University of the Philippines Los Baños
Institute of Mathematical Sciences and Physics

Multiscale Expansion Method for Periodic Homogenization

Editha C. Jose

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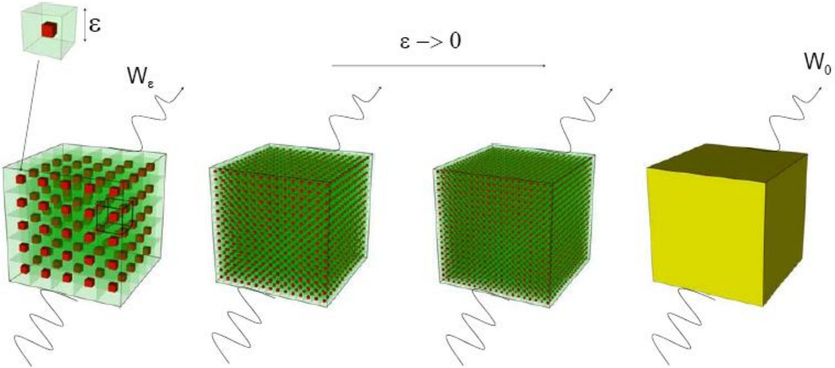
Homogenization



Reference: Meirmanov, Mathematical Methods Based on Homogenization Theory

Objectives of the Lecture

1. Show the multiscale expansion method in the one-dimensional and $n-$ dimensional cases for a model diffusion problem.

2. Derive the corresponding cell and homogenized problems.

3. Examine the properties of the cell and homogenized problems.

A **stationary** temperature field in a **non-homogeneous** rod with periodic structure:

$$\begin{cases} \dfrac{d}{dx}\left(a_\varepsilon(x)\dfrac{du}{dx}\right) = f(x), \quad x \in (0,1) \\ u(0) = g_0, u(1) = g_1. \end{cases}$$

Setting of the Problem

A **stationary** temperature field in a **non-homogeneous** rod with periodic structure:

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Assumptions:

1. The rod is of periodic structure.
2. The thermal conductivity a_ε is periodic with period $\varepsilon = \dfrac{1}{n}$ where n is a large natural number.
3. $a_\varepsilon(x) = a\left(\dfrac{x}{\varepsilon}\right)$ and $a(y)$ is a 1-periodic function which is positive and differentiable.

Setting of the Problem

$$\begin{cases} \dfrac{d}{dx}\left(a_\varepsilon(x)\dfrac{du}{dx}\right) = f, \quad x \in (0,1) \\ u(0) = g_0, u(1) = g_1. \end{cases}$$

Setting of the Problem

$$\begin{cases} \dfrac{d}{dx}\left(a_\varepsilon(x)\dfrac{du}{dx}\right) = f, \quad x \in (0,1) \\ u(0) = g_0, u(1) = g_1. \end{cases}$$

Our goal: Solve for u when $\varepsilon \to 0$.

Multiscale asymptotic expansion method

Construct the solution u in the form (also called an ansatz):

$$u = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots$$

Multiscale asymptotic expansion method

Construct the solution u in the form (also called an ansatz):

$$u = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots$$

Notes:

1. slow (or macroscopic) variable x
2. fast (or microscopic) variable $y = \dfrac{x}{\varepsilon}$
3. $u_i(x,y)$ are periodic with respect to the variable y with period 1

Multiscale asymptotic expansion method

Differentiation formula:

$$\frac{dF}{dx}(x,y) = \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F}{\partial y}.$$

Multiscale asymptotic expansion method

Differentiation formula:

$$\frac{dF}{dx}(x,y) = \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F}{\partial y}.$$

A series expansion in ε:

$$\varepsilon^{-2}\left[\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial y}\right)\right] + \varepsilon^{-1}\left[\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right)\right)\right]$$
$$+ \varepsilon^{0}\left[\frac{\partial}{\partial x}\left(a(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\right)\right]$$
$$+ \varepsilon^{1}\left[\dots\right] + \varepsilon^{2}\left[\dots\right] + \cdots = f.$$

Multiscale asymptotic expansion method

For homogenization purposes, only the $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^{0}$ terms are needed:

$$\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right)\right) = 0$$

$$\frac{\partial}{\partial x}\left(a(y)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\right) = f$$

(A system of linear differential equations for u_0, u_1, u_2 with respect to y, considering x as a parameter)

Multiscale asymptotic expansion method

The mean value (period-average) operator $\mathcal{M}$:

$$\mathcal{M}(F(x,y)) = \int_0^1 F(x,y)\,dy$$

with x and y considered as independent.

Multiscale asymptotic expansion method

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Step 1:

$$\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial y}\right) = 0 \Rightarrow u_0(x,y) = v_0(x).$$

Multiscale asymptotic expansion method

Step 2:

$$\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right)\right)=0$$

$$\Rightarrow \frac{\partial u_1}{\partial y}=\left(\frac{\widehat{a}}{a(y)}-1\right)\frac{dv_0}{dx}$$

$$\Rightarrow u_1(x,y)=N_1(y)\frac{dv_0}{dx}+v_1(x),$$

where $v_1(x)$ is an arbitrary function and

$$N_1(y)=\int_0^y\left(\frac{\widehat{a}}{a(t)}-1\right)dt,$$

with $\widehat{a}=\dfrac{1}{\mathcal{M}(a(y))^{-1}}$.

Multiscale asymptotic expansion method

Step 3:

$$\frac{\partial}{\partial x}\left(a(y)\left(\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)\right)=f$$

$$\Rightarrow \int_0^1 \left[\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_1}{\partial y}\right)+\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial x}\right)\right] dy = \int_0^1 f(x)\, dy$$

Multiscale asymptotic expansion method

Step 3:

$$\frac{\partial}{\partial x}\left(a(y)\left(\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(a(y)\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)\right)=f$$

$$\Rightarrow \int_0^1\left[\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_1}{\partial y}\right)+\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial x}\right)\right]dy=\int_0^1 f(x)\,dy$$

Substituting the value of u_0 and $\dfrac{\partial u_1}{\partial y}$ into the preceding equation gives

$$\widehat{a}\frac{d^2 v_0}{dx^2}=f(x)$$

with the boundary conditions $v_0(0)=g_0$ and $v_0(1)=g_1$.

Setting of the problem

A model problem of conductivity reads:

$$\begin{cases} -\text{div}\,(A_\varepsilon(x)\nabla u_\varepsilon) = f(x), & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

where $A_\varepsilon(x)$ and $u_\varepsilon(x)$ are the conductivity and unknown function modeling electric potential or temperature, respectively.

Setting of the problem

$$\begin{cases} -\text{div}\,(A_\varepsilon(x)\nabla u_\varepsilon) = f, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Assumptions:

1. $\Omega \subseteq \mathbb{R}^N$, is a bounded, periodic domain with period ε (very small positive number)
2. $Y = (0,1)^N$ is the rescaled unit periodic cell.
3. The conductivity varies periodically with period ε and will be the matrix $A(y)$ where $y = \dfrac{x}{\varepsilon} \in Y$. A is assumed to be bounded and positive definite, that is, for all $\xi \in \mathbb{R}^N$ and at any point $y \in Y$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2.$$

   for some positive constants $\beta \leq \alpha > 0$.
4. $A_\varepsilon(x) = A\left(\dfrac{x}{\varepsilon}\right)$

Setting of the problem

$$\begin{cases} -\text{div}\,(A_\varepsilon(x)\nabla u_\varepsilon) = f, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

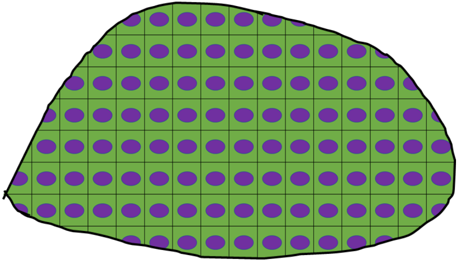

Figure: A periodic domain Ω

Setting of the problem

$$\begin{cases} -\mathsf{div}\,(A_\varepsilon(x)\nabla u_\varepsilon) = f, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Setting of the problem

$$\begin{cases} -\text{div}\,(A_\varepsilon(x)\nabla u_\varepsilon) = f, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

The PDE can be written in the form:

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\right) = f(x).$$

Setting of the problem

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$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\right) = f(x).$$

Our goal: Solve the above problem as the period $\varepsilon \to 0$ and derive some global or average behavior of the domain Ω.

Multiscale asymptotic expansion method

The solution u_ε as a power series in ε (ansatz):

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i u_i \left(x, \frac{x}{\varepsilon}\right)$$

where $u_i(x, y)$ is a $Y-$ periodic function with respect to y.

Multiscale asymptotic expansion method

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where $u_i(x, y)$ is a $Y-$ periodic function with respect to y.

Differentiation formula:

$$\nabla\left(u_i\left(x, \frac{x}{\varepsilon}\right)\right) = \left(\frac{1}{\varepsilon}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\varepsilon}\right)$$

where ∇_y and ∇_x denote the partial derivative with respect to the first and second variable of $u_i(x, y)$.

Multiscale asymptotic expansion method

A series expansion in ε:

$$
\begin{aligned}
&-\varepsilon^{-2}\left(\mathsf{div}_y A\nabla_y u_0\right)\left(x,\frac{x}{\varepsilon}\right)\\
&-\varepsilon^{-1}\left[\mathsf{div}_y A(\nabla_x u_0+\nabla_y u_1)+\mathsf{div}_x(A\nabla_y u_0)\right]\left(x,\frac{x}{\varepsilon}\right)\\
&-\varepsilon^{0}\left[\mathsf{div}_x A(\nabla_x u_0+\nabla_y u_1)+\mathsf{div}_y A(\nabla_x u_1+\nabla_y u_2)\right]\left(x,\frac{x}{\varepsilon}\right)\\
&-\sum_{i=1}^{\infty}\varepsilon^{i}\left[\mathsf{div}_x A(\nabla_x u_i+\nabla_y u_{i+1})+\mathsf{div}_y A(\nabla_x u_{i+1}+\nabla_y u_{i+2})\right]\left(x,\frac{x}{\varepsilon}\right)\\
&=f(x).
\end{aligned}
$$

Multiscale asymptotic expansion method

Order ε^{-2} equation:

$$-\mathsf{div}_y A(y)\nabla_y u_0(x,y) = 0$$

Order ε^{-1} equation:

$$-\mathsf{div}_y(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) - \mathsf{div}_x(A(y)\nabla_y u_0)(x,y) = 0$$

Order ε^{0} equation:

$$\begin{aligned} &-\mathsf{div}_x(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) \\ &\quad -\mathsf{div}_y(A(y)(\nabla_x u_1 + \nabla_y u_2))(x,y) = f(x) \end{aligned}$$

Multiscale asymptotic expansion method

Step 1:

$$-\mathsf{div}_y A(y)\nabla_y u_0(x,y) = 0 \Rightarrow u_0(x,y) = u_0(x)$$

where u_0 does not depend on y.

Step 2:

$$-\mathsf{div}_y(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) - \mathsf{div}_x(A(y)\nabla_y u_0)(x,y) = 0$$

Multiscale asymptotic expansion method

Step 2:

$$-\mathrm{div}_y(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) - \mathrm{div}_x(A(y)\nabla_y u_0)(x,y) = 0$$

Finding u_1 in terms of u_0:

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x_i}(x)\omega_i(y).$$

where $\omega_i(y) \in H^1_{per}(Y)$ (unique up to an additive constant) the solution to

$$-\mathrm{div}_y(A(y)\nabla_y \omega_i(y)) = \mathrm{div}_y(A(y)e_i)$$

with e_i the i^{th} basis vector of $\mathbb{R}^N$.

Multiscale asymptotic expansion method

Step 3:

$$
\begin{aligned}
&-\operatorname{div}_x(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) \\
&\quad -\operatorname{div}_y(A(y)(\nabla_x u_1 + \nabla_y u_2))(x,y) = f(x)
\end{aligned}
$$

Multiscale asymptotic expansion method

Step 3:

$$-\operatorname{div}_x(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) \\ - \operatorname{div}_y(A(y)(\nabla_x u_1 + \nabla_y u_2))(x,y) = f(x)$$

Integration and using the periodic boundary condition of u_1 and u_2 give:

$$-\operatorname{div}_x(A^*\nabla_x u_0) = f(x),$$

Step 3:

$$
\begin{aligned}
&-\,\mathrm{div}_x(A(y)(\nabla_x u_0 + \nabla_y u_1))(x,y) \\
&\quad -\,\mathrm{div}_y(A(y)(\nabla_x u_1 + \nabla_y u_2))(x,y) = f(x)
\end{aligned}
$$

Integration and using the periodic boundary condition of u_1 and u_2 give:

$$-\mathrm{div}_x(A^* \nabla_x u_0) = f(x),$$

where A^* is known as the effective diffusion tensor and is given by

$$A^*_{j,k} = \int_Y \left(A_{j,k}(y) + \sum_{l=1}^{N} A_{j,l} \frac{\partial \omega_k}{\partial y_l}(y) \right) \, dy.$$

Multiscale asymptotic expansion method

Using the Dirichlet boundary condition, the limit u_0 satisfies the boundary value problem:

$$\begin{cases} -\mathsf{div}_x \left(A^* \nabla_x u_0 \right) = f(x), & \text{in } \Omega, \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

The cell and homogenized problems

The cell problem:

$$-\mathsf{div}_y(A(y)\nabla_y \omega_i(y)) = \mathsf{div}_y(A(y)e_i)$$

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Remarks:

1. The **cell problem** allows us to solve u_1 in terms of u_0.
2. The **homogenized problem** gives the equation solved by u_0.
3. The diffusion tensor A^* does not depend on x.
4. The functions ω_i in the cell problem are known as the **correctors**.

Questions about the cell and homogenized problems

Questions about the cell and homogenized problems

1. Is the homogenized problem well-posed? What can be said of A^*?
2. Is the cell problem well-posed?
3. What do the correctors mean?

The correctors

$$u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)$$
$$\approx u_0(x) + \varepsilon \sum_{i=1}^{N} \frac{\partial u_0}{\partial x_i}(x)\omega_i\left(\frac{x}{\varepsilon}\right).$$

The correctors

$$u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)$$
$$\approx u_0(x) + \varepsilon \sum_{i=1}^{N} \frac{\partial u_0}{\partial x_i}(x)\omega_i\left(\frac{x}{\varepsilon}\right).$$

Remarks:

1. The correctors ω_i measures the difference of the heterogeneous solution and the homogenized solution.
2. The solution u_ε oscillates with an amplitude ε and with a profile ω_i scaled by $\dfrac{\partial u_0}{\partial x_i}(x)$.

The solvability of the cell problem

The cell problem:

$$-\mathsf{div}_y(A(y)\nabla_y \omega_i(y)) = \mathsf{div}_y(A(y)e_i)$$

The solvability of the cell problem

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$$-\mathsf{div}_y(A(y)\nabla_y \omega_i(y)) = \mathsf{div}_y(A(y)e_i)$$

The variational formulation of the cell problem:

$$\int_Y (A(y)\nabla_y \omega_i(y), \nabla_y \varphi(y))\, dy = -\int_Y (A(y)e_i, \nabla_y \varphi(y))\, dy,$$

where $\varphi \in H^1_{per}(Y)$.

The solvability of the cell problem

The cell problem:

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The variational formulation of the cell problem:

$$\int_Y (A(y)\nabla_y \omega_i(y), \nabla_y \varphi(y))\, dy = -\int_Y (A(y)e_i, \nabla_y \varphi(y))\, dy,$$

where $\varphi \in H^1_{per}(Y)$.

Using the assumption on A and the Lax-Milgram Theorem, the preceding equation has a **unique solution** in the Hilbert space

$$V = \{\psi \in H^1_{per}(Y) \text{ such that } \int_Y \psi(y)\, dy = 0\}.$$

The diffusion tensor A^*

$$\int_Y (A(y)\nabla_y \omega_i(y), \nabla_y \varphi(y))\, dy = -\int_Y (A(y)e_i, \nabla_y \varphi(y))\, dy,$$

The diffusion tensor A^*

$$\int_Y (A(y)\nabla_y \omega_i(y), \nabla_y \varphi(y))\, dy = -\int_Y (A(y)e_i, \nabla_y \varphi(y))\, dy,$$

Let $\varphi = \omega_k$ in the variational formulation, that is, ω_k is a test function. Then

$$\int_Y (A(y)(e_i + \nabla_y \omega_i(y)), \nabla_y \omega_k(y))\, dy = 0.$$

The diffusion tensor A^*

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Let $\varphi = \omega_k$ in the variational formulation, that is, ω_k is a test function. Then

$$\int_Y (A(y)(e_i + \nabla_y \omega_i(y)), \nabla_y \omega_k(y))\, dy = 0.$$

It can be shown that:

$$\begin{aligned} A^*_{j,k} &= \int_Y \left(A_{j,k}(y) + \sum_{l=1}^{N} A_{j,l} \frac{\partial \omega_k}{\partial y_l}(y) \right) dy \\ &= \int_Y (A(y)(e_k + \nabla_y \omega_k(y)), e_j + \nabla_y \omega_j)\, dy \end{aligned}$$

The diffusion tensor A^*

If $\xi \in \mathbb{R}^N$ then

$$
\begin{aligned}
(A^*\xi, \xi) &= \sum_{j,k=1}^{N} A^* \xi_j \xi_k \\
&= \int_Y \left(A(y) \left(\xi + \sum_{k=1}^{N} \xi_k \nabla_y \omega_k(y) \right), \xi + \sum_{j=1}^{N} \xi_j \nabla_y \omega_j(y) \right) \, dy \\
&\geq \left\| \xi + \sum_{k=1}^{N} \xi_k \nabla_y \omega_k(y) \right\|_{L^2(Y)}^2 ,
\end{aligned}
$$

The diffusion tensor A^*

If $\xi \in \mathbb{R}^N$ then

$$
\begin{aligned}
(A^*\xi, \xi) &= \sum_{j,k=1}^{N} A^*\xi_j\xi_k \\
&= \int_Y \left(A(y) \left(\xi + \sum_{k=1}^{N} \xi_k \nabla_y \omega_k(y) \right), \xi + \sum_{j=1}^{N} \xi_j \nabla_y \omega_j(y) \right) \, dy \\
&\geq \left\| \xi + \sum_{k=1}^{N} \xi_k \nabla_y \omega_k(y) \right\|_{L^2(Y)}^2,
\end{aligned}
$$

showing that A^* is a positive definite matrix.

The solvability of the homogenized problem

Using the property of A^* and the Lax-Milgram Theorem, the homogenized problem:

$$\begin{cases} -\text{div}_x\left(A^*\nabla_x u_0\right) = f(x), & \text{in } \Omega, \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

has a unique solution in the Hilbert space $H_0^1(\Omega)$.

The multiscale expansion method

This method is a heuristic one which allows us to find the homogenized problem but not a rigorous one. It may not be optimal and perfect. The next methods will justify our result.

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Dhanyahvaad!