Lecture 1: The Tartar's method of oscillating test functions and correctors

Patrizia Donato
University of Rouen Normandie

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The periodic model of composite materials

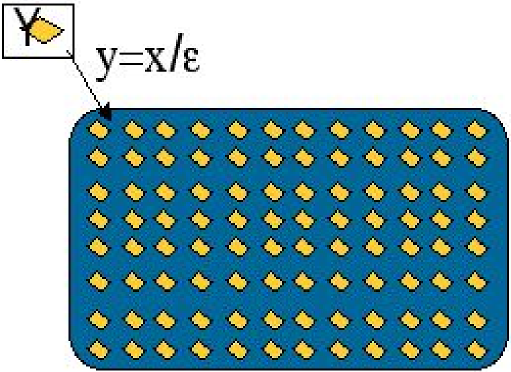

Figure: The composed material and the reference cell Y.

A model periodic elliptic problem

In the sequel:

- Ω is a bounded open set of $\mathbb{R}^N$,
- $Y =]0, \ell_1[\times \ldots \times]0, \ell_n[$ is the reference cell in $\mathbb{R}^N$,
- $\{\varepsilon\}$ is a sequence of positive real numbers converging to zero,
- for some real numbers α, β with $0 < \alpha < \beta$,
  $A(y) = (a_{ij}(y))_{1 \leq i,j \leq N}$ is a Y-periodic matrix field in
  $M(\alpha, \beta, Y)$, i.e.

$$\begin{cases} (A(y)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(y)\lambda| \leq \beta|\lambda|, \quad \text{for any } \lambda \in \mathbb{R}^N, \end{cases}$$

- $A^\varepsilon(x) = (a_{ij}(\frac{x}{\varepsilon}))_{1 \leq i,j \leq N} \quad$ a.e. on Ω.

☛ Observe that by construction $A^\varepsilon \in M(\alpha, \beta, \Omega)$.

The Tartar's method in a fixed domain The Tartar's method of

oscillating test functions is an elegant method based on the use of suitable test functions. The ideas can be also adapted to some nonlinear problems.

Let us start with the case of periodic coefficients in a fixed domain. Consider the problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where (for instance) $f \in L^2(\Omega)$.

Remark. As the figure above shows, it is not realistic to suppose A smooth, for instance continuous (in the picture A is a step function). Hence, the variational framework of weak solutions is needed.

Existence and uniqueness of a weak solution

- As already shown, there exists a unique solution $u_\varepsilon \in H^1_0(\Omega)$ such that

$$\int_\Omega A^\varepsilon \nabla u_\varepsilon \, \nabla v \; dx = \int_\Omega f \, v \; dx \quad \forall v \in H^1_0(\Omega).$$

- Moreover, one has the following a priori estimate

$$\|u_\varepsilon\|_{H^1_0(\Omega)} \leq \frac{C_\Omega}{\alpha} \|f\|_{L^2(\Omega)}.$$

$\star$ This estimate is independent of ε, which is essential in the homogenization process.

We recall the following notion

Definition

Let $\{x_n\}$ be a sequence in a Banach space E. Then, $\{x_n\}$ is said to converge weakly to x in E (denoted $x_n \rightharpoonup x$) if

$$\langle x', x_n\rangle_{E',E} \;\to\; \langle x', x\rangle_{E',E}, \quad \forall\, x' \in E'.$$

Remark

Let $1 \leq p < +\infty$. If $E = L^p(\Omega)$, due to the representation theorem of its dual, $\{f_n\}$ converges weakly to f if

$$\int_\Omega f_n(x)g(x)dx \;\to\; \int_\Omega f(x)g(x)dx, \quad \forall\, g \in L^{p'}(\Omega).$$

Theorem

Let $a(y)$ a Y-periodic function in $L^p(Y)$ $(1 \leq p < +\infty)$ and set

$$a_\varepsilon(x) = a(\frac{x}{\varepsilon}), \quad \text{a.e. in } \mathbb{R}^N.$$

Then,

$$a_\varepsilon \rightharpoonup \mathcal{M}_Y(a) \doteq \frac{1}{|Y|} \int_Y a(y)\, dy \quad \text{in } L^p(\Omega).$$

★ This convergence is never strong (except if a is constant). For exemple, set

$$\varepsilon = 1/n, \quad a(y) = \sin(y), \quad a_n(x) = \sin(nx).$$

Then, a_n weakly converges to $m_{]0,2\pi[}(a) = 0$ for any p. But, $a_n \not\to 0$ strongly in $L^p(]a,b[)$.

As a consequence of the theorem above,

$$A^{\varepsilon} \rightharpoonup \mathcal{M}_Y(A).$$

Does it describe the macroscopic behaviour?

- No, it doesn't. Engineers know very well that the average matrix fields do not describe the macroscopic behaviour of the material.
- Actually, a good description is given by the matrix (called "the effective matrix") which appears in the equation solved by the limit of the solutions, when passing to the limit in the problem.
- Mathematically, this is not so easy to prove!

Two facts about weak convergence

The following properties are very important in homogenization:

Theorem 1 If E reflexive, then bounded sequences are weakly relatively compact.

As a consequence, due to the a priori estimates on the solutions,

there exists $u_0 \in H_0^1(\Omega)$ s.t. (up a subsequence),

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega).$$

Hence,

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 \quad \text{weakly in } (L^2(\Omega))^N,$$

and, in view of the compact embedding theorem, which states that the inclusion of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, u_ε strongly converges in $L^2(\Omega)$.

Theorem 2 If
$$\begin{cases} x_n \rightharpoonup x & \text{weakly in } E \\ y_n \to y & \text{strongly in } E', \end{cases}$$
then
$$\lim_{n\to\infty} \langle y_n, x_n \rangle_{E',E} = \langle y, x \rangle_{E',E}.$$

★ This property does not hold if both convergences are weak, indeed

$\mathcal{M}_Y(a^2) \neq (\mathcal{M}_Y(a))^2.$

As a consequence,

★ there is a main difficuly when passing to the limit in the product $A^\varepsilon \nabla u_\varepsilon$ since both convergences are weak.

As we will see, the weak limit is not $\mathcal{M}_Y(A)\nabla u_0$.

Looking for the limit problem

The variational formulation for our model case is:

Find $u_\varepsilon \in H_0^1$ such that

$$\int_\Omega A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_\Omega f \, v dx \quad \forall v \in H_0^1(\Omega).$$

By weak compactness, there exists $\xi^0 \in L^2(\Omega)$, such that (up a subsequence)

$$A^\varepsilon \nabla u_\varepsilon \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N,$$

so that

$$\int_\Omega \xi^0 \nabla v \, dx = \int_\Omega f \, v dx, \quad \forall v \in H_0^1(\Omega),$$

i.e.

$$-\text{div}\, \xi^0 = f \quad \text{in } \Omega.$$

Question: the relation between ξ^0 and u_0?

The case $n = 1$ (S. Spagnolo 1967)

Let $\Omega =]c, d[$ be an interval in $\mathbb{R}$ and consider the problem

$$\begin{cases} -\dfrac{d}{dx}\Big(a^\varepsilon \dfrac{du_\varepsilon}{dx}\Big) = f \quad \text{in }]c, d[, \\ u_\varepsilon(c) = u_\varepsilon(d) = 0, \end{cases}$$

where $a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big)$, a is ℓ_1-periodic , with

$$0 < \alpha \leq a(x) \leq \beta < +\infty$$

and f is in $L^2(]c, d[)$.

As seen before, up to a subsequence

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } L^2(]c, d[) \\ \dfrac{du_\varepsilon}{dx} \rightharpoonup \dfrac{du_0}{dx} \quad \text{weakly in } L^2(]c, d[). \end{cases}$$

and

$$\xi^\varepsilon \rightharpoonup \xi^0 \quad \text{weakly in } L^2(]c,d[),$$

where $\xi^\varepsilon = a^\varepsilon \frac{du_\varepsilon}{dx}$ satisfies

$$-\frac{d\xi^\varepsilon}{dx} = f \quad \text{in }]c,d[.$$

This implies that ξ^ε is bounded in $H^1(]c,d[)$, so that, by compactness

$$\xi^\varepsilon \to \xi^0 \quad \text{strongly in } L^2(]c,d[).$$

By definition

$$\frac{du_\varepsilon}{dx} = \frac{1}{a^\varepsilon}\xi^\varepsilon$$

which contains a "weak-strong" product, with

$$\frac{1}{a^\varepsilon} \rightharpoonup \mathcal{M}_{(0,\ell_1)}\Big(\frac{1}{a}\Big) \neq 0 \quad \text{weakly* in } L^\infty(]c,d[).$$

This gives at the limit

$$\frac{du_0}{dx} = \mathcal{M}_{(0,\ell_1)}\Big(\frac{1}{a}\Big)\xi^0.$$

Since

$$-\frac{d\xi^0}{dx} = f \quad \text{in }]c, d[,$$

we deduce that u_0 is the unique solution in $H_0^1(c, d)$ of the problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_{(0,\ell_1)}\big(\frac{1}{a}\big)}\,\dfrac{du_0}{dx}\right) = f \quad \text{in }]c, d[, \\ u_0(c) = u_0(d) = 0. \end{cases}$$

Remarks.

- It works since here $\frac{d}{dx} = \mathrm{div}$.
- Clearly,
$$\frac{1}{\mathcal{M}_{(0,\ell_1)}(\frac{1}{a})}\frac{du_0}{dx} \neq \mathcal{M}_{(0,\ell_1)}(a)\frac{du_0}{dx},$$
so that
$$\lim_{\varepsilon\to 0}\Big(a^\varepsilon \frac{du_\varepsilon}{dx}\Big) \neq \big(\lim_{\varepsilon\to 0} a^\varepsilon\big)\big(\lim_{\varepsilon\to 0}\frac{du_\varepsilon}{dx}\big).$$
- For layered materials in $\mathbb{R}^N$ (coefficients oscillating only in one direction), one still obtains an explicit algebric expression for the homogenized matrix.
- This is not true anymore in the general case.

The general answer

One has

$$\xi^0 = A^0 \nabla u_0$$

where the homogenized matrix A^0 is constant, positive definite and given by

$$A^0\lambda = \mathcal{M}_Y(A\,\nabla \widehat{w}_\lambda), \qquad \text{for any } \lambda \in \mathbb{R}^N,$$

with $\widehat{w}_\lambda = \lambda \cdot x - \widehat{\chi}_\lambda$ and $\widehat{\chi}_\lambda$ solution of

$$\begin{cases} \text{- div } (A\nabla\widehat{\chi}_\lambda) = \text{-div } (A\lambda) \quad \text{in } Y, \\ \widehat{\chi}_\lambda \; Y\text{- periodic}, \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0, \end{cases}$$

whose variational formulation is

$$\int_Y A(y)\nabla\widehat{\chi}_\lambda \, \nabla v \, dy = \int_Y A(y)\, \lambda \, \nabla v \, dy, \quad \forall v \in H^1_{per}(Y),$$

By linearity, one can explicity compute the element of A^0, as

$$a^0_{ij} = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\Big(\sum_{k=1}^n a_{ik}\frac{\partial \widehat{\chi}^j}{\partial y_k}\Big),$$

where $\widehat{\chi}_j = \widehat{\chi}_{e_j}$ and $(e_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbb{R}^N$.

★ Hence, to compute A^0 one has only to solve n problems with constat coefficients in the reference cell (for instance by FEM)

The functions $\widehat{\chi}_j$ (or the corresponding $\widehat{w}_i$) $(j = 1, \dots, n)$ are usually referred to as the correctors due to the corerctor result that we will prove later..

Hence, u_0 is the unique solution of

$$\begin{cases} -\text{div}\,(A^0 \nabla u_0) = f \quad \text{in } \Omega, \\ \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

so that, the whole sequences converge

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\ \\ A^\varepsilon \nabla u_\varepsilon \rightharpoonup A^0 \nabla u_0 \text{ weakly in } L^2(\Omega). \end{cases}$$

★ As already mentioned, A^0 gives a good description of the macroscopic behaviour of the material, and it is cleary easier to compute, since is constant. More generally, one can treat many different kind of PDS's.

The Tartar's method of oscillating test functions gives the relation $\xi^0 = A^0 \nabla u_0$ between ξ^0 and u_0.

It is based on

- the presence of two scales ε and x/ε allowing us to pass from the domain Ω covered by εY cells, to the cell Y and viceversa, by $y = \varepsilon x$, respectively $x = \varepsilon y$,
- the use of the **adjoint problem** on the cell Y.

The method lies on the following result, which states that the homogenized matrix A^0 can also given by in terms of the periodic solutions related to the transposed matrix field ${}^tA$. The proof makes use of a suitable choice of test functions in the periodic problems.

Theorem *Let B^0 be the matrix defined by*

$$B^0\lambda = \mathcal{M}_Y({}^tA\nabla w_\lambda), \quad \forall\lambda \in \mathbb{R}^N,$$

and A^0 be the homogenized matrix defined above. Then

$$A^0 = {}^tB^0,$$

that is

$${}^tA^0\lambda = \mathcal{M}_Y({}^tA\nabla w_\lambda),$$

where

$$w_\lambda = \lambda \cdot x - \chi_\lambda$$

and χ_λ is the solution of the adjoint problem

$$\begin{cases} \text{- div } ({}^tA\nabla\chi_\lambda) = \text{-div } ({}^tA\lambda) \quad \text{in } Y, \\ \chi_\lambda \ Y\text{- periodic}, \\ \mathcal{M}_Y(\chi_\lambda) = 0. \end{cases}$$

Proof. To prove the result it is sufficient to show that

$$B^0\lambda\mu = A^0\mu\lambda, \quad \forall\lambda,\mu \in \mathbb{R}^N.$$

From the definition of B^0 and w_λ one has

$$\begin{aligned} B^0\lambda\,\mu &= \frac{1}{|Y|}\int_Y {}^tA\nabla w_\lambda\,\mu\,dy = \frac{1}{|Y|}\int_Y {}^tA(\lambda - \nabla\chi_\lambda)\,\mu\,dy \\ &= \frac{1}{|Y|}\int_Y A\mu\,\lambda\,dy - \frac{1}{|Y|}\int_Y A\mu\,\nabla\chi_\lambda\,dy. \end{aligned}$$

From this equality, by choosing $v = \chi_\lambda$ in the variational formulation of the problem satisfied by $\widehat{\chi}_\mu$ one has by transposition

$$\begin{aligned} B^0\lambda\,\mu &= \frac{1}{|Y|}\int_Y A\mu\lambda - \frac{1}{|Y|}\int_Y A\nabla\widehat{\chi}_\mu\nabla\chi_\lambda\,dy \\ &= \frac{1}{|Y|}\int_Y A\mu\lambda\,dy - \frac{1}{|Y|}\int_Y {}^tA\nabla\chi_\lambda\nabla\widehat{\chi}_\mu\,dy. \end{aligned}$$

From this relation, by choosing $v = \widehat{\chi}_\mu$ in in the variational formulation of the problem satisfied by χ_λ one has by transposition, using the definition of w_μ and A^0,

$$\begin{aligned} B^0\lambda\mu &= \frac{1}{|Y|}\int_Y A\mu\lambda\, dy - \frac{1}{|Y|}\int_Y {}^t A\lambda\nabla\widehat{\chi}_\mu\, dy \\ &= \frac{1}{|Y|}\int_Y A\mu\lambda\, dy - \frac{1}{|Y|}\int_Y A\nabla\widehat{\chi}_\mu\lambda\, dy \\ &= \frac{1}{|Y|}\int_Y A(\mu - \nabla\widehat{\chi}_\mu)\lambda\, dy \\ &= \frac{1}{|Y|}\int_Y A\nabla\big(\mu\cdot y - \widehat{\chi}_\mu\, dy \\ &= \frac{1}{|Y|}\int_Y A\nabla\widehat{w}_\mu\lambda\, dy = A^0\mu\,\lambda, \end{aligned}$$

which ends the proof. □

The following important property is straightforward from the previous theorem:

Corollary

Let A be a matrix satisfying (6.4) and A^0 be the corresponding homogenized matrix given by Theorem 6.1. Then, the homogenized matrix $({}^tA)^0$ corresponding to ${}^tA$, is given by

$$({}^tA)^0 = {}^t(A^0).$$

This implies that if A is symmetric, then A^0 is also symmetric.

Proof of the homogenization result

Under the notations above, set

$$w_\lambda^\varepsilon(x) = \varepsilon w_\lambda(\frac{x}{\varepsilon}) = \lambda \cdot x - \varepsilon \chi_\lambda(\frac{x}{\varepsilon}).$$

As seen before, $\chi_\lambda(\frac{x}{\varepsilon})$ is bounded in $L^2(\Omega)$. Then,

$$w_\lambda^\varepsilon \longrightarrow \lambda \cdot x \quad \text{strongly in } L^2(\Omega).$$

By the derivation rule,

$$\nabla w_\lambda^\varepsilon(x) = \lambda - \nabla_y \chi_\lambda(\frac{x}{\varepsilon}),$$

so that w_λ^ε is bounded in $H^1(\Omega)$ and

$$w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\Omega).$$

Introduce now the (flux) vector $\eta_\lambda^\varepsilon = {}^t\!A^\varepsilon \nabla w_\lambda^\varepsilon$. By construction,

$$ {}^t\!A^\varepsilon \nabla w_\lambda^\varepsilon(x) = \big({}^t\!A \nabla w_\lambda\big)\big(\frac{x}{\varepsilon}\big), $$

so that

$$ \eta_\lambda^\varepsilon \rightharpoonup {}^t\!A^0 \lambda \quad \text{weakly in } (L^2(\Omega))^n. $$

Moreover, extending by periodicity ${}^t\!A \nabla w_\lambda$, a change of scale gives

$$ (\star) \qquad \int_\Omega \eta_\lambda^\varepsilon \nabla v \; dx \; = 0, \quad \forall v \in H_0^1(\Omega). $$

Let now $\varphi \in \mathcal{D}(\Omega)$ and choose

$$ \varphi w_\lambda^\varepsilon $$

as test function in the problem solved by u_ε and

$$ \varphi u_\varepsilon $$

in equation $(\star)$, solved by η_λ^ε.

Setting $\xi^\varepsilon = A^\varepsilon \nabla u_\varepsilon$, this gives

$$\int_\Omega \xi^\varepsilon (\nabla \varphi)\, w_\lambda^\varepsilon \, dx + \int_\Omega \xi^\varepsilon (\nabla w_\lambda^\varepsilon)\, \varphi \, dx - \int_\Omega \eta_\lambda^\varepsilon (\nabla \varphi)\, u_\varepsilon \, dx$$
$$- \int_\Omega \eta_\lambda^\varepsilon (\nabla u_\varepsilon)\, \varphi \, dx = \int_\Omega f\, \varphi\, w_\varepsilon^j \, dx,$$

where, at first glance, we are not able to pass to the limit in the red terms. But

$$\xi^\varepsilon \nabla w_\lambda^\varepsilon = A^\varepsilon \nabla u_\varepsilon \nabla w_\lambda^\varepsilon = {}^t\!A^\varepsilon \nabla w_\lambda^\varepsilon \nabla u_\varepsilon = \eta_\lambda^\varepsilon \nabla u_\varepsilon =$$

Wonderful, the bad terms cancel!

The remaining terms pass easily to the limit to give

$$\int_\Omega \xi^0 \cdot \nabla \varphi\, (\lambda \cdot x)\, dx - \int_\Omega {}^t\!A^0 \lambda \cdot \nabla \varphi\; u_0 \, dx = \int_\Omega f(\lambda \cdot x) \varphi \, dx$$

for every $\varphi \in \mathcal{D}(\Omega)$.

This implies, using the equation of ξ^0,

$$\int_\Omega \xi^0 \cdot \lambda \, \varphi \, dx = - \int_\Omega {}^t A^0 \lambda \cdot \nabla \varphi \; u_0 \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Since ${}^t A^0 \lambda$ is constant, we get

$$\int_\Omega \xi^0 \cdot \lambda \, \varphi \, dx = \int_\Omega {}^t A^0 \lambda \cdot \nabla u_0 \, \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence,

$$\xi^0 \cdot \lambda = {}^t A^0 \lambda \; \nabla u_0 = A^0 \nabla u_0 \cdot \lambda,$$

which gives

$$\xi^0 = A^0 \nabla u_0$$

since λ is arbitrary in $\mathbb{R}^N$.

Proof of the ellipticity of A^0

Let us prove that A^0 is elliptic of constant α.
Observe first that using the problem satisfied by $\widehat{\chi}_j$ and doing some algebric computations one can derive the following equality:

$$
\begin{aligned}
a^0_{ij} &= \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial \widehat{w}_j}{\partial y_\ell} \frac{\partial \widehat{w}_i}{\partial y_k}\, dy \\
&= \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial (y_j - \widehat{\chi}_j)}{\partial y_\ell} \frac{\partial (y_i - \widehat{\chi}_i)}{\partial y_k}\, dy, \quad \forall\, i,j = 1, \dots, N.
\end{aligned}
$$

Then, for every $\xi \in \mathbb{R}^N$ one has that

$$
\sum_{i,j=1}^{N} a^0_{ij} \xi_i \xi_j = \frac{1}{|Y|} \sum_{i,j=1}^{N} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell}\, \xi_i \frac{\partial (y_i - \widehat{\chi}_i)}{\partial y_k}\, \xi_j \frac{\partial (y_j - \widehat{\chi}_j)}{\partial y_\ell}\, dy.
$$

Setting $\zeta = \sum_{i=1}^{N} \xi_i (y_i - \widehat{\chi}_i)$ and using the ellipticity of A we get

$$\sum_{i,j=1}^{N} a^0_{ij} \xi_i \xi_j \geq \frac{\alpha}{|Y|} \int_Y |\nabla \zeta|^2 \, dy \geq 0, \quad \text{for any } \xi \in \mathbb{R}^N.$$

Let us show that this inequality implies that

$$\sum_{i,j=1}^{N} a^0_{ij} \xi_i \xi_j > 0, \quad \text{for any } \xi \in \mathbb{R}^N, \ \xi \neq 0.$$

Indeed, if this is not true, one would have some $\xi \neq 0$ such that

$$|\nabla \zeta| = 0.$$

This means that

$$\zeta = \sum_{i=1}^{N} \xi_i (y_i - \widehat{\chi}_i) = \text{constant},$$

that is,

$$\sum_{i=1}^{N} \xi_i\, y_i = \sum_{i=1}^{N} \xi_i\, \widehat{\chi}_i + \text{constant},$$

and this is impossible since the right-hand side function is periodic by definition and $\xi \neq 0$.

To finish the proof of ellipticity, let h be the following function:

$$h(\zeta, \zeta) = \sum_{i,j=1}^{N} a^0_{ij}(y)\zeta_i\zeta_j.$$

This function is continuous on the unit sphere S_1 which is a compact set of $\mathbb{R}^N$, so that it achieves its minimum on S_1 and, due to the previous result, this minimum is positive. So, $\exists\ \alpha_0 > 0$ such that

$$h(\zeta, \zeta) \geq \alpha_0, \quad \forall \zeta \in S_1.$$

Consequently,

$$\sum_{i,j=1}^{N} a^0_{ij}\xi_i\xi_j = |\xi|^2 \sum_{i,j=1}^{N} a^0_{ij}\, \frac{\xi_i}{|\xi|}\, \frac{\xi_j}{|\xi|} \geq \alpha_0|\xi|^2\,, \quad \text{for any } \xi \in \mathbb{R}^2,\ \xi \neq 0.$$

$\square$

Convergence of the energy

One interesting consequence of the homogenization result is the convergence of the energy associated to the ε-problem namely of the quantity

$$E^\varepsilon(u^\varepsilon) = \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx.$$

Proposition. *The following convergence holds true:*

$$E^\varepsilon(u^\varepsilon) \longrightarrow E^0(u^0) = \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx.$$

Proof. From the variational formulation for u^ε and the homogenization result one has

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx = \int_\Omega f \, u^\varepsilon dx \to \int_\Omega f \, u^0 dx.$$

On the other hand, from the variational formulation for u^0,

$$\int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx = \langle f, u^0 \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

$\square$

From the homogenization result we have the weak convergence

$$\nabla u^\varepsilon - \nabla u^0 \rightharpoonup 0 \quad \text{weakly in } (L^2(\Omega))^N.$$

☞ As can be easily seen in the one-dimensional case, in general this convergence is not strong.

The idea is then to introduce the so-called corrector, in order to have a strong approximation of ∇u^ε, by adjusting the term ∇u^0.

Let us define the corrector matrix field $C^\varepsilon = \left(C^\varepsilon_{ij}\right)_{1\le i,j\le n}$, by

$$\begin{cases} C^\varepsilon_{ij}(x) = C_{ij}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega, \\ C_{ij}(y) = \dfrac{\partial \widehat{w}_j}{\partial y_i}(y), \quad i,j=1,\dots,n & \text{a.e. on } Y, \end{cases}$$

where $\{e_j\}_{j=1}^N$ is the canonical basis of R^N and $\widehat{w}_j = e_j \cdot x - \widehat{\chi}_j$.

☞ The j^{th} column of C is the gradient of $\widehat{w}_j$ (which is related to A).

Some interesting properties of the corrector matrix C^ε are below.

Proposition. The following convergences hold:

$$\begin{cases} i. \qquad C^\varepsilon \rightharpoonup I \quad \text{weakly in } (L^2(\Omega))^{N\times N}, \\ \\ ii. \quad A^\varepsilon C^\varepsilon \rightharpoonup A^0 \quad \text{weakly in } (L^2(\Omega))^N, \end{cases}$$

where I is the unit $N \times N$ matrix.

Proof. Introduce, for $i = 1, \dots, N$ the functions

$$\widehat{w}_i^\varepsilon(x) = \varepsilon \widehat{w}_i(\frac{x}{\varepsilon}) = x_i - \varepsilon \,\widehat{\chi}_i(\frac{x}{\varepsilon}).$$

The same argument used before gives

$$\begin{cases} i. \qquad \widehat{w}_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega). \\ \\ ii. \qquad \widehat{w}_i^\varepsilon \to x_i \quad \text{strongly in } L^2(\Omega). \end{cases}$$

These proves (i), since by definition,

$$C_{ij}^\varepsilon(x) = \frac{\partial \widehat{w}_j}{\partial y_i}(\frac{x}{\varepsilon}) = \frac{\partial \widehat{w}_j^\varepsilon}{\partial x_i}(x).$$

To prove (ii), let us now introduce the vector function

$$\widehat{\eta}_i^\varepsilon = \Big(\sum_{j=1}^N a_{1j}^\varepsilon \frac{\partial \widehat{w}_i^\varepsilon}{\partial x_j}, \dots, \sum_{j=1}^N a_{Nj}^\varepsilon \frac{\partial \widehat{w}_i^\varepsilon}{\partial x_j}\Big) = A^\varepsilon \nabla \widehat{w}_i^\varepsilon.$$

Arguing as done previously for η_i^ε, one can show that

$$\widehat{\eta}_i^\varepsilon \rightharpoonup \mathcal{M}_Y(A\nabla \widehat{w}_i) = A^0 e_i, \quad \text{weakly in } (L^2(\Omega))^N.$$

To conclude, observe that for any $i = 1, \dots, N$, $\widehat{\eta}_i^\varepsilon = A^\varepsilon C^\varepsilon e_i$. $\square$

☛ Similarly to before, one shows that $\widehat{\eta}_i^\varepsilon$ satisfies

$$\int_\Omega \widehat{\eta}_i^\varepsilon \nabla v = 0, \quad \forall v \in H_0^1(\Omega).$$

This will be used in the next proof.

The corrector result

Fro the previous proposition,

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \rightharpoonup 0 \quad \text{weakly in } (L^1(\Omega))^N.$$

Actually, one can prove that this convergence is strong.

Theorem. *Under the previous notations,*

$$\lim_{\varepsilon\to 0} ||\nabla u_\varepsilon - C^\varepsilon \nabla u_0||_{\left(L^1(\Omega)\right)^N} = 0.$$

Moreover, if $C \in (L^r(Y))^{N\times N}$ *for some* r *such that* $2 \leq r \leq \infty$, *and* $\nabla u^0 \in (L^s(\Omega))^N$ *for some* s *such that* $2 \leq s < \infty$, *then*

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0 \quad \textit{strongly in } (L^t(\Omega))^N,$$

where

$$t = \min\left\{2, \frac{r\,s}{r+s}\right\}.$$

The proof of this result is based on the following proposition, whose proof is rather technical (but interesting !):

Lemma. *There exists a positive constant c independent of ε, such that for any $\Phi \in (\mathcal{D}(\Omega))^N$, one has*

$$\limsup_{\varepsilon \to 0} \|\nabla u^\varepsilon - C^\varepsilon \Phi\|_{L^2(\Omega)} \leq c\|\nabla u^0 - \Phi\|_{L^2(\Omega)}.$$

Proof. Let $\Phi = (\Phi_1, \ldots \Phi_N) \in (\mathcal{D}(\Omega))^N$. By ellipticity,

$$\begin{aligned}
\alpha\|\nabla u^\varepsilon - C^\varepsilon \Phi\|^2_{L^2(\Omega)} &\leq \int_\Omega A^\varepsilon \left(\nabla u^\varepsilon - C^\varepsilon \Phi\right)\left(\nabla u^\varepsilon - C^\varepsilon \Phi\right)\, dx \\
&= \int_\Omega A^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon \, dx - \int_\Omega A^\varepsilon \nabla u^\varepsilon \left(C^\varepsilon \Phi\right)\, dx \\
&\quad - \int_\Omega A^\varepsilon \left(C^\varepsilon \Phi\right) \nabla u^\varepsilon \, dx + \int_\Omega A^\varepsilon \left(C^\varepsilon \Phi\right)\left(C^\varepsilon \Phi\right)\, dx.
\end{aligned}$$

We want to pass to the limit in all the terms from the right-hand side of this inequality.

The first term is nothing else than the energy, so that

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx \longrightarrow \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx.$$

For the second term, from the definition of C^ε, one can write

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, (C^\varepsilon \Phi) \, dx &= \sum_{i=1}^N \lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, (\Phi_i \nabla \widehat{w}_i^\varepsilon) \, dx \\ &= \sum_{i=1}^N \lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla (\Phi_i \, \widehat{w}_i^\varepsilon) \, dx \\ &\quad - \sum_{i=1}^N \lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla \Phi_i \, \widehat{w}_i^\varepsilon \, dx. \end{aligned}$$

We use the equation to remove the weak-weak product to have

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla (\Phi_i \, \widehat{w}_i^\varepsilon) \, dx = \int_\Omega f \, \Phi_i \, \widehat{w}_i^\varepsilon \, dx,$$

so that we can pass to the limit in every sum.

We obtain

$$\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla u^\varepsilon \ (C^\varepsilon \Phi) \ dx =$$

$$= \sum_{i=1}^N \Big(\lim_{\varepsilon\to 0}\int_\Omega f\, \Phi_i \, \widehat{w}_i^\varepsilon \ dx - \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla u^\varepsilon \ \nabla \Phi_i \ \widehat{w}_i^\varepsilon \ dx\Big)$$

$$= \sum_{i=1}^N \int_\Omega f\, \Phi_i \boldsymbol{x}_i \ dx - \int_\Omega A^0 \nabla u^0 \ \nabla \Phi_i \ x_i \ dx\Big)$$

$$= \sum_{i=1}^N \int_\Omega f \Phi_i \, \widehat{w}_i^\varepsilon \ dx - \sum_{i=1}^N \int_\Omega A^0 \nabla u^0 \ \nabla (\Phi_i \ x_i) \ dx$$

$$+ \int_\Omega A^0 \nabla u^0 \ \nabla \Phi_i \ x_i \ dx$$

where we used $\Phi_i \ x_i$ as test function in the limit problem. We finally get

$$\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla u^\varepsilon \ (C^\varepsilon \Phi) \ dx = \int_\Omega A^0 \nabla u^0 \ \Phi \ dx.$$

To treat the third term of the right-hand side we use the equation satisfied by $\widehat{\eta}_i^\varepsilon = A^\varepsilon C^\varepsilon e_i$, taking $\Phi_i\, u^\varepsilon$ as test function.

We obtain, by taking into account all the convergences,

$$\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\,(C^\varepsilon\Phi)\;\nabla u^\varepsilon\,dx = \sum_{i=1}^N \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla\widehat{w}_i^\varepsilon\;\nabla u^\varepsilon\;\Phi_i\,dx$$

$$= \sum_{i=1}^N \lim_{\varepsilon\to 0}\Big(\int_\Omega \widehat{\eta}_i^\varepsilon\cdot\nabla\,(\Phi_i u^\varepsilon)\;dx - \int_\Omega \widehat{\eta}_i^\varepsilon\cdot\nabla\Phi_i\;u^\varepsilon\;dx\Big)$$

$$= 0 - \sum_{i=1}^N \lim_{\varepsilon\to 0}\int_\Omega \widehat{\eta}_i^\varepsilon\cdot\nabla\Phi_i\;u^\varepsilon\;dx =$$

$$-\sum_{i=1}^N \int_\Omega A^0 e_i\cdot\nabla\Phi_i\;u^0\;dx = \int_\Omega A^0\Phi\cdot\nabla u^0\;dx.$$

For the last term, we use again the equation satisfied by $\widehat{\eta}_i^\varepsilon$, choosing here $\Phi_i \Phi_j \widehat{w}_j^\varepsilon$ as test function.

Using the properties of the corrector, we have

$$\lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \left(C^\varepsilon \Phi\right)\left(C^\varepsilon \Phi\right)\, dx = \sum_{i,j=1}^N \lim_{\varepsilon\to 0} \int_\Omega A^\varepsilon \nabla \widehat{w}_i^\varepsilon \, \nabla \widehat{w}_j^\varepsilon \; \Phi_i \; \Phi_j \; dx$$

$$= \sum_{i,j=1}^N \lim_{\varepsilon\to 0} \Big(\int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla(\Phi_i \; \Phi_j \; \widehat{w}_j^\varepsilon)\, dx - \int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla(\Phi_i \; \Phi_j) \; \widehat{w}_j^\varepsilon \; dx \Big)$$

$$= 0 - \sum_{i,j=1}^N \int_\Omega A^0 e_i \cdot \nabla(\Phi_i \Phi_j)\, x_j \, dx = \int_\Omega A^0 \, \Phi \, \Phi \, dx.$$

Collecting these limits we have

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \alpha \|\nabla u^\varepsilon - C^\varepsilon \Phi\|^2_{L^2(\Omega)} & \\
&\leq \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \; dx - \int_\Omega A^0 \nabla u^0 \; \Phi \; dx \\
&\quad - \int_\Omega A^0 \Phi \cdot \nabla u^0 \; dx + \int_\Omega A^0 \, \Phi \; \Phi \; dx \\
&= \int_\Omega A^0 \left(\nabla u^0 - \Phi\right) \left(\nabla u^0 - \Phi\right) \; dx \\
&\leq c \|\nabla u^0 - \Phi\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we used the boundedness of A^0 (actually constant).
This ends the proof. □

☛ Actually, one can compute the upper bound of A^0, showing that

$$A^0 \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega).$$

Proof of the corrector result

The L^1-convergence makes use of a density argument. Let $\delta > 0$ and $\Phi_\delta \in (\mathcal{D}(\Omega))^N$ such that

$$\|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} \leq \delta.$$

Consequently, by the triangular inequality, using the lemma and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon \nabla u^0\|_{L^1(\Omega)} \\
&\leq \limsup_{\varepsilon\to 0} \left[\|\nabla u^\varepsilon - C^\varepsilon \Phi_\delta\|_{L^1(\Omega)} + \|C^\varepsilon \Phi_\delta - C^\varepsilon \nabla u^0\|_{L^1(\Omega)}\right] \\
&\leq \limsup_{\varepsilon\to 0} c_1 \|\nabla u^\varepsilon - C^\varepsilon \Phi_\delta\|_{L^2(\Omega)} + c_2 \|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} \\
&\leq c\, c_1 \|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} + c_2\, \delta \leq c_3\, \delta.
\end{aligned}$$

The second statement follows similarly, using Hölder's inequality with

$$p = \frac{r+s}{s}, \quad p' = \frac{r+s}{r}.$$

□

Thanks for your attention!