Lecture 2: Tartar's method and correctors in perforated domains

Patrizia Donato
University of Rouen Normandie

Multi-Scale Analysis and Theory of Homogenization
Discussion-Meeting, ICTS Bangalore

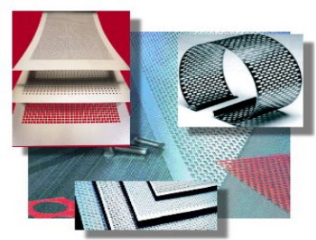
26 August 2019 - 6 September 2019

Motivations

Oscillations can also came from the geometry of the domain, as for example for

- perforated domains,
- reticulated structures, trusses,
- oscillating boundaries.

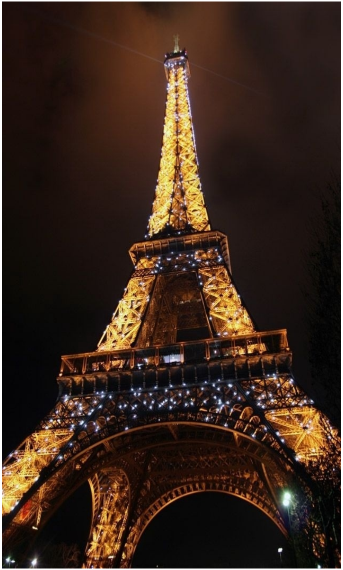
Perforated materials in industry ..




Related materials : reticulated structures, trusses ...

Figure: Roof trusses.

A well known truss ...



Oscillating boundaries ..

s

More oscillating boundaries ..

Periodically perforated domains

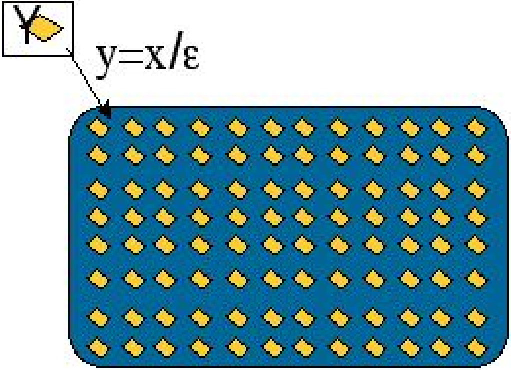


Figure: The perforated domain, the reference cell Y and the reference hole (in yellow).

We introduce the periodically perforated domain as follows.
Let Ω be a bounded connected open set of $\mathbb{R}^N$, $N \geq 2$ and let as before

$$Y =]0, \ell_1[\times ... \times]0, \ell_N[$$

be the reference cell. The reference hole T is a nonempty compact set such that $T \subset Y$ and we set $Y^* = Y \setminus T$. For any $k \in \mathbb{Z}^N$, let

$$T_i^k := k_l + T, \quad \text{where } k_l = (k_1 l_1, ..., k_N l_N)$$

As before, we denote by ε a positive parameter taking its values in a decreasing positive sequence which tends to zero and set, for every ε,

$$K_\varepsilon := \{k \in \mathbb{Z}^N | \varepsilon T_i^k \cap \Omega \neq \emptyset\}.$$

Here we assume that the holes do not intersect the boundary of Ω, that is

$$\partial\Omega \cap (\bigcup_{k \in \mathbb{Z}^N} (\varepsilon T_i^k)) = \phi.$$

This is not necessary foor Dirichlet conditions.

Then, for any ε, we define the perforated domain Ω_ε by

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon, \qquad \text{where } T_\varepsilon = \bigcup_{k \in \mathbb{K}_\varepsilon} (\varepsilon T_i^k).$$

Observe that

$$\partial\Omega_\varepsilon = \partial\Omega \cup \partial T_\varepsilon, \qquad \partial\Omega \cap \partial T_\varepsilon = \emptyset.$$

☛ Here, the holes have the same size as the period. Other situations can be studied, and present very different behaviours.

From now on, we follow the usual notations:

- χ_ω the characteristic function of any open set $\omega \subset \mathbb{R}^N$,
- $m_\omega(v) = \frac{1}{|\omega|} \int_\omega v \, dx$, the mean value over a measurable set ω,
- $\widetilde{v}$ the zero extension to $\mathbb{R}^N$ of any function v defined on Ω or T.

☛ Since $\chi_{\Omega_\varepsilon}(x) = \chi_{Y^*}(\frac{x}{\varepsilon})$, one has

$$\chi_{\Omega_\varepsilon} \rightharpoonup \theta := \frac{|Y^*|}{|Y|}, \qquad \text{weakly in } L^2(\Omega),$$

By construction, $0 < \theta < 1$, and this convergence is not strong.

Consider the problem

$$\begin{cases} -\text{div } (A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \\ + \text{ some boundary conditions on } \partial T_\varepsilon, \end{cases}$$

where $f \in L^2(\Omega)$, $A^\varepsilon(x) = (a_{ij}(\frac{x}{\varepsilon}))_{1 \leq i,j \leq N}$ a.e. on Ω and $A(y)$ is a Y-periodic matrix field in $M(\alpha, \beta, Y)$

Remarks.

- The main difficulty cames from the fact that the domain (and consequently the related functional spaces) depend on ε. Convergence of solutions need a commun fixed space.
- The case of the laplacian (i.e. $A = I$) present the same difficulties.
- The Dirichlet condition on $\partial\Omega$ provides a Poincaré inequality.
- According on different boundary conditions we obtain different homogenized problems.

Boundary conditions

Let n be the outward unit normal to $\partial\Omega_\varepsilon$.

We consider the following boundary conditions on the boundary ∂T_ε of the holes:

- $u_\varepsilon = 0,$ (Homogeneous Dirichlet condition)
- $A^\varepsilon \nabla u \cdot n = 0,$ (Homogeneous Neumann condition)
- $A^\varepsilon \nabla u \cdot n = g,$ (Nonhomogeneous Neumann condition)
- $A^\varepsilon \nabla u \cdot n + hu = g,$ (Nonhomogeneus Robin condition)

where φ, h and g are given functions defined on ∂T_ε.

For a complete discussion on the subject we refer to

D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences 136, Springer-Verlag New York, 1999.

Homogeneous Dirichlet boundary conditions

Under the above notations, consider

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{cases}$$

We know that in this case for every ε there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon} f\, v \, dx \quad \forall v \in H_0^1(\Omega_\varepsilon).$$

Moreover, in view of the results on H_0^1, the zero extension $\widetilde{u}_\varepsilon$ verifies

$$\alpha \|\widetilde{u}_\varepsilon\|^2_{H_0^1(\Omega)} = \alpha \|u_\varepsilon\|^2_{H_0^1(\Omega_\varepsilon)} \leq \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx = \int_{\Omega_\varepsilon} f\, u_\varepsilon \, dx$$
$$\leq \|\widetilde{u}_\varepsilon\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \leq C_\Omega \|\widetilde{u}_\varepsilon\|_{H_0^1(\Omega)} \|f\|_{L^2(\Omega)},$$

where C_Ω is the Poincaré constant in Ω.

Consequently one has the following a priori estimate

$$\|\widetilde{u}_\varepsilon\|_{H_0^1(\Omega)} \leq c,$$

where c is independent of ε.
Then, by compactness there exists $U \in H_0^1(\Omega)$ s.t. (up a subsequence),

$$\widetilde{u}_\varepsilon \rightharpoonup U \quad \text{weakly in } H_0^1(\Omega).$$
$$\widetilde{u}_\varepsilon \to U \quad \text{strongly in } L^2(\Omega).$$

Proposition (Cioranescu-Saint Jean Paulin). One has $U = 0$ and the above convergence is strong, that is:

$$\widetilde{u}_\varepsilon \to 0 \quad \text{strongly in } H_0^1(\Omega).$$

Proof. Let us first prove that $U = 0$. If χ_{T_ε} denotes the characteristic function of the holes, then $\widetilde{u}_\varepsilon \, \chi_{T_\varepsilon} = 0$, a.e. in Ω.
Passing to the limit as $\varepsilon \to 0$, from the above convergences one has $U(1-\theta) = 0$, which implies $U = 0$ since $\theta \neq 1$.

Hence, from the above estimates one has

$$\alpha \|\widetilde{u}_\varepsilon\|^2_{H^1_0(\Omega)} = \int_{\Omega_\varepsilon} f\, u_\varepsilon\, dx = \int_\Omega f\, \widetilde{u}_\varepsilon\, dx \to 0.$$

Since the norm tends to zero, this ends the proof. □

Actually, one can go further in the analysis of the problem. Indeed, one can prove the following Poincaré estimate in Ω_ε and improve the estimates.

Theorem. *Let Ω be a bounded open set in $\mathbb{R}^N$. There exists a constant C, independent of ε such that*

$$\|u\|_{L^2(\Omega_\varepsilon)} \leq C\, \varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)}, \quad \forall u \in H^1_0(\Omega_\varepsilon).$$

The following theorem is very interesting (and surprising!).

Theorem (J.L. Lions, L. Tartar).

One has

$$\begin{cases} \varepsilon^{-1}\widetilde{u}_\varepsilon \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega), \\ \varepsilon^{-2}\widetilde{u}_\varepsilon \rightharpoonup f\ w_0 \quad \text{weakly in } L^2(\Omega), \end{cases}$$

where $w_0 = m_Y(w)$, *with* w *the unique solution of the cell problem*

$$\begin{cases} -div\,({}^tA\nabla w) = 1 \quad \text{in } Y^* \\ w = 0 \quad \text{on } \partial T, \\ w\ \ Y\text{-periodic}, \end{cases}$$

☛ Here no limit equation, but the limits above are explicit.

Proof. A similar computation as above gives

$$\alpha\|u_\varepsilon\|^2_{H^1_0(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} f\, u_\varepsilon\, dx \leq \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|f\|_{L^2(\Omega)}$$
$$\leq c\varepsilon\|u_\varepsilon\|_{H^1_0(\Omega_\varepsilon)}\|f\|_{L^2(\Omega)},$$

as a consequence of the Poincaré inequality in Ω_ε. This implies that the sequence $\{\|\varepsilon^{-1}u_\varepsilon\|_{H^1_0(\Omega_\varepsilon)}\}$ is bounded, and using again the Poincaré's inequality that the sequence $\{\|\varepsilon^{-2}u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\}$ is also bounded. Hence, $\{\varepsilon^{-1}\widetilde{u}_\varepsilon\}$ is bounded in $H^1_0(\Omega)$ and $\{\varepsilon^{-2}\widetilde{u}_\varepsilon\}$ is bounded in $L^2(\Omega)$.

By compactness there exists a subsequence (still denoted ε) and two functions $u* \in H^1_0(\Omega)$, and $u_0 \in L^2(\Omega)$ such that

$$\begin{cases} \varepsilon(\varepsilon^{-2}\widetilde{u}_\varepsilon) = \varepsilon^{-1}\widetilde{u}_\varepsilon \rightharpoonup u^* & \text{weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega), \\ \varepsilon^{-2}\widetilde{u}_\varepsilon \rightharpoonup u_0 & \text{weakly in } L^2(\Omega). \end{cases}$$

Comparing these two convergences, we deduce that $u^* = 0$.

To identify now the limit u_0 we use again the Tartar method as follow. Set

$$w_\varepsilon(x) = w(\frac{x}{\varepsilon}), \quad \text{a.e. in } \mathbb{R}^N,$$

were w has been extended by zero to the whole T. It easy to check that $\{\varepsilon w_\varepsilon\}$ is bounded in $H_0^1(\Omega)$ and $\{w_\varepsilon\}$ is bounded in $L^2(\Omega)$, and due to periodicity,

$$\begin{cases} \varepsilon w_\varepsilon \rightharpoonup 0 & \text{weakly in } H_0^1(\Omega), \\ w_\varepsilon \rightharpoonup w_0 & \text{weakly in } L^2(\Omega). \end{cases}$$

Moreover, by a change of scale,

$$\begin{cases} -\text{div}\,({}^tA^\varepsilon \nabla w_\varepsilon) = \varepsilon^{-2} & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 \quad \text{on } \partial T_\varepsilon, \end{cases}$$

so that

$$\int_{\Omega_\varepsilon} {}^tA^\varepsilon \nabla w_\varepsilon \, \nabla v \; dx \; = \int_{\Omega_\varepsilon} \varepsilon^{-2} v \; dx \quad \text{for every } \in H_0^1(\Omega_\varepsilon).$$

Let now $\varphi \in \mathcal{D}(\Omega)$ and choose

$$\varphi w_\varepsilon$$

as test function in the problem solved by u_ε and

$$\varphi u_\varepsilon$$

in the equation solved by w_ε.

We obtain

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \varphi \, w^\varepsilon \, dx + \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \, \nabla w_\varepsilon \, \varphi \, dx - \int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla w_\varepsilon \nabla \varphi \, u_\varepsilon \, dx$$
$$- \int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla w_\varepsilon \nabla u_\varepsilon \, \varphi \, dx = \int_{\Omega_\varepsilon} f \, \varphi \, w_\varepsilon \, dx - \int_{\Omega_\varepsilon} \varepsilon^{-2} \varphi \, u_\varepsilon \, dx,$$

were here again the bad terms cancel and we are able to pass to the limit in the others.

Taking into account the a priori estimates, the remaining two terms in the left-hand side go to zero.

Hence, we have, using the above convergences

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon} f\,\varphi\, w_\varepsilon\ dx - \int_{\Omega_\varepsilon}\varepsilon^{-2}\,\varphi\, u_\varepsilon\ dx\right)\\
&= \lim_{\varepsilon\to 0}\left(\int_{\Omega} f\,\varphi\, w_\varepsilon\ dx - \int_{\Omega}\varepsilon^{-2}\,\varphi\, \widetilde{u}_\varepsilon\ dx\right)\\
&= \int_{\Omega} f\,\varphi\, w_0\ dx - \int_{\Omega}\varphi\,\widetilde{u}_0\ dx.
\end{aligned}
$$

That is,

$$
\int_\Omega (f\ w_0 - \widetilde{u}_0)\varphi\ \ dx, \quad \text{for every } \varphi\in\mathcal{D}(\Omega).
$$

This gives by know results that

$$
u_0 = f\ w_0, \quad \text{a.e. on } \Omega,
$$

and ends the proof. □

Under the above notations and assumptions, consider

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot n = 0, \quad \text{on } \partial T_\varepsilon \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Before introducing the variational formulation of this problem, let us define the space

$$V^\varepsilon \doteq \{v \in \Omega_\varepsilon |\ v = 0 \text{ on } \partial\Omega\},$$

in the sense of the trace.

☛ The zero extension of a function in V^ε is not in $H_0^1(\Omega)$. Then, we need some suitable extension operators.

Assumption. From now on we suppose that

The boundary ∂T of the reference hole T is Lipschitz continuous.

Let us recall the following extensions results due to D. Cioranescu and J. Saint Jean Paulin:

Theorem. *There exists a linear continuous extension operator P belonging to $\mathcal{L}(H^1(Y^*); H^1(Y)) \cap \mathcal{L}(L^2(Y^*); L^2(Y))$, such that for some positive constant C*

$$\|Pv\|_{L^2(Y)} \leq C\|v\|_{L^2(Y^*)}$$

and

$$\|\nabla Pv\|_{L^2(Y)} \leq C\|\nabla v\|_{L^2(Y^*)},$$

for every v in $H^1(Y^*)$.

☛ The main feature of this result is to estimate the gradient of the extension operator only by the gradient (and not by the H^1 norm).

By a change of scale, one has

Theorem. *There exists a linear continuous extension operator* P^ε *belonging to* $\mathcal{L}(\Omega_\varepsilon; L^2(\Omega_\varepsilon)) \cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ *such that, for some positive constant* C *(independent of* ε*)*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega_\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega_\varepsilon)},$$

for every v *in* $H^1(\Omega_\varepsilon)$*.*

As a consequence, we have a Poincaré inequality holds in V^ε independently of ε, i.e.

Theorem. *There exists a positive constant c, independent of ε, satisfying*

$$\|v\|_{L^2(\Omega_\varepsilon)} \leq c\|\nabla v_1\|_{L^2(\Omega_\varepsilon)},$$

Proof. Indeed, using the Poincaré inequality in Ω one has

$$\|v\|_{L^2(\Omega_\varepsilon)} \leq \|Pv\|_{L^2(\Omega)} \leq \|\nabla C_\Omega P^\varepsilon v\|_{L^2(\Omega)} \leq CC_\Omega\|\nabla v_1\|_{L^2(\Omega_\varepsilon))},$$

for every v in V^ε.

Corollary. *The quantity*

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{H^1(\Omega_\varepsilon)}$$

is a norm on V^ε, which is equivalent to the norm $H^1(\Omega_\varepsilon)$.

A variational solution and a priori estimates

The variational formulation of the above Neumann problem is:

Find $u_\varepsilon \in V^\varepsilon$ such that

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon} f \, v \, dx \quad \forall v \in V^\varepsilon.$$

Thanks to the above uniform Poincaré inequality, one can still apply the Lax-Milgram theorem to prove that for every ε there exists a unique weak solution $u_\varepsilon \in V^\varepsilon$.

Moreover, one has the following a priori estimate

$$\|u_\varepsilon\|_{V^\varepsilon} \leq \frac{c}{\alpha} \|f\|_{L^2(\Omega_\varepsilon)},$$

where c is a constant independent on ε.

Then, using the extension operator, we deduce that there exists a constant c independent on ε, such that

$$\|P^\varepsilon u_\varepsilon\|_{H_0^1(\Omega)} \leq \frac{c}{\alpha}\|f\|_{L^2(\Omega_\varepsilon)}.$$

Then, there exists $u_0 \in H_0^1(\Omega)$ s.t. (up to a subsequence),

$$P^\varepsilon u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega),$$

and, in view of the compact embedding theorem (the inclusion of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact), $P^\varepsilon u_\varepsilon$ strongly converges in $L^2(\Omega)$.

☛ Clearly, one can also consider the zero extension of u_ε, which only belong to $L^2(\Omega)$ and is bounded.
Hence there exists $U \in L^2(\Omega)$ s.t. (up to a subsequence),

$$\widetilde{u_\varepsilon} \rightharpoonup U \quad \text{weakly in } L^2(\Omega),$$

The following results compare u_0 with U.

Proposition. *Suppose that*

$$P^\varepsilon u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega),$$

then

$$\widetilde{u}_\varepsilon \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega),$$

where $\theta = \frac{|Y^*|}{|Y|}$ *is the proportion of material.*

Proof. If $\chi_{\Omega_\varepsilon}$ denotes the characteristic function of Ω_ε, then

$$\widetilde{u}_\varepsilon = P^\varepsilon u_\varepsilon \, \chi_{\Omega_\varepsilon}, \quad \text{a.e. in } \Omega.$$

Passing to the limit as $\varepsilon \to 0$, from the above convergences one has $U = \theta u_0$. □

The homogenization result for the Neumann problem

We describe now the homogenization result for this case.
We proceed in a similar way as for a fixed domain, but we have to pay paying attention on how to extend the different involved functions to the whole of Ω. Set

$$\xi^\varepsilon = A^\varepsilon \nabla u_\varepsilon.$$

By the boundedness of A^ε and the estimates on u_ε we deduce that there exists $\xi^0 \in L^2(\Omega)$, such that (up a subsequence)

$$\widetilde{\xi^\varepsilon} \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N.$$

Let $v \in H_0^1(\Omega)$. From the variational formulation of u_ε,

$$\int_\Omega \widetilde{\xi^\varepsilon}\, \nabla v \, dx = \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \, \nabla v \, dx = \int_{\Omega_\varepsilon} f \, v \, dx = \int_\Omega \chi_{\Omega_\varepsilon} f \, v \, dx$$

which gives at the limit

$$\int_\Omega \xi^0 \, \nabla v \, dx \; = \theta \int_\Omega f \, v dx, \quad \forall v \in H_0^1(\Omega),$$

that is, $$-\text{div}\ \xi^0 = \theta f \quad \text{in } \Omega.$$

Same question as before: the relation between ξ^0 and u_0?

The answer (Cioranescu-Saint Jean Paulin, 1979 : One has

$$\xi^0 = A^0 \nabla u_0$$

where the homogenized matrix A^0 is constant, positive definite and given by

$$A^0\lambda = \mathcal{M}_Y(\widetilde{A\,\nabla\widehat{w}_\lambda}), \qquad \text{for any } \lambda \in \mathbb{R}^N,$$

with $\widehat{w}_\lambda = \lambda \cdot x - \widehat{\chi}_\lambda$ and $\widehat{\chi}_\lambda$ solution of

$$\begin{cases} \text{- div } (A\nabla\widehat{\chi}_\lambda) = \text{-div } (A\lambda) \quad \text{in } Y^* \\ A\nabla(\widehat{\chi}_\lambda - \lambda \cdot y) \cdot n = 0 \quad \text{on } \partial T, \\ \widehat{\chi}_\lambda \ Y\text{- periodic}, \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0, \end{cases}$$

whose variational formulation is

$$\int_{Y^*} A(y)\nabla\widehat{\chi}_\lambda \, \nabla v \, dy = \int_{Y^*} A(y) \, \lambda \, \nabla v \, dy, \quad \forall v \in H^1_{per}(Y^*),$$

Hence, u_0 is the unique solution of

$$\begin{cases} -\mathrm{div}\,(A^0 \nabla u_0) = \theta f \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

so that, the whole sequences converge

$$\begin{cases} P^\varepsilon u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega), \\ \widetilde{u_\varepsilon} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega), \\ A^\varepsilon \nabla u_\varepsilon \rightharpoonup A^0 \nabla u_0 \text{ weakly in } L^2(\Omega). \end{cases}$$

We have a similar result as before (with a similar proof).

Theorem

Let $\lambda \in \mathbb{R}^N$, *and* A^0 *as above. Then*

$$ {}^tA^0\lambda = \mathcal{M}_Y(\widetilde{{}^tA\nabla w_\lambda}), $$

where

$$ w_\lambda = \lambda \cdot x - \chi_\lambda $$

and χ_λ *is the solution of the adjoint problem*

$$ \begin{cases} \text{- } div\ ({}^tA\nabla\chi_\lambda) = \text{-}div\ ({}^tA\lambda) \quad in\ Y^* \\ A\nabla(\chi_\lambda - \lambda \cdot y) \cdot n = 0 \quad on\ \partial T, \\ \chi_\lambda\ Y\text{- } periodic, \\ \mathcal{M}_Y(\chi_\lambda) = 0, \end{cases} $$

We follows the same ides as for a fixed domain, pointing out only the main differences.

Under the notations above, we define w_λ^ε on teh whole Ω by

$$w_\lambda^\varepsilon(x) = \lambda \cdot x - \varepsilon (P\chi_\lambda)(\frac{x}{\varepsilon}).$$

Then,

$$w_\lambda^\varepsilon \longrightarrow \lambda \cdot x \quad \text{strongly in } L^2(\Omega).$$

Moreover, it is to check that w_λ^ε is bounded in $H^1(\Omega)$ so that

$$w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\Omega).$$

Introduce now the (flux) vector $\eta_\lambda^\varepsilon = {}^t A^\varepsilon \nabla w_\lambda^\varepsilon$, so that

$$\widetilde{\eta}_\lambda^\varepsilon \rightharpoonup {}^t A^0 \lambda \quad \text{weakly in } (L^2(\Omega))^n.$$

We also have

$$(\star) \qquad \int_\Omega \widetilde{\eta}_\lambda^\varepsilon \nabla v \, dx \; = 0, \quad \forall v \in H_0^1(\Omega).$$

Let now $\varphi \in \mathcal{D}(\Omega)$ and choose $\varphi w_\lambda^\varepsilon$ as test function in the problem solved by u_ε and φu_ε in equation $(\star)$, solved by η_λ^ε.

As for a fixed domain, the bad terms cancel! We obtain

$$\int_{\Omega_\varepsilon} \xi^\varepsilon(\nabla\varphi)\, w_\lambda^\varepsilon \, dx \; - \int_{\Omega_\varepsilon} \eta_\lambda^\varepsilon(\nabla\varphi)\, u_\varepsilon \, dx = \int_{\Omega_\varepsilon} f\, \varphi\, w_\varepsilon^j \, dx,$$

which can be rewritten as

$$\int_{\Omega} \widetilde{\xi^\varepsilon}(\nabla\varphi)\, w_\lambda^\varepsilon \, dx \; - \int_{\Omega} \widetilde{\eta_\lambda^\varepsilon}(\nabla\varphi)\, P^\varepsilon u_\varepsilon \, dx = \int_{\Omega} \chi_{\Omega_\varepsilon} f\, \varphi\, w_\varepsilon^j \, dx.$$

The remaining part of the proof, as well as the proof of the ellipticity follow along the lines those for the case of a fixed domain. □

Proposition. *The following convergence holds true:*

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx \longrightarrow \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx.$$

Moreover, the following corrector result holds:

Proposition. *if we set*

$$C^\varepsilon_{ij}(x) = \frac{\partial \widehat{w}_j}{\partial y_i}\left(\frac{x}{\varepsilon}\right) \qquad a.e.\ on\ \Omega_\varepsilon,$$

then

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u_0\|_{L^1(\Omega^*_\varepsilon)^N} = 0.$$

The proof is similar to that for a fixed domain, using the equations and the extension operators.

NonhomogeneousRobin boundary conditions

Consider of the following problem (Cioranescu-D, 1988):

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot n + \varepsilon^\gamma \rho_\varepsilon u_\varepsilon = g_\varepsilon, \quad \text{on } \partial T_\varepsilon \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where A^ε and f are as before, $\gamma \geq 1$, and

$$\rho_\varepsilon(x) = \rho(\frac{x}{\varepsilon}),$$

with ρ a Y-periodic positive function in $L^\infty(\partial T)$.
Moreover, we suppose that g is a Y-periodic function in $L^2(\partial T)$, and we set

$$g_\varepsilon(x) = \begin{cases} \varepsilon g(\frac{x}{\varepsilon}) & \text{if} \quad \mathcal{M}_{\partial T}(g) \neq 0, \\ g(\frac{x}{\varepsilon}) & \text{if} \quad \mathcal{M}_{\partial T}(g) = 0. \end{cases}$$

The variational formulation of problem is:

Find $u_\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx + \varepsilon^\gamma \int_{\partial T_\varepsilon} \rho_\varepsilon v \, d\sigma = \int_{\Omega_\varepsilon} f_\varepsilon v \, dx + \int_{\partial T_\varepsilon} g_\varepsilon v \, d\sigma, \ \forall \, v \in V_\varepsilon.$$

The existence and uniqueness of a solution can still be proved using the Lax-Milgram Theorem and the trace.

Moreover the solutions are bounded in V^ε (due to the assumptions on g_ε), so that we can argue as before to show that there exists $u_0 \in H_0^1(\Omega)$ s.t. (up to a subsequence),

$$P^\varepsilon u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega).$$

☛ In this framework, we assume that ∂T is of class C^2.

The homogenization result for the Robin problem

We have as before

$$\begin{cases} P^\varepsilon u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\ \widetilde{u_\varepsilon} \rightharpoonup \theta u_0 \quad \text{ weakly in } L^2(\Omega), \\ A^\varepsilon \nabla u_\varepsilon \rightharpoonup A^0 \nabla u_0 \text{ weakly in } L^2(\Omega). \end{cases}$$

Here, the limit function u_0 is the unique solution of the problem

$$\begin{cases} -\text{div } (A^0 \nabla u_0) + K(\partial T) \mathcal{M}_{\partial T}(\rho) u_0 = \theta f + \dfrac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \quad \text{in } \Omega, \\ v_0 = 0 \end{cases}$$

where A^0 as in the homogeneus Neumann problem, and

$$K(\gamma) = \begin{cases} \dfrac{|\partial T|}{|Y|} \quad \text{if } \gamma = 1, \\ 0 \qquad \text{if } \gamma > 1. \end{cases}$$

Sketch of the proof

One can use again the Tartar method, but here there is an additional difficulty related to the boundary term, where the integral is over an oscillating surface.

To overcome this difficulty, we use an auxiliary problem introduced by Vanninathan for the homogenization of eigenvalues problems in perforated domains.

Lemma. *For $h \in L^2(\partial T)$, set*

$$c_h \doteq \frac{1}{|Y^*|} \int_{\partial T} h(y) d\sigma_y$$

and let $\psi_h \in H^1(Y^*)$ be the unique solution of the problem

$$\begin{cases} -\triangle \psi_h = -c_h & \text{in } Y^*, \\ \nabla \psi_h \cdot n = h & \text{on } \partial T \end{cases}$$

with a zero mean value on Y^*.

Then, still denoting ψ_h the extension by periodicity of ψ_h one has

$$\varepsilon \int_{\partial T_\varepsilon} h(x/\varepsilon)v(x)d\sigma = \int_{\Omega_\varepsilon} \nabla_y \psi_h(x/\varepsilon)\nabla_x v(x)\, dx + c_h \int_{\Omega_\varepsilon} v(x)dx.$$

for every v in V^ε.

As a consequence, on can prove the following

Theorem. *Let*

$$\mu_h^\varepsilon : v \in H_0^1(\Omega) \to \varepsilon \int_{\partial T_\varepsilon} h(x/\varepsilon)v(x)d\sigma.$$

Then, $\mu_h^\varepsilon \in H^{-1}(\Omega) = (H_0^1(\Omega))'$ and

$$\mu_h^\varepsilon \to \mu_h \quad \textit{strongly in } H^{-1}(\Omega),$$

where

$$\mu_h : v \in H_0^1(\Omega) \to \theta c_h \int_\Omega v(x)d\sigma.$$

This allows to threat the case where the mean value of g is different from zero, but not the case where the mean value of g is zero and g is different from zero.

Indeed, if one defines in this case

$$\nu_h^\varepsilon : v \in H_0^1(\Omega) \to \int_{\partial T_\varepsilon} h(x/\varepsilon)v(x)d\sigma,$$

one has only a weak convergence, which is not a priori sufficient.

Neverthless, some suitable argument allow to conclude the proof also in this case.

Convergence of the energy and correctors

The results are different in the two cases $\mathcal{M}_{\partial T}(g) \neq 0$ and $\mathcal{M}_{\partial T}(g) = 0$. We only describe here the corrector result.

Let C^ε be the corrector of teh Neumann problem.

• In the case $\mathcal{M}_{\partial T}(g) \neq 0$ or or $g = 0$ we have (as for Neumann):

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u_0\|_{L^1(\Omega_\varepsilon^*)^N} = 0.$$

• The case $\mathcal{M}_{\partial T}(g) = 0$ (with $g \neq 0$) was an open problem for longtime, and was only recently solved by I. Chourabi and P.D. in 2015, by using the periodic unfolding method.

The result (a bit surprising) state that in this case,

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u_0 - \nabla_y \widehat{\chi_g}(\frac{\cdot}{\varepsilon})\|_{L^1(\Omega_\varepsilon^*)^N} = 0,$$

where $\widehat{\chi_g}$ is the solution of the following problem:

$$\begin{cases} -\mathrm{div}(A\nabla \widehat{\chi_g}) = 0 \quad \text{in } Y\setminus\overline{T}, \\ A\nabla\widehat{\chi_g}\cdot n = g \quad \text{on } \partial T, \\ \widehat{\chi_g} \quad \text{Y-periodic}, \\ \mathcal{M}_{Y^*}(\widehat{\chi_g}) = 0. \end{cases}$$

Holes of a smaller size than the period

We now describe, the so called "strange term" phenomenon introduced in

D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs, in *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. II and III*, ed. by H. Brezis and J.-L. Lions, Research Notes in Mathematics, 60 and 70, Pitman, London, 1982, 93-138 and 154-178.

☛ Since then, a huge number of different sort of homogenization problems in this kind of domains have been studied.

We consider here in the simplest model case , altougth the problem was originally studied in a general abstract framework.
Consider the simplest case where Ω is a domain in $\mathbb{R}^N$ ($N \geq 3$) (the case $N = 2$ can also be treated but is a bit more delicate).
Let Ω be a bounded connected open set of $\mathbb{R}^N$, $N \geq 3$ (the case $N = 2$ can also be treated but is a bit more delicate).

Choose here as reference cell the cube

$$Y =]-1,1[^N$$

be the reference cell and as reference hole the unit ball B_1 centered at the origin.
As before, we denote by ε a positive parameter taking its values in a decreasing positive sequence which tends to zero and let a_ε be, ofr every ε, a positive number such that $0 < a_\varepsilon < \varepsilon$ with

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon} = 0.$$

For every ε and $k \in \mathbb{Z}^N$ we set

$$S_\varepsilon^k = a_\varepsilon B_1 + k\varepsilon Y$$

Then, for any ε, we define the perforated domain Ω_ε by

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon, \qquad \text{where } T_\varepsilon = \bigcup_{k \in \mathbb{Z}^N} (S_\varepsilon^k).$$

Observe that here, the holes have a size a_ε which is of order smaller then the period 2ε.

The homogenization result

Consider for $f \in L^2(\Omega)$, the problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

If for some constant C_0,

$$a_\varepsilon = C_0\, \varepsilon^{N/(N-2)},$$

then the sequence $\widetilde{u_\varepsilon}$ satisfies

$$\begin{cases} \widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ \displaystyle\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \to \int_\Omega |\nabla u|^2 dx + \mu \int_\Omega |u|^2 dx, \end{cases}$$

where u is the unique solution of

$$\begin{cases} -\Delta u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

☛ A zero order terme, the "strange term" appears et the limit.

Moreover, μ is the constant given by

$$\mu = \frac{C_0^{N-2}}{2^N}\, Cap\,(B_1, \mathbb{R}^N)$$

where

$$Cap\,(B_1, \mathbb{R}^N) = \inf_{\substack{\varphi \in \mathcal{D}(\mathbb{R}^n) \\ \varphi = 1 \text{ on } B_1}} \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx$$

is the capacity in $\mathbb{R}^N$ of the closed set B_1.

Remarks

- The size $a_\varepsilon = C_0\, \varepsilon^{N/(N-2)}$, is called the critical size.

  One can prove that if a_ε is of order bigger the the critical size, then the limit of the solution is zero.

  On the other hand, if the size is smaller the the critical size, then the limit problem is simply

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- The case of $-\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f$ was not originally studied by Cioranescu-Murat, since the method used there do not extend to oscillating coefficients.
  It was successively studied by Dal Maso-Murat but the proof was very complicated.
  A nice and simpler proof has been more recently given in

  D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei; *The periodic unfolding method for perforated domains and Neumann sieve models*, J. Math. Pures Appl. 89 (2008), 248-277,

  using the periodic unfolding method.
- In the general abstract framework, μ is a measure.

Thanks for your attention!