divisor and* $\dim |n(C+K)| > 0$*. Then the following assertions hold.*

(1) *The* $\mathbf{P}^1$*-fibration* $\pi : W \to B$ *is relatively minimal.*

(2) *The divisor* C *is written as*

$$C = H + \sum_i d_i F_i \ ,$$

where H *is a sum of horizontal components and the* F_i *are the fibers of* π*. Each component of* H *has coefficient* 1*, and* H *is either a*

2-section or consists of two cross-sections. The divisor H has only normal crossings.

(3) *We have the following formula:*

$$C + K_W = \pi^*(K_W + \delta) + \sum_i d_i F_i \ ,$$

where $t := \deg \delta$ is equal to a sum of the number of double points of H and one half of the number of branch points of $\pi \mid_H$ if H is a 2-section and t is equal to $(H_1 \cdot H_2)$ if $H = H_1 + H_2$ with the cross-sections H_1 and H_2.

(4) *For a sufficiently large integer n as above and with the nd_i integers, we have*

$$h^0(W, n(C + K_W)) = n(2g - 2 + t) + \sum_i nd_i + 1 - g \ .$$

Consider the case of Kodaira dimension 0.

Theorem 4.11 *Let $(\overline{V}, \overline{D})$ be a relatively minimal log projective surface with $\kappa(V - D) = 0$. Suppose that there are no superfluous exceptional components in D and no log exceptional curves of the second kind whose proper transforms on V are (-1) curves lying outside* Supp $[D^{\#}]$. *Then the following assertions hold true:*

(1) *Suppose $\kappa(V) = 0$. Then $D^{\#} = 0$ and V is relatively minimal. In particular, D consists of (-2) rods and (-2) forks. If $D \neq 0$ then V is either a K3-surface or an Enriques surface.*

(2) *Suppose V is an irrational ruled surface. Then V is a ruled surface over an elliptic curve, say B, V is relatively minimal and D consists of a 2-section C which is an étale covering of B or a sum $C_1 + C_2$, where C_1 and C_2 are mutually disjoint cross-sections. Furthermore, $2(C + K_V) \sim 0$ or $C_1 + C_2 + K_V \sim 0$.*

(3) *Suppose that V is a rational surface and that D contains either an irrational component C or a cycle of rational curves, i.e., a collection of nonsingular rational curves $C_1 + C_2 + \cdots + C_r$ such that $(C_i \cdot C_{i+1}) = (C_r \cdot C_1) = 1$ for $1 \leq i < r$. Then either $[D^{\#}] = C$ is an elliptic curve with $C + K_V \sim 0$ or $[D^{\#}] = C_1 + C_2 + \cdots + C_r$, which is a cycle of rational curves, and $C_1 + C_2 + \cdots + C_r + K_V \sim 0$. Furthermore,*

Supp (Bk (D)) *is disjoint from* $[D^{\#}]$ *and consists of* (-2) *rods or* (-2) *forks.*

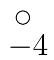
(4) *Suppose that* V *is a rational surface and that* D *is a tree of rational curves. Then* $n(D^{\#}+K_V) \sim 0$ *with* $n > 1$.

In the case (4) of the above theorem, we have the following result.

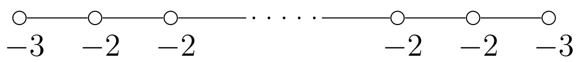
Theorem 4.12 *Let* $(\overline{V}, \overline{D})$ *and* (V, D) *be the same as in Theorem 4.11. Suppose that* V *is a rational surface and the dual graph of* D *is a* connected *tree of rational curves. Then the following assertions hold true.*

(1) *Suppose* $[D^{\#}] = 0$. *Then* $D^{\#} = (1/2)D$ *and* $2(D^{\#}+K_V) \sim 0$. *The divisor* D *has one of the next dual graphs:*

Case 1

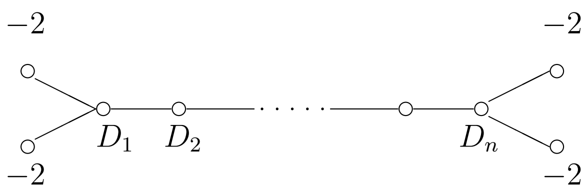


Case 2



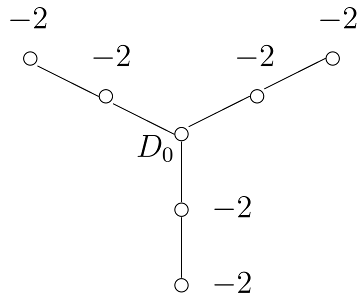
(2) *Suppose* $[D^{\#}] \neq 0$. *Then* $[D^{\#}]$ *is either a linear chain* $D_1 + \cdots + D_n$ *or a single component* D_0. *If* $[D^{\#}]$ *is a linear chain, then* $2(D^{\#}+K_V) \sim 0$ *and the dual graph of* D *is given as below:*

Case 3

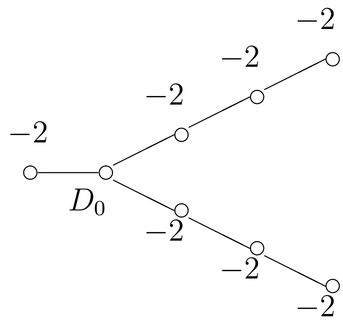


(3) *Suppose that* $[D^{\#}]$ *is a single component* D_0. *Then the divisor* D *has one of the next dual graphs, and* $3(D^{\#}+K_V) \sim 0$ *in Case 4,* $4(D^{\#}+K_V) \sim 0$ *in Case 5 and* $6(D^{\#}+K_V) \sim 0$ *in Case 6.*

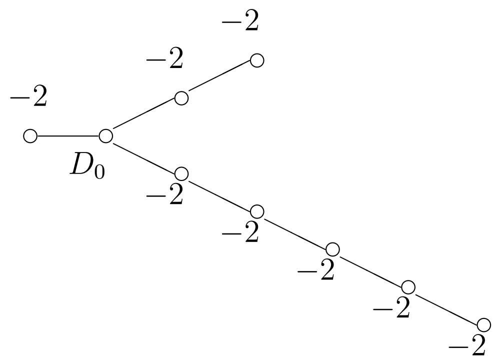
Case 4



Case 5



Case 6



(4) *Let* $X := V - D$. *Suppose* rank Pic $(X) = 0$. *Then* X *is affine and Cases* 1 *and* 2 *do not occur. Furthermore, the irreducible components of* D *are numerically independent in Cases* $4, 5$ *and* 6. *We have* $\sum_{i=1}^{n}({D_i}^2) \geq 6 - 3n$ *in Case* 3, $({D_0}^2) = 1$ *in Case* 4, $({D_0}^2) = 0$ *in Case* 5, *and* $({D_0}^2) = -1$ *in Case* 6.

Finally, we note, in the case of Kodaira dimension 2, the following beautiful result of Miyaoka-Yau type, which was proved by R. Kobayashi but has not yet proved in the algebro-geometric method.

Theorem 4.13 *Let* V *be a nonsingular minimal algebraic surface defined over the complex field* $\mathbf{C}$. *Suppose that* $\kappa(V) = 2$. *Then we have* $({K_V}^2) \leq 3e(V)$, *where* $e(V)$ *is the topological Euler number.*