University of the Philippines Los Baños
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Controls and their limits

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26 August- 6 September 2019
2019 Multi-scale Analysis and Theory of Homogenization (MATH)
International Center for Theoretical Sciences, Bangalore

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"..if every instrument could accomplish its own work, obeying or anticipating the will of others...if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."

Politics by Aristotle

Control Systems

Control Systems

- simple machines such as the bathroom water tank
- heating, ventilation and air conditioning systems with thermo-fluid and electro-mechanical subsystems
- pH control in chemical reactions
- nuclear security
- control of robots
- electrical plants and distribution networks

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- Control theory may be considered as a theoretical support to control engineering which design systems with desired behaviors. It is a rich crossing point of Engineering and Mathematics.
- R. Bellman (dynamic programming); R. Kalman (filtering techniques) and L. Pontrayagin (maximum principle) established the foundations of modern control theory.

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A.K. Nandakumaran (2014) Introduction to Exact Controllability and Observability, Variational Approach and Hilbert Uniqueness Method, *Lecture on the NPDE-TCA Advanced Level Workshop on Partial Differential Equations*, IISER, Trivandrum

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  (c) Look for controls which stabilizes the state or observations (stabilization problem).
- The control can be the right-hand side of the equation (internal control), the data in the boundary condition (boundary control), an initial condition, a coefficient or parameter in the equation.

A mathematical control problem

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$$A(y) = f(v) \tag{1}$$

where y is the state, the unknown which we want to control and v is the control.

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The operator A determines the equation that must be satisfied by the state variable y while the function f indicates the way the control v acts on the system governing the state.

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The main goal of control theory is to find the controls $v \in U_{ad}$ such that the associated states $y = y(v)$, that is, the solutions of the corresponding controlled system, lead to a desired situation.

Controllability vs. Optimal Control

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- (Optimal control) Fix a cost function $J = J(v)$, say,

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 + \frac{\mu}{2}\|v\|_U^2, \ \forall v \in U_{ad}.$$

Then look for a minimizer u of J.

Introductory References

- A.K. Nandakumaran. (2014), Introduction to Exact Controllability and Observability, Variational Approach and Hilbert Uniqueness Method, Lecture on the NPDE-TCA Advanced Level Workshop on Partial Differential Equations, IISER, Trivandrum
- E. Zuazua and E. Fernandez-Cara. (2003), Control Theory: History, Mathematical Achievements and Perspectives, in Bol. SEMA (Sociedad Espanola de Matematica Aplicada) 26, 79-140.
- E. Zuazua.(2002), Controllability of Partial Differential Equations and its Semi-discrete Approximations, Discrete and Continuous Dynamical Systems, 8 (2), 469-513.

Control problems for PDEs

Control problems for PDEs

There are several types of control problems for PDEs depending on their class (elliptic, parabolic or hyperbolic):

- exact controllability (we reach a desired state)
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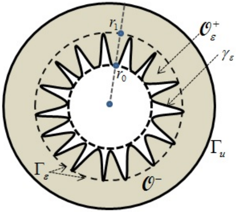
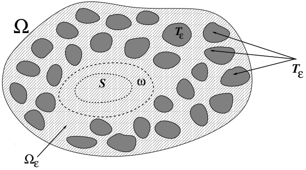
Remark: We want to act locally (on an arbitrary zone and not everywhere).

Related collaborative works

- S. Aiyappan, E. Jose, I. C. Lomerio and A. Nandakumaran. (2019), Control Problem on a Rough Circular Domain and Homogenization, *Asymptotic Analysis*, DOI 10.3233/ASY-191526



- C. Conca, P. Donato, E. Jose and I. Mishra. (2016), Asymptotic analysis of optimal controls of a semilinear problem in a perforated domain, *J. Ramanujan Math. Soc.*, 31 (3), 265-305



The Heat Equation

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Given $T > 0$, the nonhomogeneous heat equation is given by:

$$\begin{cases} u_t - \Delta u = f\chi_\omega & \text{in } \Omega \times (0,T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases}$$

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where $u = u(x,t)$ is the state and $f = f(x,t)$ is the control function with a support localized in ω.

Note: If $f \in L^2(\omega \times (0,T))$ and $u^0 \in L^2(\Omega)$ then the above equation has a unique weak solution $u \in C([0,T], L^2(\Omega))$.

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The variational approach

Following an idea by J.-L. Lions, the approximate control can be constructed as the solutions of a related transposed (backward) problem, having as final data the (unique) minimum point of a suitable functional.

J.-L. Lions, *Remarques sur la controlabilité approchée*. in Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos, octubre 1990, Grupo de Análisis Matemático Aplicado de la University of Malaga, Spain (1991), 77-87.

The approximate controllability result for a model case

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A model approximate controllability problem for the heat equation reads:

Given $w \in L^2(\Omega)$ and $\delta > 0$, find $\varphi \in L^2(\Omega)$ such that for a given $u^0 \in L^2(\Omega)$, the solution u of

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verifies the following approximate controllability:

$$\|u(x,T) - w\|_{L^2(\Omega)} \leq \delta.$$

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In the variational approach of J.-L. Lions, φ (the internal control) is obtained as the solution of the following transposed (adjoint) problem:

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Here, the final data φ^T is the (unique) minimum point of the functional J on $L^2(\Omega)$ given by

$$J(\varphi^T) = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 \, dx \, dt + \delta\|\varphi^T\|_{L^2(\Omega)} - \int_\Omega w\varphi^T \, dx,$$

where φ is the solution of the adjoint problem with final data φ^T.

Control and its Limit: The case of oscillating coefficients

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In this case for every ε, one can construct a control for the problem:

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Suppose now that:

$$\begin{cases} (i) \quad u_\varepsilon^0 \to u^0 & \text{strongly in } L^2(\Omega), \\ (ii) \quad w_\varepsilon \to w & \text{strongly in } L^2(\Omega), \end{cases}$$

for some u^0 and w in $L^2(\Omega)$.

An interesting question is:

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Do the control and the corresponding solution of the ε-problem converge (as $\varepsilon \to 0$) to a control of the homogenized problem and to the corresponding solution, respectively?

A positive answer is given in

E. Zuazua, *Approximate Controllability for Linear Parabolic Equations with Rapidly Oscillating Coefficients*. Control Cybernet, 4 (1994), 793-801.

A heat equation in a composite with interfacial resistances

A heat equation in a composite with interfacial resistances

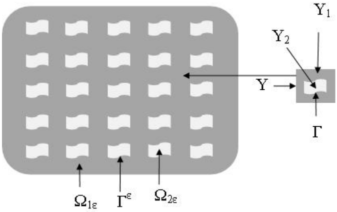
We consider a more complicated case where the domain is a two-component composite.

On the periodic interface, a jump of the solution is prescribed, which is proportional to the conormal derivative via a parameter $\gamma \in \mathbb{R}$, and a Dirichlet condition is imposed on the exterior boundary $\partial\Omega$.

This problem models the heat diffusion in a two-component composite with an imperfect contact on the interface (see below for a physical justification of the model)

H.S.Carslaw, J.C. Jaeger, *Conduction of Heat in Solids*. The Clarendon Press, Oxford, 1947.

The domain



Y is the reference cell where

- $Y = Y_1 \cup \overline{Y_2}$, with $\overline{Y_2} \subset Y$
- $\Gamma := \partial Y_2$ Lipschitz continuous.

The domain in $\mathbb{R}^n$:

$$\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}},$$

where

- $\Omega_{1\varepsilon}$ is a connected union of ε^{-n} periodic translated sets of εY_1,
- $\Omega_{2\varepsilon}$ is a union of ε^{-n} periodic disjoint translated sets of εY_2,
- $\Gamma_\varepsilon := \partial\Omega_{2\varepsilon}$ is the interface between the two components, with $\partial\Omega \cap \Gamma_\varepsilon = \emptyset$.

The $\varepsilon-$problem in the two-component domain

Consider for $\gamma \in \mathbb{R}$ the following parabolic system of equations:

$$\begin{cases} {u_{1\varepsilon}}' - \text{div}(A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon}) = \chi_{\omega_{1\varepsilon}}\varphi_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0,T)\,, \\ {u_{2\varepsilon}}' - \text{div}(A(\frac{x}{\varepsilon})\nabla u_{2\varepsilon}) = \chi_{\omega_{2\varepsilon}}\varphi_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0,T)\,, \\ A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A(\frac{x}{\varepsilon})\nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A(\frac{x}{\varepsilon})\nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h(\frac{x}{\varepsilon})(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i = 1,2)$, ω is a given open non-empty subset of Ω, and we set

$$\omega_{i\varepsilon} = \omega \cap \Omega_{i\varepsilon},\ i = 1,2.$$

Assumptions

- A is an $n \times n$ matrix field which is $Y-$ periodic, symmetric and of class $C^1(\bar{Y})$ such that for some $0 < \alpha < \beta$, one has

$$\begin{cases} (A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(y)\lambda| \leq \beta|\lambda|. \end{cases}$$

$\forall\ \lambda \in \mathbb{R}^n$ and a.e. in Y.

- h is a Y- periodic function in $L^\infty(\Gamma)$ such that $\exists\ \ h_0 \in \mathbb{R}$ with $0 < h_0 < h(y), \quad y$ a.e. in Γ.
- $(U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$,
- $(\varphi_{1\varepsilon}, \varphi_{2\varepsilon}) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$.

Three questions to be answered

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We give here positive answers to all these three questions.

We describe here the case $\gamma = 1$, which is the most interesting case since the homogenized problem is a coupled system of a P.D.E. and an O.D.E., giving rise to a memory effect.

However, our results concern all the value of $\gamma \in \mathbb{R}$.

The related controllability problem (first question)

Given $w_{i\varepsilon} \in L^2(\Omega_{i\varepsilon}), i = 1, 2$ $\delta_1 > 0$ and $\delta_2 > 0$, find a control $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ such that the solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ of the above problem verifies the estimates

$$\begin{cases} (i) \quad \|u_{1\varepsilon}(T) - w_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \leq \delta_1 \\ \\ (ii) \quad \|u_{2\varepsilon}(T) - w_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \leq \delta_2. \end{cases}$$

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Important tools

• The corrector results play an important role in the proofs.

• Unique continuation results are needed for the two problems, in particular for the homogenized coupled problem (F. Ammar Khodja).

Some references

- L. Faella and S. Monsurrò, *Memory effects arising in the homogenization of composites with inclusions*. In: Topics on Mathematics for Smart Systems, 107-121. World Sci. Publ., Hackensack, NJ, 2007.
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- F. Ammar Khodja, personal communication.

The variational approach to the controllability problem

Let $(w_{1\varepsilon}, w_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ and $\varphi^0 \in L^2(\Omega)$ and define the functional J_ε by

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_{\omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 \, dx \, dt \int_0^T \int_{\omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 \, dx \, dt\right) +$$

$$+\delta\left(\sqrt{\theta_1}\|\varphi^0\|_{L^2(\Omega_{1\varepsilon})} + \sqrt{\theta_2}\|\varphi^0\|_{L^2(\Omega_{2\varepsilon})}\right)$$

$$-\int_{\Omega_{1\varepsilon}} (w_{1\varepsilon} - v_{1\varepsilon}(T))\varphi^0 \, dx - \int_{\Omega_{2\varepsilon}} (w_{2\varepsilon} - v_{2\varepsilon}(T))\varphi^0 \, dx,$$

where $\theta_i = \dfrac{|Y_i|}{|Y|}$ for $i = 1, 2$.

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$$J_\varepsilon(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_{\omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 \, dx \, dt \int_0^T \int_{\omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 \, dx \, dt\right) +$$

$$+\delta\left(\sqrt{\theta_1}\|\varphi^0\|_{L^2(\Omega_{1\varepsilon})} + \sqrt{\theta_2}\|\varphi^0\|_{L^2(\Omega_{2\varepsilon})}\right)$$

$$-\int_{\Omega_{1\varepsilon}} (w_{1\varepsilon} - v_{1\varepsilon}(T))\varphi^0 \, dx - \int_{\Omega_{2\varepsilon}} (w_{2\varepsilon} - v_{2\varepsilon}(T))\varphi^0 \, dx,$$

where $\theta_i = \dfrac{|Y_i|}{|Y|}$ for $i = 1, 2$.

⋆ Observe that $\chi_{\Omega_{i\varepsilon}} \rightharpoonup \theta_i$ in L^∞ only weakly*. Then, one difficulty in this study is that the convergence of a function v_ε to some v do not imply the convergence of $\chi_{\Omega_{i\varepsilon}} v_\varepsilon$ to $\theta_i v$.

The variational approach to the controllability problem

In the defined functional, $\varphi_\varepsilon = (\varphi_{1\varepsilon}, \varphi_{2\varepsilon})$ is the solution of the transposed problem of the system given by

$$\begin{cases} -{\varphi_{1\varepsilon}}' - \mathsf{div}(A^\varepsilon \nabla \varphi_{1\varepsilon}) = 0 & \text{in } \Omega_{1\varepsilon} \times (0,T)\,, \\ -{\varphi_{2\varepsilon}}' - \mathsf{div}(A^\varepsilon \nabla \varphi_{2\varepsilon}) = 0 & \text{in } \Omega_{2\varepsilon} \times (0,T)\,, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \varphi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A^\varepsilon \nabla \varphi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h_\varepsilon (\varphi_{1\varepsilon} - \varphi_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ \varphi_{1\varepsilon}(x,T) = \varphi^0|_{\Omega_{1\varepsilon}} & \text{in } \Omega_{1\varepsilon}, \\ \varphi_{2\varepsilon}(x,T) = \varphi^0|_{\Omega_{2\varepsilon}} & \text{in } \Omega_{2\varepsilon}. \end{cases}$$

On the other hand, $v_\varepsilon = (v_{1\varepsilon}, v_{2\varepsilon})$ is the solution of the auxiliary problem

$$\begin{cases} {v_{1\varepsilon}}' - \mathrm{div}(A^\varepsilon \nabla v_{1\varepsilon}) = 0 & \text{in } \Omega_{1\varepsilon} \times (0,T)\,, \\ {v_{2\varepsilon}}' - \mathrm{div}(A^\varepsilon \nabla v_{2\varepsilon}) = 0 & \text{in } \Omega_{2\varepsilon} \times (0,T)\,, \\ A^\varepsilon \nabla v_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla v_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A^\varepsilon \nabla v_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h_\varepsilon (v_{1\varepsilon} - v_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ v_{1\varepsilon}(x,0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ v_{2\varepsilon}(x,0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases}$$

where n_i is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i = 1, 2)$ and $(U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$.

The controllability result for fixed ε

Theorem 1 [Donato-E.J. (2015)] Let $T, \delta_1, \delta_2 > 0$ be given real numbers and U_ε^0 be in $L^2(\Omega)$. Fix $w_\varepsilon = (w_{1\varepsilon}, w_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$.

Let $\widehat{\varphi}_\varepsilon^0$ be the unique minimum point of the functional J_ε and $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ the solution of the transposed problem with final data $\widehat{\varphi}_\varepsilon^0$. Then the solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ of the following system:

$$\begin{cases} {u_{1\varepsilon}}' - \text{div}(A^\varepsilon \nabla u_{1\varepsilon}) = \chi_{\omega_{1\varepsilon}} \widehat{\varphi}_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0,T)\,, \\ {u_{2\varepsilon}}' - \text{div}(A^\varepsilon \nabla u_{2\varepsilon}) = \chi_{\omega_{2\varepsilon}} \widehat{\varphi}_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0,T)\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_2 & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_1 = -\varepsilon^\gamma h_\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_{1\varepsilon}(x,0) = U_{1\varepsilon}^0 & \text{in } \Omega_{1\varepsilon}\,, \\ u_{2\varepsilon}(x,0) = U_{2\varepsilon}^0 & \text{in } \Omega_{2\varepsilon}\,, \end{cases}$$

satisfies the following estimate:

$$\begin{cases} (i)\ \|u_{1\varepsilon}(T) - w_\varepsilon\|_{L^2(\Omega_{1\varepsilon})} \leq \delta_1 \\ (ii)\ \|u_{2\varepsilon}(T) - w_\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \leq \delta_2. \end{cases}$$

A review of the homogenization and correctors results

⋆ From now on, we consider $\gamma = 1$, the most interesting case.

⋆ From now on, we consider $\gamma = 1$, the most interesting case.

The homogenization result [E.J. (2009)]

Let A^ε and h^ε be as before and $z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon})$ be the solution of

$$\begin{cases} {z_{i\varepsilon}}' - \text{div}(A^\varepsilon \nabla z_{i\varepsilon}) = g_{i\varepsilon} & \text{in } \Omega_{i\varepsilon} \times (0,T)\,,\ i = 1,2, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla z_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ A^\varepsilon \nabla z_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon\, h^\varepsilon (z_{1\varepsilon} - z_{2\varepsilon}) & \text{on } \Gamma_\varepsilon \times (0,T)\,, \\ z_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ z_{i\varepsilon}(x,0) = Z^0_\varepsilon|_{\Omega_{i\varepsilon}} & \text{in } \Omega_{i\varepsilon},\ i = 1,2, \end{cases}$$

where $Z^0_\varepsilon \in L^2(\Omega)$ and $(g_{1\varepsilon}, g_{2\varepsilon}) \in [L^2(0,T;L^2(\Omega))]^2$. Suppose that

$$\begin{cases} (i) \quad (\chi_{\Omega_{1\varepsilon}} Z^0_\varepsilon, \chi_{\Omega_{2\varepsilon}} Z^0_\varepsilon) \rightharpoonup (\theta_1 Z^0_1, \theta_2 Z^0_2) & \text{weakly in } [L^2(\Omega)]^2, \\ (ii) \quad (\chi_{\Omega_{1\varepsilon}} g_{1\varepsilon}, \chi_{\Omega_{1\varepsilon}} g_{2\varepsilon}) \rightharpoonup (\theta_1 g_1, \theta_2 g_2) & \text{weakly in } [L^2(0,T;L^2(\Omega))]^2, \end{cases}$$

Then

$$\begin{cases} (i) \quad \widetilde{z_{1\varepsilon}} \rightharpoonup \theta_1 z_1 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (ii) \quad \widetilde{z_{2\varepsilon}} \rightharpoonup z_2 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (iii) \quad \varepsilon^{\frac{1}{2}} \|z_{1\varepsilon} - z_{2\varepsilon}\|_{L^2(0,T;L^2(\Gamma_\varepsilon))} < c, \end{cases}$$

where $\sim$ denotes the zero extension to the whole of Ω.
Furthermore,

$$\begin{cases} (i) \quad A^\varepsilon \widetilde{\nabla z_{1\varepsilon}} \rightharpoonup A^0 \nabla z_1 & \text{weakly in } L^2(0,T;[L^2(\Omega)]^n), \\ (ii) \quad A^\varepsilon \widetilde{\nabla z_{2\varepsilon}} \rightharpoonup 0 & \text{weakly `in } L^2(0,T;[L^2(\Omega)]^n), \end{cases}$$

where $A^0\lambda := m_Y(A\widetilde{\widehat{w}_\lambda})$, the function $\widehat{w}_\lambda \in H^1(Y_1)$ being for any $\lambda \in \mathbb{R}^n$, the unique solution of the problem

$$\begin{cases} -\text{div}\,(A\nabla \widehat{w}_\lambda) = 0 & \text{in } Y_1, \\ (A\nabla \widehat{w}_\lambda) \cdot n_1 = 0 & \text{in } \Gamma, \\ \widehat{w}_\lambda - \lambda \cdot y \;\; Y\text{-periodic and } \; m_{Y_1}(\widehat{w}_\lambda - \lambda \cdot y) = 0. \end{cases}$$

The pair
$(z_1, z_2) \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \times C^0([0,T]; L^2(\Omega))$
with $z_1' \in L^2(0,T; \mathsf{H}^{-1}(\Omega))$ is the unique solution of the homogenized coupled system

$$\begin{cases} \theta_1 z_1' - \text{ div } (A^0 \nabla z_1) + c_h(\theta_2 z_1 - z_2) = \theta_1 g_1 & \text{in } \Omega \times (0,T)\,, \\ z_2' - c_h(\theta_2 z_1 - z_2) = \theta_2 g_2 & \text{in } \Omega \times (0,T)\,, \\ z_1 = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ z_1(0) = Z_1^0,\ z_2(0) = \theta_2 Z_2^0 & \text{in } \Omega, \end{cases}$$

where $c_h = \dfrac{1}{|Y_2|} \displaystyle\int_\Gamma h(y)\, \mathrm{d}\sigma_y$.

The corrector result [Donato-E.J. (2010)]

Under the assumption of the homogenization theorem, suppose further that

for $Z_\varepsilon^0 \in L^2(\Omega)$ and $g_{i\varepsilon} \in L^2(0,T;L^2(\Omega)),\ i = 1,2$ one has

$$(\chi_{\Omega_{1\varepsilon}} Z_\varepsilon^0, \chi_{\Omega_{2\varepsilon}} Z_\varepsilon^0) \rightharpoonup (\theta_1 Z_1^0, \theta_2 Z_2^0) \quad \text{weakly in } [L^2(\Omega)]^2$$

and

$$\begin{cases} (i) \quad g_{i\varepsilon} \to g_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \\ (ii) \quad \|Z_\varepsilon^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|Z_\varepsilon^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|Z_1^0\|^2_{L^2(\Omega)} + \theta_2 \|Z_2^0\|^2_{L^2(\Omega)}. \end{cases}$$

If $(e_j)_{j=1,\dots,n}$ is the canonical basis of $\mathbb{R}^n$ and $\widehat{w}_j$ is the solution of the cell problem written for $\lambda = e_j$, $j = 1, ..., n$, let

$$C^\varepsilon = (C_{ij}^\varepsilon)_{1 \le i,j \le n}$$

be the corrector matrix defined, for $i, j = 1, ..., n$, by

$$C_{ij}(y) := \frac{\partial \widehat{w}_j}{\partial y_i}(y), \quad \text{a.e. on } Y_1, \qquad C_{ij}^\varepsilon(x) = \widetilde{C_{ij}}\left(\frac{x}{\varepsilon}\right) \text{a.e. on } \Omega.$$

Assuming that Γ is of class $\mathcal{C}^2$, the following corrector results hold true:

$$\begin{cases} (i) \quad \lim_{\varepsilon \to 0} \|z_{1\varepsilon} - z_1\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (ii) \quad \lim_{\varepsilon \to 0} \|z_{2\varepsilon} - \theta_2^{-1} z_2\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (iii) \quad \lim_{\varepsilon \to 0} \|\nabla z_{1\varepsilon} - C^{\varepsilon} \nabla z_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iv) \quad \lim_{\varepsilon \to 0} \|\nabla z_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0. \end{cases}$$

Assuming that Γ is of class $\mathcal{C}^2$, the following corrector results hold true:

$$\begin{cases} (i) \quad \lim_{\varepsilon\to 0} \|z_{1\varepsilon} - z_1\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (ii) \quad \lim_{\varepsilon\to 0} \|z_{2\varepsilon} - \theta_2^{-1} z_2\|_{C^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (iii) \quad \lim_{\varepsilon\to 0} \|\nabla z_{1\varepsilon} - C^\varepsilon \nabla z_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iv) \quad \lim_{\varepsilon\to 0} \|\nabla z_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0. \end{cases}$$

Remark: In particular, assumption (i) holds if for $i = 1, 2$, $g_{i\varepsilon} = g_\varepsilon|_{\Omega_{i\varepsilon}}$ and

$$g_\varepsilon \to g \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

On the other hand, the assumptions on the initial data hold if for $i = 1, 2$, one has for instance $\chi_{\Omega_{i\varepsilon}} Z_\varepsilon^0 = Z_{i\varepsilon}^0|_{\Omega_{i\varepsilon}}$ for some $Z_{i\varepsilon}^0 \in L^2(\Omega)$ such that $Z_{i\varepsilon}^0 \to Z_i^0$ strongly in $L^2(\Omega)$ (as will be applied in the control problem).

Construction of the control for the homogenized problem

Let $T, \delta_1, \delta_2 > 0$ be given and w,U_1^0, U_2^0 be in $L^2(\Omega)$.

For a given $w \in L^2(\Omega)$, we define the functional J_0 on $L^2(\Omega) \times L^2(\Omega)$ by

$$J_0(\Phi^0, \Psi^0) = \frac{1}{2}\theta_1 \int_0^T \int_\omega |\varphi_1|^2 \, dx \, dt + \frac{1}{2}\theta_2^{-1} \int_0^T \int_\omega |\varphi_2|^2 \, dx \, dt$$

$$+\delta_1 \sqrt{\theta_1} \|\Phi^0\|_{L^2(\Omega)} + \delta_2 \sqrt{\theta_2} \|\Psi^0\|_{L^2(\Omega)}$$

$$- \theta_1 \int_\Omega (w - v_1(T)) \, \Phi^0 \, dx - \theta_2 \int_\Omega \left(w - \theta_2^{-1} v_2(T)\right) \Psi^0 \, dx,$$

where (φ_1, φ_2) is the solution of the following homogeneous transposed problem:

$$\begin{cases} -\theta_1 {\varphi_1}' - \mathrm{div}(A^0 \nabla \varphi_1) + c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times (0,T)\,, \\ -{\varphi_2}' - c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times (0,T)\,, \\ \varphi_1 = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ \varphi_1(x,T) = \Phi^0, \quad \varphi_2(x,T) = \theta_2 \Psi^0 & \text{in } \Omega. \end{cases} \tag{2}$$

and (v_1, v_2) is the solution of the problem

$$\begin{cases} \theta_1 {v_1}' - \mathrm{div}(A^0 \nabla v_1) + c_h(\theta_2 v_1 - v_2) = 0 & \text{in } \Omega \times (0,T)\,, \\ {v_2}' - c_h(\theta_2 v_1 - v_2) = 0 & \text{in } \Omega \times (0,T)\,, \\ v_1 = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ v_1(x,0) = U_1^0, \quad v_2(x,0) = \theta_2 U_2^0 & \text{in } \Omega. \end{cases}$$

The controllability result for the homogenized system

Theorem 2 [Donato-E.J.] (2015) Let $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ be the unique minimum point of the functional J_0 and $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ the solution of (2) with final data $(\widehat{\Phi}^0, \theta_2\widehat{\Psi}^0)$.

Then if (u_1, u_2) is the solution of

$$\begin{cases} \theta_1 {u_1}' - \text{div}(A^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \chi_\omega \theta_1 \widehat{\varphi}_1 & \text{in } \Omega \times (0,T)\,, \\ {u_2}' - c_h(\theta_2 u_1 - u_2) = \chi_\omega \widehat{\varphi}_2 & \text{in } \Omega \times (0,T)\,, \\ u_1 = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_1(x,0) = U_1^0, \quad u_2(x,0) = \theta_2 U_2^0 & \text{in } \Omega, \end{cases}$$

we have the following approximate controllability:

$$\|\theta_1 u_1(x,T) + u_2(x,T) - w\|_{L^2(\Omega)} \leq \delta_1 \sqrt{\theta_1} + \delta_2 \sqrt{\theta_2}.$$

Asymptotic behaviour of the control problem

Theorem 3 [Donato-E.J. (2015)] Suppose that $T, \delta_1, \delta_2 > 0$ and that Γ is of class $\mathcal{C}^2$. Let w_ε and U_ε^0 be given in $L^2(\Omega)$.

Let $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ and $\widehat{\varphi}_\varepsilon = (\widehat{\varphi}_{1\varepsilon}, \widehat{\varphi}_{2\varepsilon})$ the related solution and the approximate control given by Theorem 1, respectively.

For $\{w_\varepsilon\}_\varepsilon \subset L^2(\Omega)$ and $\{U_\varepsilon^0\}_\varepsilon \subset L^2(\Omega)$, we suppose that for some $U_i^0, i = 1, 2$ and w in $L^2(\Omega)$, they satisfy the following assumptions:

$$\begin{cases} (i) \quad \chi_{\Omega_{i\varepsilon}} U_\varepsilon^0 \rightharpoonup \theta_i U_i^0 \quad \text{weakly in } L^2(\Omega), \\ (ii) \quad \|U_\varepsilon^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|U_\varepsilon^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|U_1^0\|^2_{L^2(\Omega)} + \theta_2 \|U_2^0\|^2_{L^2(\Omega)}, \\ (iii) \quad w_\varepsilon \to w \quad \text{strongly in } L^2(\Omega). \end{cases}$$

(Recall that in particular we can suppose that $\chi_{\Omega_{i\varepsilon}} U_\varepsilon^0 = U_{i\varepsilon}^0|_{\Omega_{i\varepsilon}}$ with $U_{i\varepsilon}^0 \to U_i^0$ strongly in $L^2(\Omega)$.)

Then as $\varepsilon \to 0$, one has

$$\begin{cases} (i) \quad \chi_{\omega_{1\varepsilon}} \widetilde{\widehat{\varphi}_{1\varepsilon}} \rightharpoonup \chi_{\omega} \theta_1 \widehat{\varphi}_1 & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ (ii) \quad \chi_{\omega_{2\varepsilon}} \widetilde{\widehat{\varphi}_{2\varepsilon}} \rightharpoonup \chi_{\omega} \widehat{\varphi}_2 & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ (iii) \quad (\chi_{\Omega_{1\varepsilon}} \widehat{\varphi}^0_{\varepsilon}, \chi_{\Omega_{2\varepsilon}} \widehat{\varphi}^0_{\varepsilon}) \rightharpoonup (\theta_1 \widehat{\Phi}^0, \theta_2 \widehat{\Psi}^0) & \text{weakly in } [L^2(\Omega)]^2, \end{cases}$$

where $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ is the solution of the trasposed problem with final data $(\widehat{\Phi}^0, \theta_2 \widehat{\Psi}^0)$.

Here, $(\widehat{\Phi}^0, \widehat{\Psi}^0)$ is the unique minimum point of the functional J_0.

Moreover,

$$\begin{cases} (i) \quad \widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ (ii) \quad \widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \end{cases}$$

where the couple (u_1, u_2) satisfies

$$\begin{cases} \theta_1 {u_1}' - \text{div}(A^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 \chi_\omega \widehat{\varphi}_1 & \text{in } \Omega \times (0,T)\,, \\ {u_2}' - c_h(\theta_2 u_1 - u_2) = \chi_\omega \widehat{\varphi}_2 & \text{in } \Omega \times (0,T)\,, \\ u_1 = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_1(x,0) = U_1^0, \quad u_2(x,0) = \theta_2 U_2^0 & \text{in } \Omega. \end{cases} \tag{3}$$

The couple $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ is an approximate control for the homogenized problem (3) corresponding to w and the constants δ_1 and δ_2, that is

$$\|\theta_1 u_1(x,T) + u_2(x,T) - w\|_{L^2(\Omega)} \leq \delta_1 \sqrt{\theta_1} + \delta_2 \sqrt{\theta_2}.$$

Thank you for your attention!