Lecture 3: The periodic unfolding method in perforated domains

Patrizia Donato
University of Rouen Normandie

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The aim of the periodic unfolding

Original aim: to give an elementary proof for classical periodic homogenization problems, in particular for cases with several micro-scales.

Actually, one has more ... Indeed,

- it can be adapted to many different situations
- gives corrector results under minimal regularity assumptions
- particularly well adapted for perforated domains

The method is based on two ingredients:

- The unfolding operator
- A decomposition of functions separating the scales.

The origins of the unfolding method

- Two-scale convergence (1989)
  - ⋆ by G. Nguetseng, G. Allaire,
  - ⋆ see also D. Lukkassen, G. Nguetseng and P. Wall
  - ⋆ and many other authors ...

  Generalization to multi-scale problems
  - ⋆ A. I. Ene and J. Saint Jean Paulin,
  - ⋆ G. Allaire and M. Briane,
  - ⋆ J. L. Lions, D. Lukkassen, L. Persson
  - ⋆ P. Wall, D. Lukkassen, G. Nguetseng and P. Wall.
- Dilation transformation in L^p (1990)
  - ⋆ T. Arbogast, J. Douglas and U. Hornung (1990)
  - ⋆ A. Bourgeat, S. Luckhaus and A. Mikelic (1996)
  - ⋆ G. Allaire and C. Conca (Bloch waves, 1998)

Some references

The periodic unfolding method

Doina Cioranescu, A. Damlamian and G. Griso (2002) developped the idea of a dilation operator and combined it for functions in $W^{1,p}(\Omega)$ with ideas of the Finite Element Method used in Georges Griso's thesis (1991).

We refer to

- C. R. Acad. Sci. Paris, Série 1, 335 (2002), 99–104.
- The periodic unfolding method in homogenization, SIAM J. of Math. Anal. Vol. 40, 4 (2008), 1585–1620.

The periodic unfolding method for perforated domains

- D. Cioranescu, P. Donato and R. Zaki, The periodic unfolding method in perforated domains, Portugaliae Mathematica, 63, 4 (2006), 467–496.

- D. Cioranescu, P. Donato and R. Zaki, Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions, Asymptotic Analysis, 53, 4 (2007), 209–235.

For a general presentation see

- D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, he periodic unfolding method in domains with holes, SIAM J. of Math. Anal., 44 (2) (2012), 718-760.

- see also
    - M. Ghergu, G. Griso, H. Mechkour, B. Miara, D. Onofrei,
    - D. Cioranescu, A. Damlamian, G. Griso and D. Onofrei (Dirichlet small holes, Neuman sieve),
    - D. Cioranescu and Amar Ould-Hammouda (small holes Neumann+ small holes Dirichlet).
    - D. Cioranescu, A. Damlamian, G. Griso, R. Zaki (Stokes)

and

- F. Gaveau, P.D. - Zhanying Yang (unfolding for evolution problems),
- B. Cabarrubias - P. D. (quasilinear elliptic, evolution Dirichlet small holes),
- P.D-K.H. Le Nguyen (unfolding for problems with jump), Zhanying Yang (evolution with jump).
- R. Bunoiou - P.D. (Dirichlet for doubly periodic holes)
- P.D - O. Guibé - A. Oropeza (renormalised solutions)
- P.D. -S. Monsurrò - F. Raimondi (singular problems)
- The german school: Radu-Neuss, Peter, et al (singular problems)

For oscillating boundaries

- A. Damlamian K. Petterson, Nandakumaran et al, J. Arrieta- M. Villanueva-Pesqueira, O. Guibé - A. Gaudiello

The method in perforated domains

Actually, here it is even better ... Indeed,

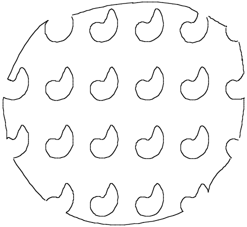
- The unfolding method is particularly well adapted for perforated domains defined as follows:

\- Ω is a fixed domain in $\mathbb{R}^n$,
\- T is a closed reference hole , $T \subset Y$.
Then the perforated domain Ω^*_ε, is obtained by removing from Ω all the ε-periodic translates of εT.

- It is the "fixed domain" character of the unfolding method that makes it so simple for perforated domains. Indeed, by the unfolding operator, functions defined on Ω^*_ε (strongly depending on ε) are mapped into functions defined on the fixed domain $\Omega \times Y^*$ where $Y \setminus T$.

Consequences

- one can investigate the convergence properties of the unfoldings of functions defined on Ω_ε^*,
- the method avoids the use of extension operators in the holes (so no need to require the regularity of ∂T proving the existence of extensions),
- it also avoids the hypothesis that the perforated domains have no holes intersecting $\partial\Omega$.



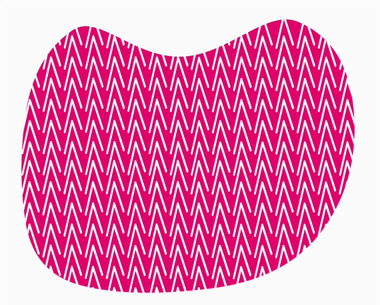
☛ Some advantages of the method are

- more cases can be treated
- proof really simpler
- one can put together several kind of holes with deffernt boundary condition (impossible using test fuctions)
- some assumptions on correctors can be weaker
- nice for some linear problems

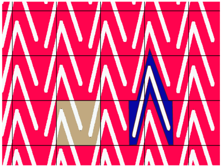
Remarks

- The unfolding operator is defined in the case
  - when the unit hole is a compact subset of the open unit cell
  - also when this is impossible to achieve (this can occur in particular in dimensions larger than 2).
- When S is not compact in Y, an extra condition in terms of a Poincaré-Wirtinger inequality is required for the union of the unit cell and its translates by a period.
- One can consider also the situation where no choice of the basis of periods gives a parallelotop Y such that $Y^* = Y \setminus T$ is connected (condition necessary for the validity of the Poincaré-Wirtinger inequality). The method applies if there exists a reference cell Y having the paving property and such that Y^* is connected.

Exemple



No choice of parallelotop gives a connected Y^*, while there are many possible Y's that give a connected Y^*, an example being the following one



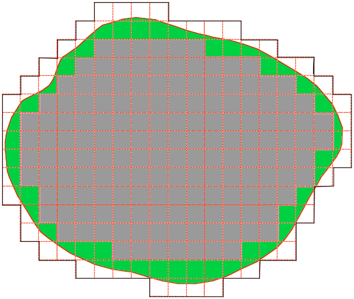
The periodic unfolding method in perforated domains

Let now $Y = \prod_{i=1}^{N} \ell_i$ be a reference cell, Ω an open subset of $\mathbb{R}^N$ and

$$\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N, \varepsilon(\xi + Y) \subset \Omega\}.$$

We introduce the sets

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big) \Big\}, \qquad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$



The sets $\widehat{\Omega}_\varepsilon$ (in grey) and Λ_ε (in green)

By construction,

- $\widehat{\Omega}_\varepsilon$ is the largest union of $\varepsilon(\xi + \overline{Y})$ cells ($\xi \in \mathbb{Z}^n$) included in Ω,
- Λ_ε is the union of the cells intersecting the boundary $\partial\Omega$, so that for every bounded open set $\omega \subset \mathbb{R}^n$, the Lebesgue measure $|\Lambda_\varepsilon \cap \omega| \to 0$.

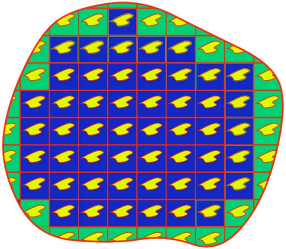
Let now T be a strict closed subset of Y, $Y^* = Y \setminus T$ (the part occupied by the material). Let $\tau(\varepsilon\overline{T})$ be the set of all translates of $\varepsilon\overline{T}$ of the form $\varepsilon(\ell k + \overline{T}),\ k \in Z^n, k\ell = (k_1\ell_1, ..., k_n\ell_n)$.

The perforated domain Ω_ε^* is obtained by removing from Ω the set of holes $S_\varepsilon = \Omega \cap \tau(\varepsilon\overline{T})$

$$\Omega_\varepsilon^* = \Omega \setminus T_\varepsilon.$$

Introduce now the sets

$$\widehat{\Omega}^*_\varepsilon = \widehat{\Omega}_\varepsilon \setminus T_\varepsilon, \qquad \Lambda^*_\varepsilon = \Omega^*_\varepsilon \setminus \widehat{\Omega}^*_\varepsilon, \qquad \Gamma^\varepsilon_1 = \partial\widehat{\Omega^*_\varepsilon} \cap \partial T_\varepsilon,$$



The sets Ω^*_ε, $\widehat{\Omega}^*_\varepsilon$ (in dark blue) and Λ^*_ε (in light green)

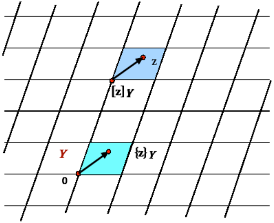
The boundary of the set of holes in Ω is $\partial T_\varepsilon \cap \Omega$ while Γ^ε_1 denotes the boundary of the holes that are included in $\widehat{\Omega}_\varepsilon$.

The first ingredient: the unfolding operator $\mathcal{T}_\varepsilon^*$.

Let $Y = \prod_{i=1}^{N} \ell_i$ be a reference cell Ω an open subset of $\mathbb{R}^N$.

By analogy with the 1-D case, for $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^{N} k_j b_j$ of the periods such that $z - [z]_Y \in Y$, and set

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for} \quad z \in \mathbb{R}^N.$$



Then for each $x \in \mathbb{R}^n$, one has

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. for} \quad x \in \mathbb{R}^n.$$

Definition *For any function ϕ Lebesgue-measurable on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined by*

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \textit{a.e. for} \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \textit{a.e. for} \quad (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Remark

It reduces to $\mathcal{T}_\varepsilon$, i.e. to that of the case without holes if $S = \emptyset$.
The relationship between $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$ is given for w defined on Ω_ε^*, by

$$\mathcal{T}_\varepsilon^*(w) = \mathcal{T}_\varepsilon(\widetilde{w})_{|_{\Omega\times Y^*}}.$$

For w defined on Ω, one has

$$\mathcal{T}_\varepsilon^*(w_{|_{\Omega_\varepsilon^*}}) = \mathcal{T}_\varepsilon(w)_{|_{\Omega\times Y^*}}.$$

Remarks

- It maps a function v in $L^p(\Omega)$ into a function $\mathcal{T}_\varepsilon^*(v)$ in the fixed space $L^p(\Omega \times Y^*)$.

- As in classical periodic homogenization, two different scales appear in the definition of $\mathcal{T}_\varepsilon^*$:

  - the "macroscopic" scale x gives the position of a point in the domain Ω,
  - the "microscopic" scale $y \ (= x/\varepsilon)$ gives the position of a point in the perforated cell Y^*.

  The unfolding operator doubles the dimension of the space and put all the oscillations in the second variable, in this way separating the two scales.

Some properties of $\mathcal{T}_\varepsilon^*$

- For every ϕ in $L^1(\Omega_\varepsilon^*)$ and w in $L^p(\Omega_\varepsilon^*)(p \in [1,+\infty[)$,

$$\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\,dx\,dy = \int_{\widehat{\Omega}_\varepsilon^*} \phi(x)\,dx = \int_{\Omega_\varepsilon^*} \phi(x)\,dx - \int_{\Lambda_\varepsilon^*} \phi(x)\,dx,$$

$$\|\mathcal{T}_\varepsilon^*(w)\|_{L^p(\Omega\times Y^*)} = |\ Y\ |^{1/p}\ \|w\,1_{\widehat{\Omega}_\varepsilon^*}\|_{L^p(\Omega_\varepsilon^*)} \le |\ Y\ |^{1/p}\ \|w\|_{L^p(\Omega_\varepsilon^*)},$$

- Let $||\phi_\varepsilon||_{L^2(\Omega_\varepsilon^*)}$ be bounded. Then from the Hölder inequality,

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon\,dx \ \sim\ \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)\,dxdy.$$

- If f is $Y-$periodic and $f_\varepsilon(x) = f\left(\dfrac{x}{\varepsilon}\right)$ a.e. for $x \in \mathbb{R}^n$, then

$$\mathcal{T}^*_\varepsilon(f_\varepsilon)(x,y) = \begin{cases} f(y) & \text{a.e. for } (x,y) \in \widehat{\Omega^*_\varepsilon} \times Y^*, \\ 0 & \text{a.e. for } (x,y) \in \Lambda^*_\varepsilon \times Y^*. \end{cases}$$

Moreover

$$\mathcal{T}_\varepsilon(f_\varepsilon|_{\Omega^*_\varepsilon}) \to f \quad \text{strongly in } L^p(\Omega \times Y^*).$$

- It is easily seen that the two-scale convergence of $\{\widetilde{v_\varepsilon}\}$ where $\{v_\varepsilon\}$ is a sequence of functions in $L^p(\Omega^*_\varepsilon)$ with bounded norms, is equivalent to the weak convergence of the sequence of unfolded functions $\{\mathcal{T}^*_\varepsilon(v_\varepsilon)\}$ in $L^p(\Omega \times Y^*)$.

This last convergence is easy to check due to density properties.

The second ingredient: the macro-micro decomposition.

It consists of separating the characteristic scales by decomposing every function φ belonging to $W^{1,p}(\Omega_\varepsilon^*)$ in two parts:

- a macroscopic part $Q_\varepsilon^*(\varphi)$, defined on Ω and designed not to capture the oscillations of order ε (if there are any),

- a microscopic part $R_\varepsilon^*(\varphi)$, is designed to do so.

Thanks to the decomposition,

$$w^\varepsilon = Q_\varepsilon^*(w^\varepsilon) + R_\varepsilon^*(w^\varepsilon)$$

one can prove the main results of the method.

The problem

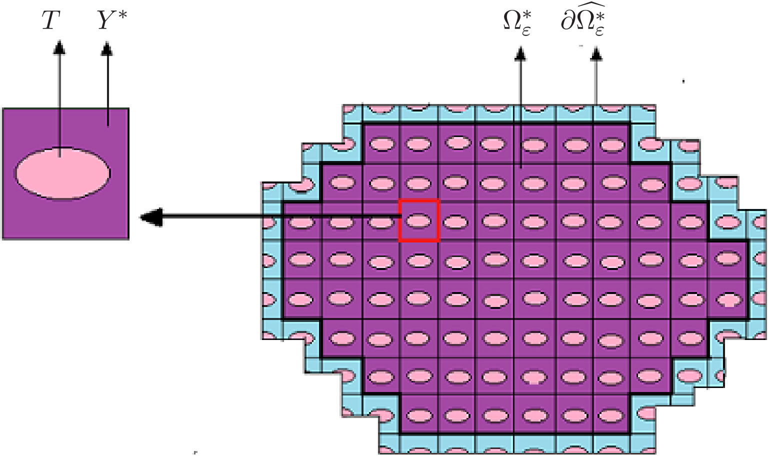
We decompose (but more general situation can be studies) the boundary of the perforated domain Ω_ε^* as follows

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon, \quad \text{where } \Gamma_1^\varepsilon = \partial\widehat{\Omega_\varepsilon^*} \cap \partial T_\varepsilon \quad \text{and} \quad \Gamma_0^\varepsilon = \partial\Omega_\varepsilon^* \setminus \Gamma_1^\varepsilon.$$

and we consider the problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \\ + \text{ some boundary conditions on } \Gamma_1^\varepsilon, \end{cases}$$

where $f \in L^2(\Omega)$, $A^\varepsilon(x) = (a_{ij}(\frac{x}{\varepsilon}))_{1 \leq i,j \leq N}$ a.e. on Ω and $A(y)$ is a Y-periodic matrix field in $M(\alpha, \beta, Y)$ (more general are possible).


T
Y^*
Ω_ε^*
$\partial\widehat{\Omega_\varepsilon^*}$

Neumann conditions

Problem 1: Homogeneous Neumann conditions on the holes

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega^*_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma^\varepsilon_0, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = 0, \quad \text{on } \Gamma^\varepsilon_1, \end{cases}$$

Its variational formulation is: Find $u_\varepsilon \in V^\varepsilon$ such that

$$\int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \; dx = \int_{\Omega^*_\varepsilon} f \; v \; dx \quad \forall v \in V^\varepsilon,$$

where here

$$V^\varepsilon \doteq \{v \in \Omega_\varepsilon |\; v = 0 \text{ on } \Gamma^\varepsilon_0\},$$

equipped with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{H^1(\Omega^*_\varepsilon)}.$$

Theorem (adapted to this geometry)

Suppose that w_ε *belongs to* V^ε *for every* ε *and satisfies*

$$\|\nabla w_\varepsilon\|_{V^\varepsilon} \leq C,$$

where C *is independent of* ε*.*

Then, there exist a subsequence, w *in* $H_0^1(\Omega)$ *and* $\widehat{w}$ *in* $L^2(\Omega; H^1_{per}(Y^*))$*, such that*

$$\begin{aligned} &\mathcal{T}^*_\varepsilon(w_\varepsilon) \to w \quad \textit{strongly in } L^2(\Omega; H^1(Y^*)),\\ &\mathcal{T}^*_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \textit{weakly in } L^2(\Omega \times Y^*),\\ &\|w_\varepsilon - w\|_{L^2(\Omega^*_\varepsilon)} \to 0. \end{aligned}$$

The homogenization results for Neumann conditions

Theorem Let u_ε the solution of Problem 1. Then,

$$\mathcal{T}^*_\varepsilon(u_\varepsilon) \to u \quad \text{strongly in } L^2(\Omega; H^1(Y^*)),$$

$$\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*),$$

$$\|u_\varepsilon - u\|_{L^2(\Omega^*_\varepsilon)} \to 0.$$

where the pair $(u_0, \widehat{u})$ is the unique solution in

$$H_0^1(\Omega) \cap L^2(\Omega; H^1_{\text{per}}(Y^*)) \text{ with } \mathcal{M}_{Y^*}(\widehat{u}) = 0$$

of the limit equation

$$\begin{cases} \forall\, \phi \in H_0^1(\Omega),\ \forall\, \Psi \in L^2(\Omega; H^1_{\text{per}}(Y^*)) \\ \displaystyle\int_{\Omega \times Y^*} A(y)\,(\nabla u_0 + \nabla_y \widehat{u})\,(\nabla\phi(x) + \nabla_y \Psi(x,y))\ dx\, dy \\ = |Y| \displaystyle\int_\Omega f\phi\, dx. \end{cases}$$

By a standard computation

$$\widehat{u}(x) = -\sum_{i=1}^{n} \widehat{\chi}_i(y) \frac{\partial u_0}{\partial x_i}(x)$$

where $\widehat{\chi}_i$ are the already defined periodic functions.

Then,

Corollary The fonction u_0 is the unique solution of

$$\begin{cases} -\text{div } (A^0 \nabla u_0) = \theta f \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Robin conditions

Problem 2: Quasilinear problem with nonhomogeneous Robin condition

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot n + \varepsilon^\gamma \rho_\varepsilon u_\varepsilon = g_\varepsilon, \quad \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where A^ε and f are as before, $\gamma \geq 1$, and

$$\rho_\varepsilon(x) = \rho(\frac{x}{\varepsilon}),$$

with ρ a Y-periodic positive function in $L^\infty(\partial T)$, and g is a Y-periodic function in $L^2(\partial T)$, and we set

$$g_\varepsilon(x) = \begin{cases} \varepsilon g(\frac{x}{\varepsilon}) & \text{if} \quad \mathcal{M}_{\partial T}(g) \neq 0, \\ g(\frac{x}{\varepsilon}) & \text{if} \quad \mathcal{M}_{\partial T}(g) = 0. \end{cases}$$

The variational formulation of problem is:

Find $u_\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon v \, d\sigma = \int_{\Omega_\varepsilon^*} f_\varepsilon v \, dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon v \, d\sigma, \ \forall\, v \in V_\varepsilon.$$

As before, there exist a subsequence, a function u in $H_0^1(\Omega)$ and $\widehat{u}$ in $L^2(\Omega; H^1_{per}(Y^*))$, such that

$$\begin{aligned} &\mathcal{T}_\varepsilon^*(u_\varepsilon) \to u \quad \text{strongly in } L^2(\Omega; H^1(Y^*)),\\ &\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*),\\ &\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^*)} \to 0. \end{aligned}$$

The homogenization results for Robin conditions

Theorem Let u_ε the solution of Problem 2. Then,

$$\mathcal{T}_\varepsilon^*(u_\varepsilon) \to u \quad \text{strongly in } L^2(\Omega; H^1(Y^*)),$$

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*),$$

$$\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^*)} \to 0.$$

If $\mathcal{M}_{\partial T}(g) \neq 0$ the pair $(u_0, \widehat{u})$ is the unique solution in

$$H_0^1(\Omega) \cap L^2(\Omega; H^1_{\mathrm{per}}(Y^*)) \text{ with } \mathcal{M}_{Y^*}(\widehat{u}) = 0$$

of the limit equation

$$\begin{cases} \forall\, \phi \in H_0^1(\Omega),\ \forall\, \Psi \in L^2(\Omega; H^1_{\mathrm{per}}(Y^*)) \\ \displaystyle\int_{\Omega \times Y^*} A(y, u_0)\,(\nabla u_0 + \nabla_y \widehat{u})\,(\nabla \phi(x) + \nabla_y \Psi(x,y))\ dx\ dy \\ \displaystyle +|Y| K(\gamma) \mathcal{M}_{\partial T}(\rho) \int_\Omega u_0 \phi\ dx \\ \displaystyle = |Y|\ \int_\Omega f \phi\ dx + |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega \phi(x)\ dx. \end{cases}$$

If $\mathcal{M}_{\partial T}(g) = 0$ (and $g \neq 0$), the pair $(u_0, \widehat{u})$ is the unique solution in

$$H_0^1(\Omega) \cap L^2(\Omega; H^1_{\text{per}}(Y^*)) \text{ with } \mathcal{M}_{Y^*}(\widehat{u}) = 0$$

of the limit equation

$$\begin{cases} \forall\, \phi \in H_0^1(\Omega),\ \forall\, \Psi \in L^2(\Omega; H^1_{\text{per}}(Y^*)) \\ \displaystyle\int_{\Omega\times Y^*} A(y, u_0)\, (\nabla u_0 + \nabla_y \widehat{u})\, (\nabla \phi(x) + \nabla_y \Psi(x, y))\ dx\, dy \\ \displaystyle +|Y| K(\gamma) \mathcal{M}_{\partial T}(\rho) \int_\Omega u_0 \phi\, dx \\ \displaystyle = |Y| \int_\Omega f\phi\, dx + \int_{\Omega\times\partial T} g(y) \Psi(x, y)\, dx\, dy. \end{cases}$$

Again,

$$\widehat{u}(x) = -\sum_{i=1}^{n} \widehat{\chi}_i(y) \frac{\partial u_0}{\partial x_i}(x)$$

where $\widehat{\chi}_i$ are the already defined periodic functions.

Corollary Then, as ε tends to 0,

$$i) \quad \widetilde{u_\varepsilon} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega),$$

$$ii) \quad A^\varepsilon(x, u_\varepsilon)\widetilde{\nabla u_\varepsilon} \rightharpoonup A^0(u_0)\nabla u_0 \quad \text{weakly in } (L^2(\Omega))^N.$$

The function u_0 is the unique solution of the problem

$$\begin{cases} -(A^0(u_0)\nabla u_0) + K(\gamma)\mathcal{M}_{\partial T}(\rho)u_0 = f + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks for your attention!