Khovanov Homology and Virtual Knot Cobordism

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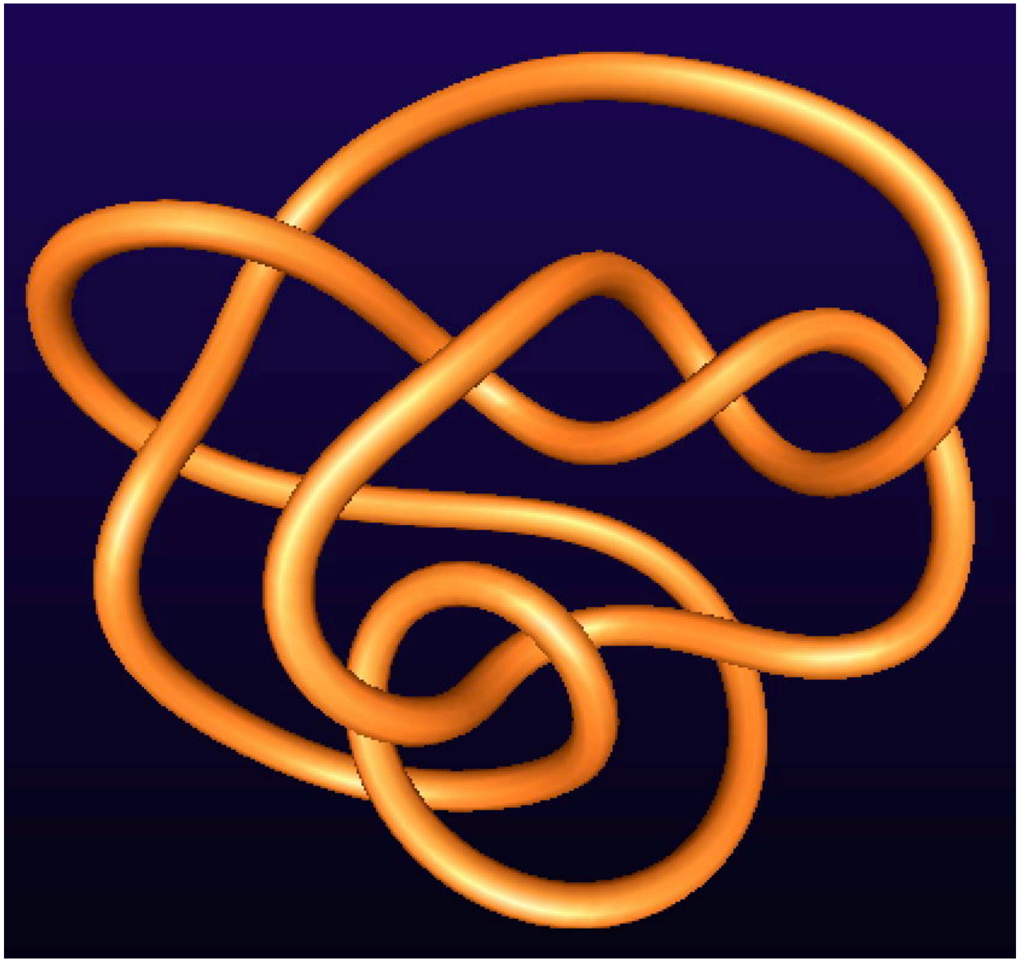
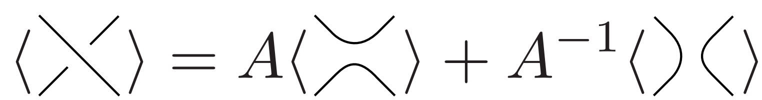
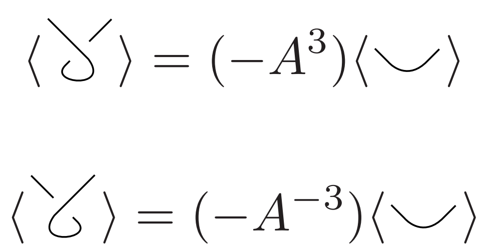


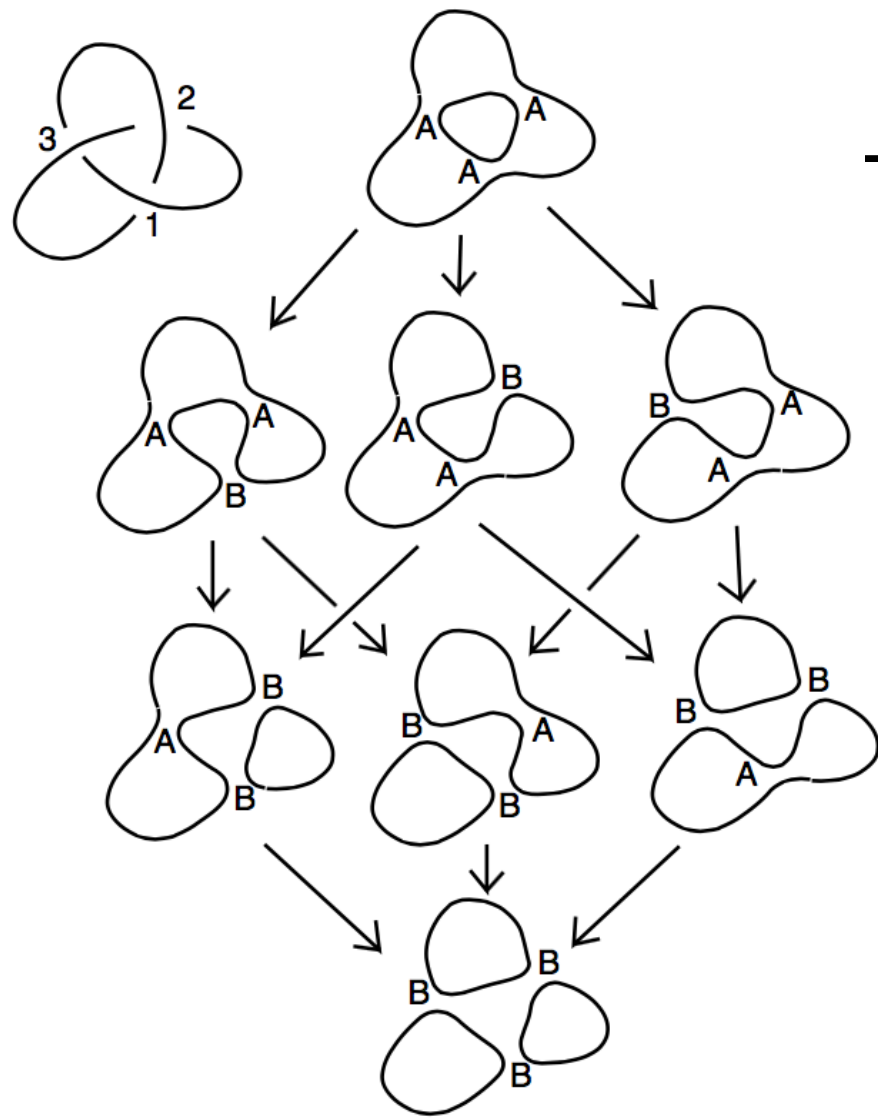
Khovanov Homology

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology theory associated with $\langle K \rangle$.



$$\langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$





Cubism

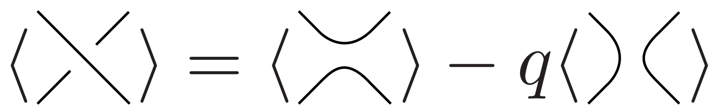
The bracket states form a category. How can we obtain topological information from this category?

Exploration: Examine the Bracket Polynomial for Clues.

Let c(K) = number of crossings on link K.

Form $A^{-c(K)}$ <K> and replace A by $-q^{-1}$.

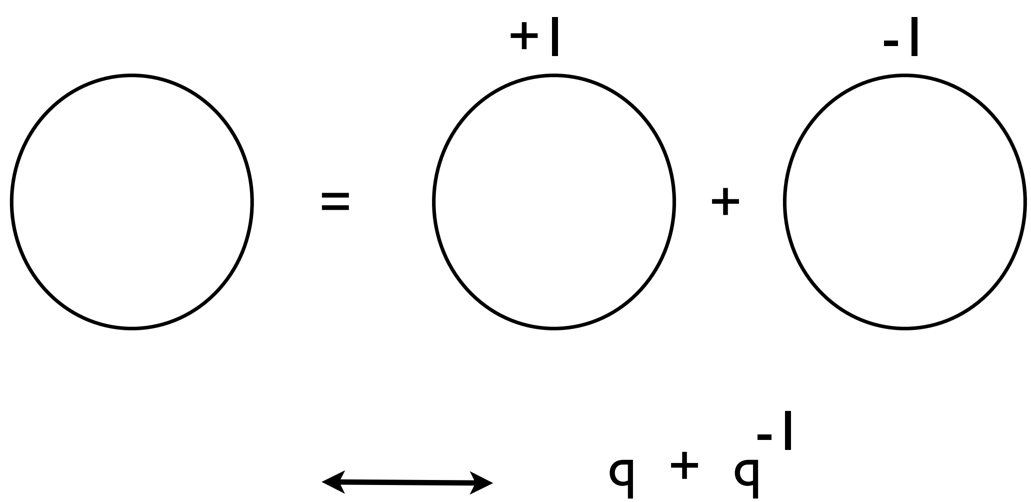
Then the skein relation for <K> will be replaced by:

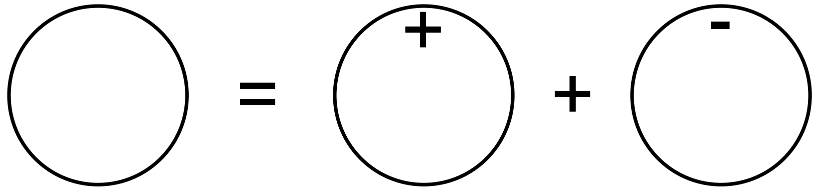


$$\langle \bigcirc \rangle = q + q^{-1}$$

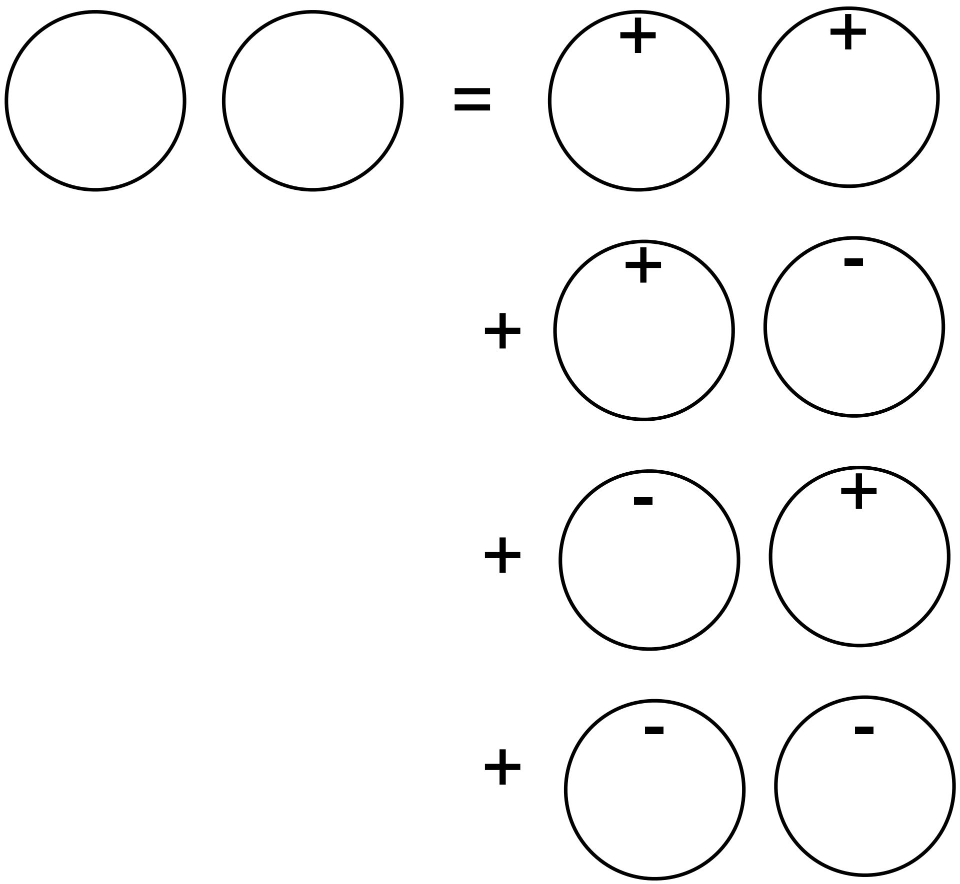
$$\langle K \bigcirc \rangle = (q + q^{-1}) \langle K \rangle$$

Use enhanced states by labeling each loop with
+1 or -1.





$$= q + q^{-1}$$



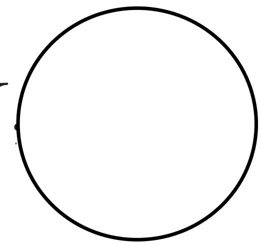
$$= qq + qq^{-1} + q^{-1}q + q^{-1}q^{-1}$$

$$= qq + 2 + (qq)^{-1}$$

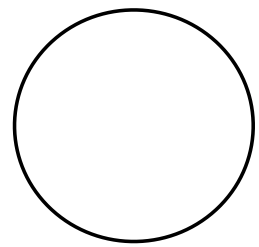
$$= (q + q^{-1})^2$$

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

$$\langle K \rangle = \sum_{s} (-1)^{n_B(s)} q^{j(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

$n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

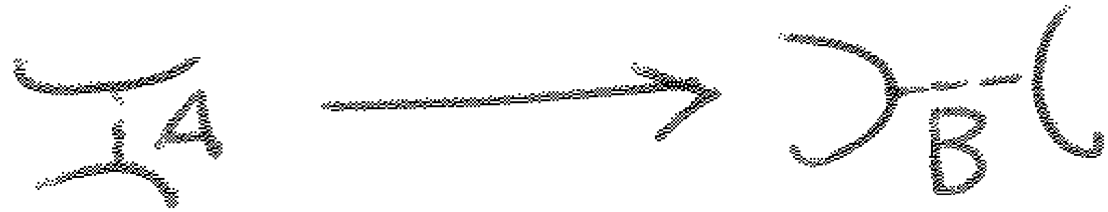
Wanted: differential acting in the form

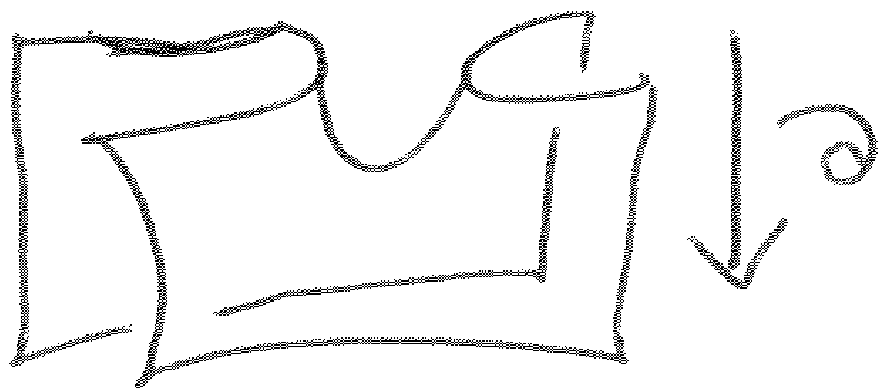
$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.

∂ :





The differential should increase the homological grading i by 1 and leave fixed the quantum grading j.

Then we would have

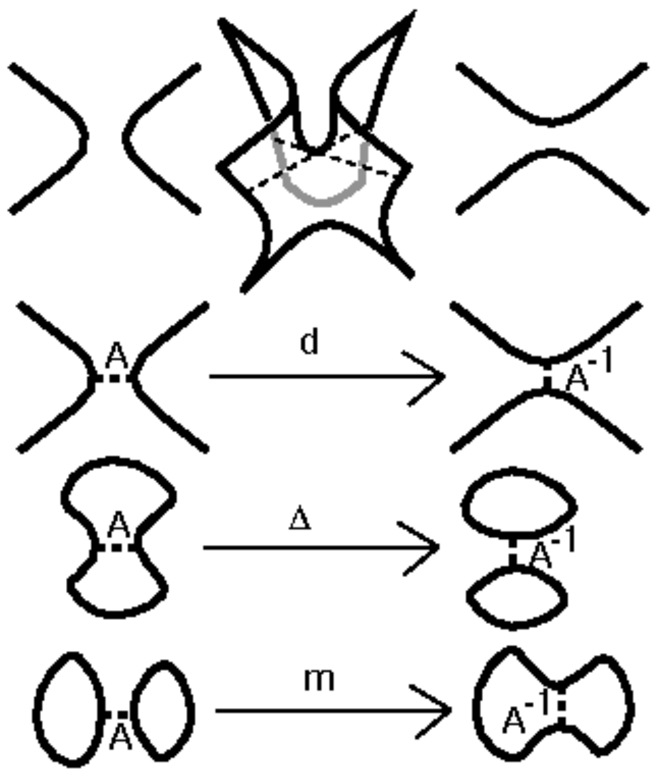
$$\langle K\rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet\, j})$$

$$\chi(H(\mathcal{C}^{\bullet\, j})) = \chi(\mathcal{C}^{\bullet\, j})$$

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

The boundary is a sum of partial differentials corresponding to resmoothings on the states.

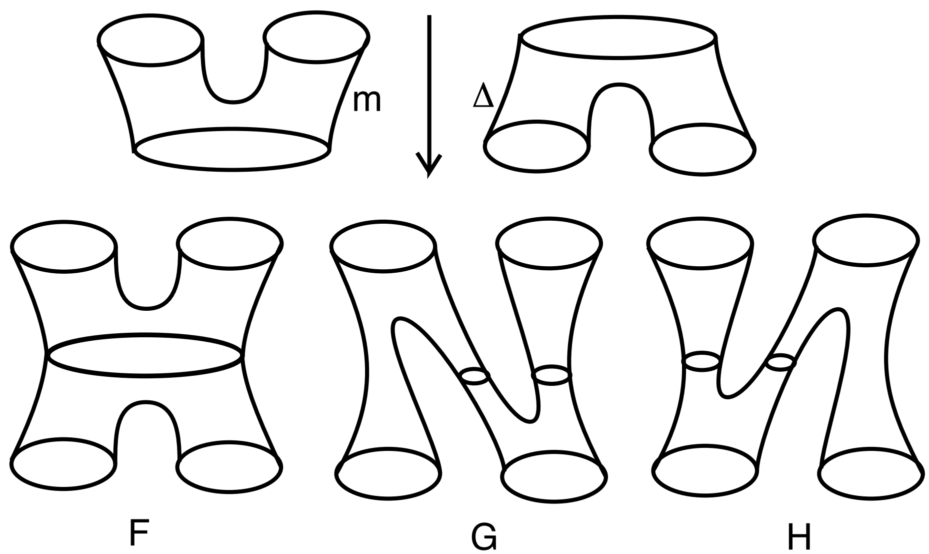


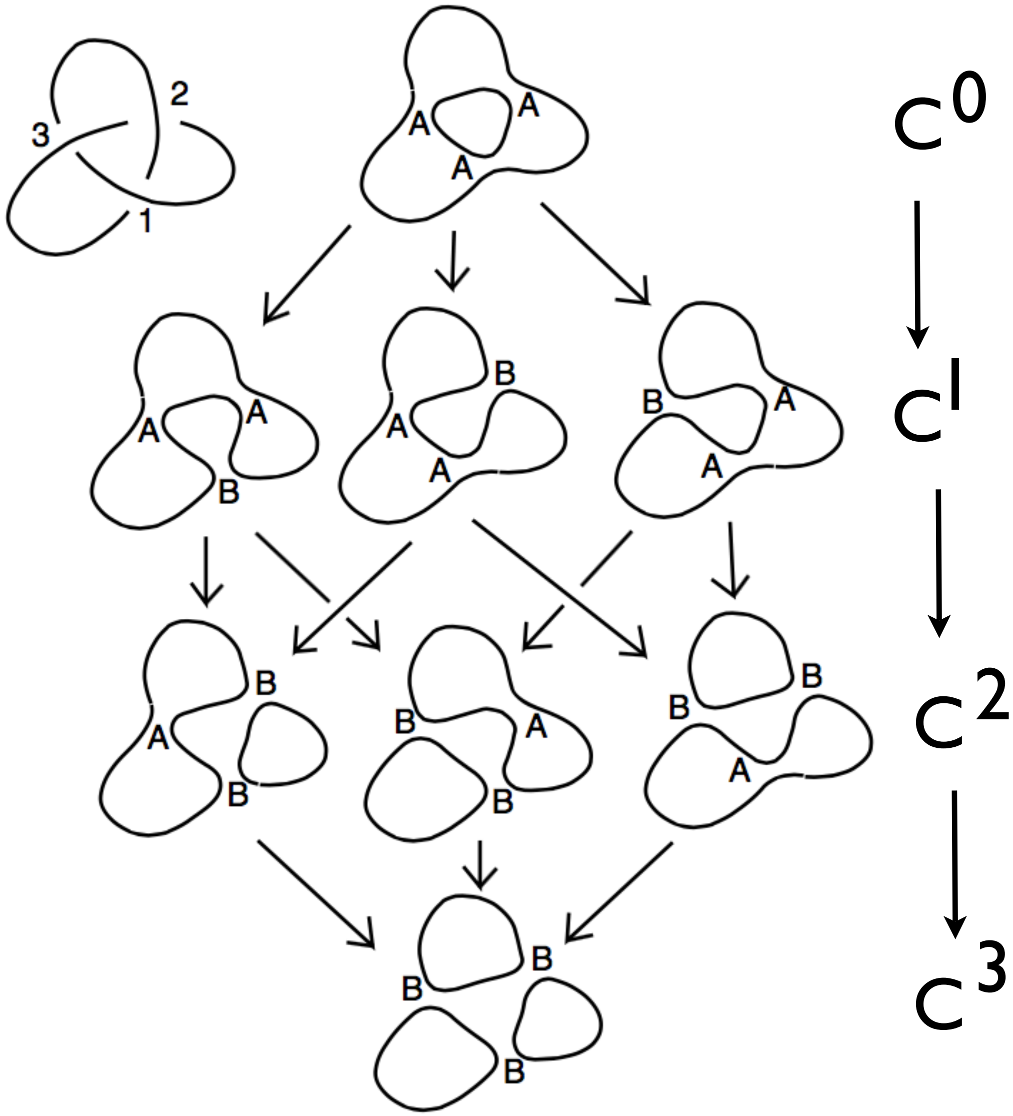
$\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1.$

$X^2 = 0$

Proposition. The partial differentials $\partial_\tau(s)$ are uniquely determined by the condition that $j(s') = j(s)$ for all s' involved in the action of the partial differential on the enhanced state s. This unique form of the partial differential can be described by the following structures of multiplication and comultiplication on the algebra A = $k[X]/(X^2)$ where $k = Z/2Z$ for mod-2 coefficients, or $k = Z$ for integral coefficients.

1. The element 1 is a multiplicative unit and $X^2 = 0$.
2. $\Delta(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$.





Bracket states
form a
category that
assembles
itself into a chain
complex.

Levels in the chain
complex are
direct sums of modules
corresponding to
states with a constant
number of B
smoothings.

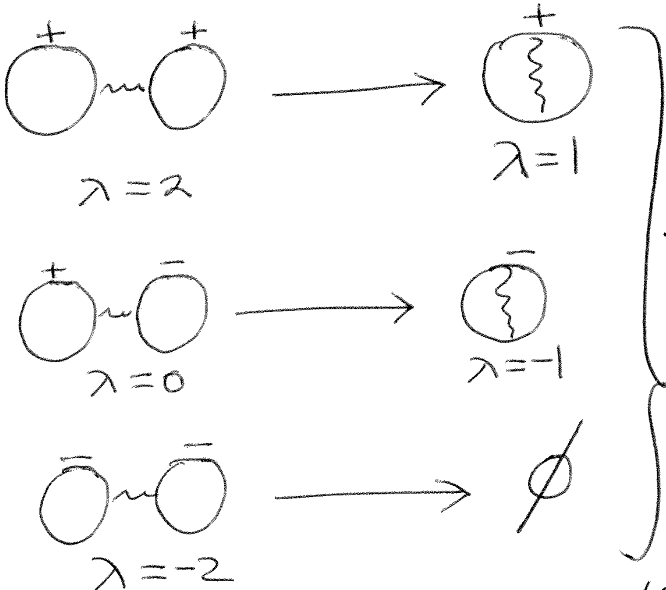
$\partial : C^{i,j} \longrightarrow C^{i+1,j}$

For j to remain fixed, we need $\boxed{\lambda \overset{\partial}{\longmapsto} \lambda - 1}$

where

$$\lambda(s) = \#(+1 \text{ loops}) - \#(-1 \text{ loops}).$$

The ∂ is determined by this condition.



$x \equiv -$

$1 \equiv +$

Then

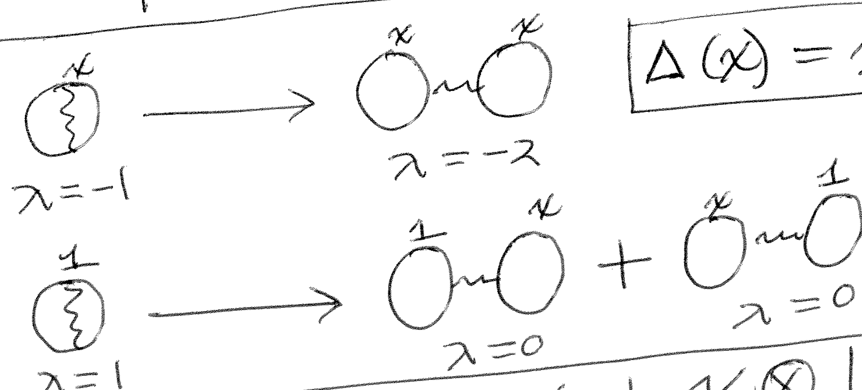
$x^2 = 0$

$1 \times 1 = 1$

$1 \times x = x \times 1 = x$

So far $V = \mathbb{Z}[x]/(x^2)$.

For product $m : V \otimes V \longrightarrow V$.

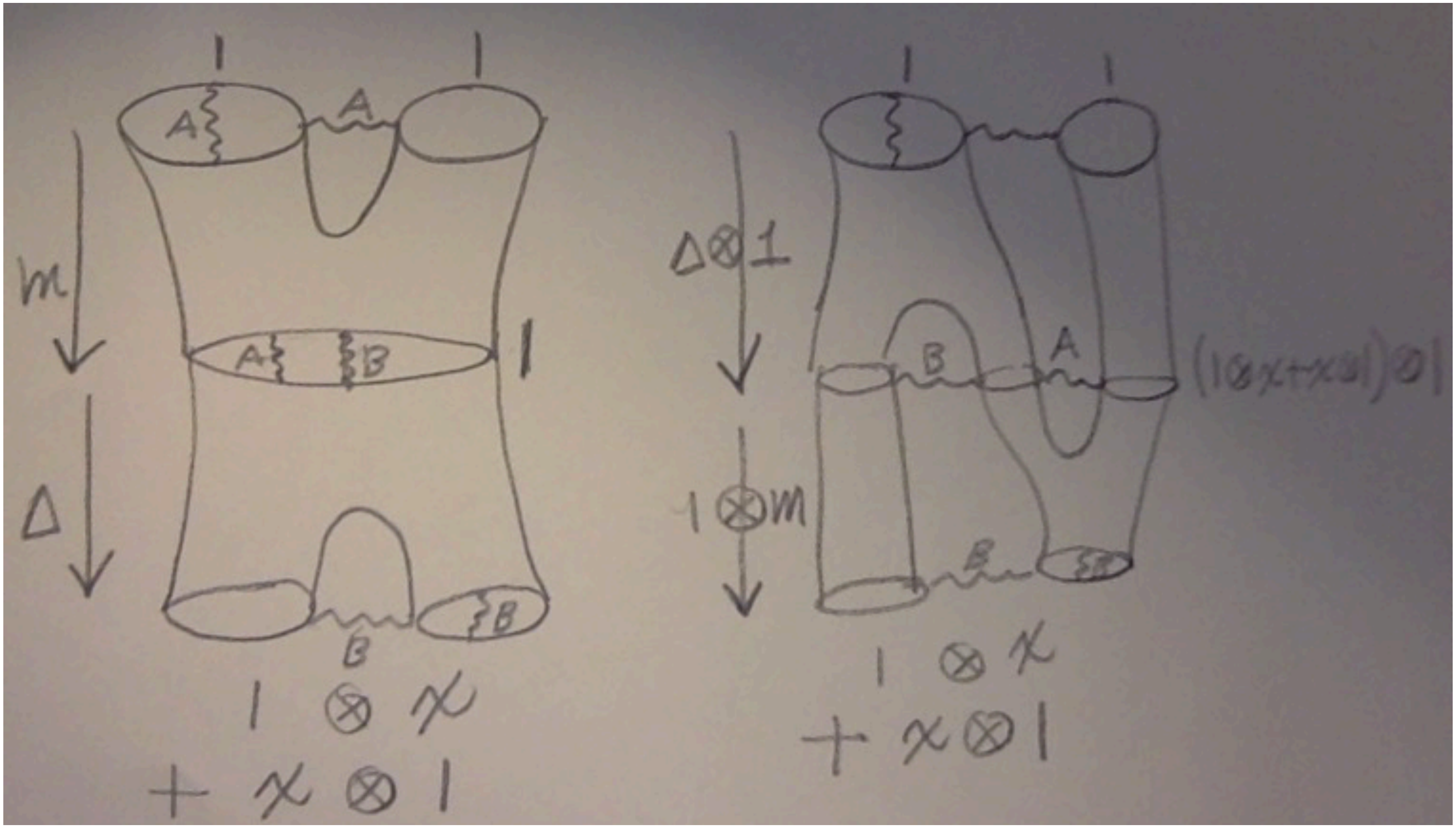


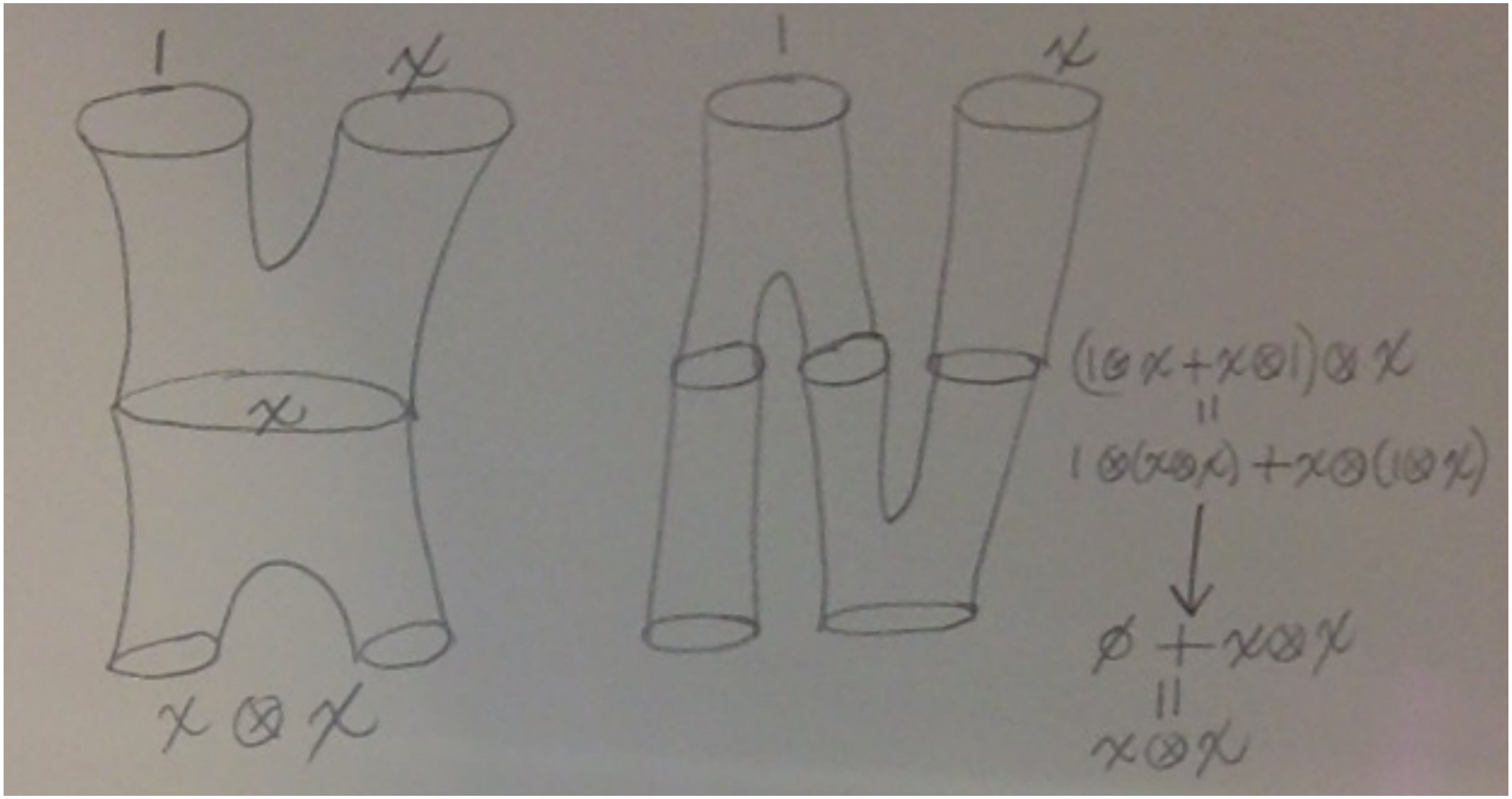
$\Delta(x) = x \otimes x$

$\Delta(1) = 1 \otimes x + x \otimes 1$

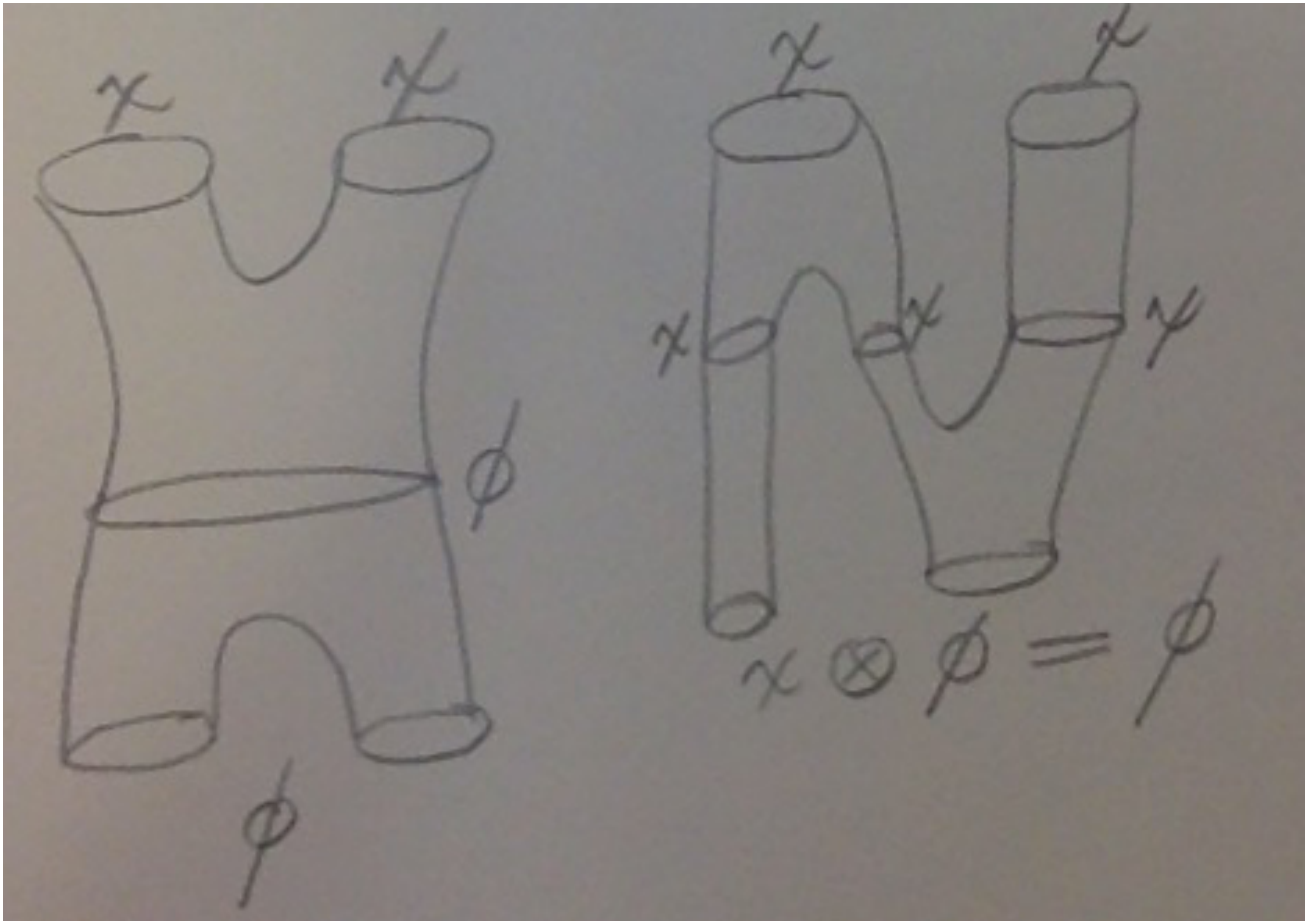
Enhanced States
Plus
Boundary
Requirement
Yields
Frobenius Algebra.

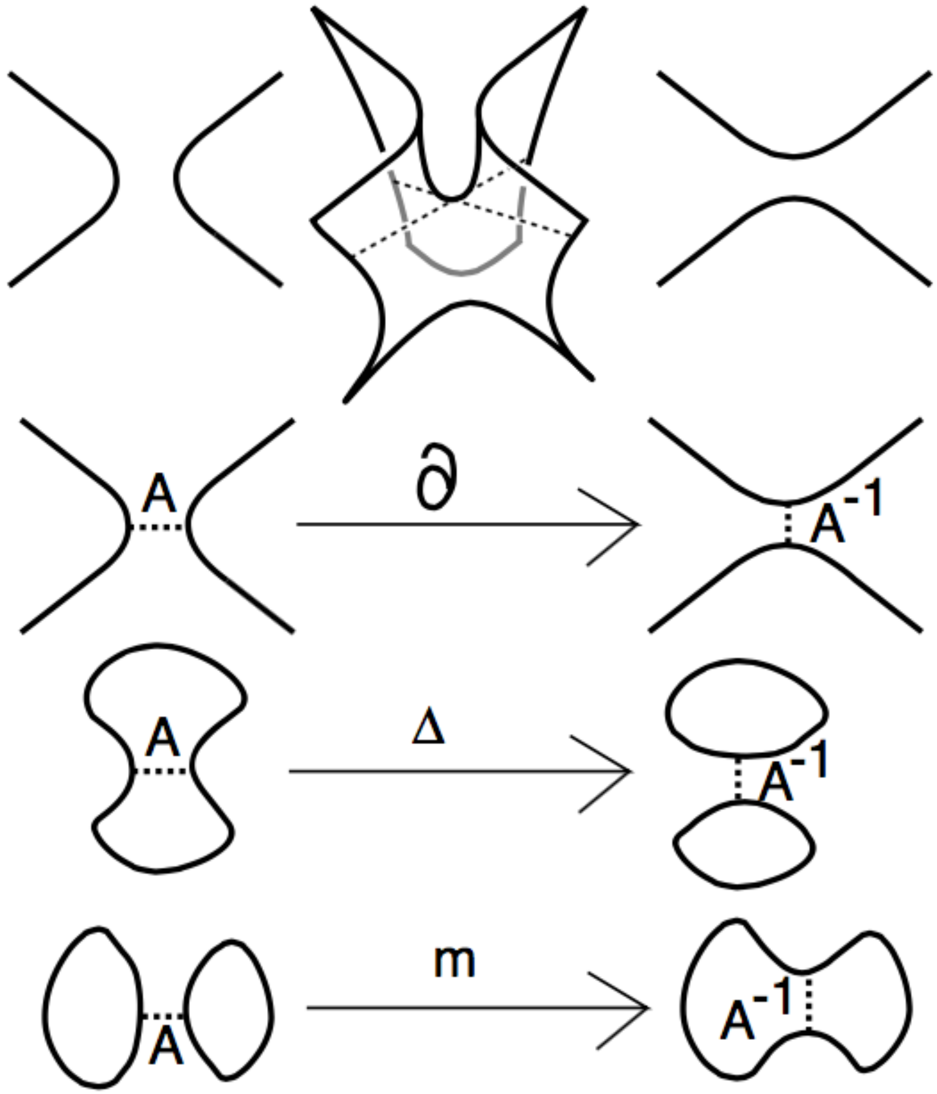
Checking Order Compatibility



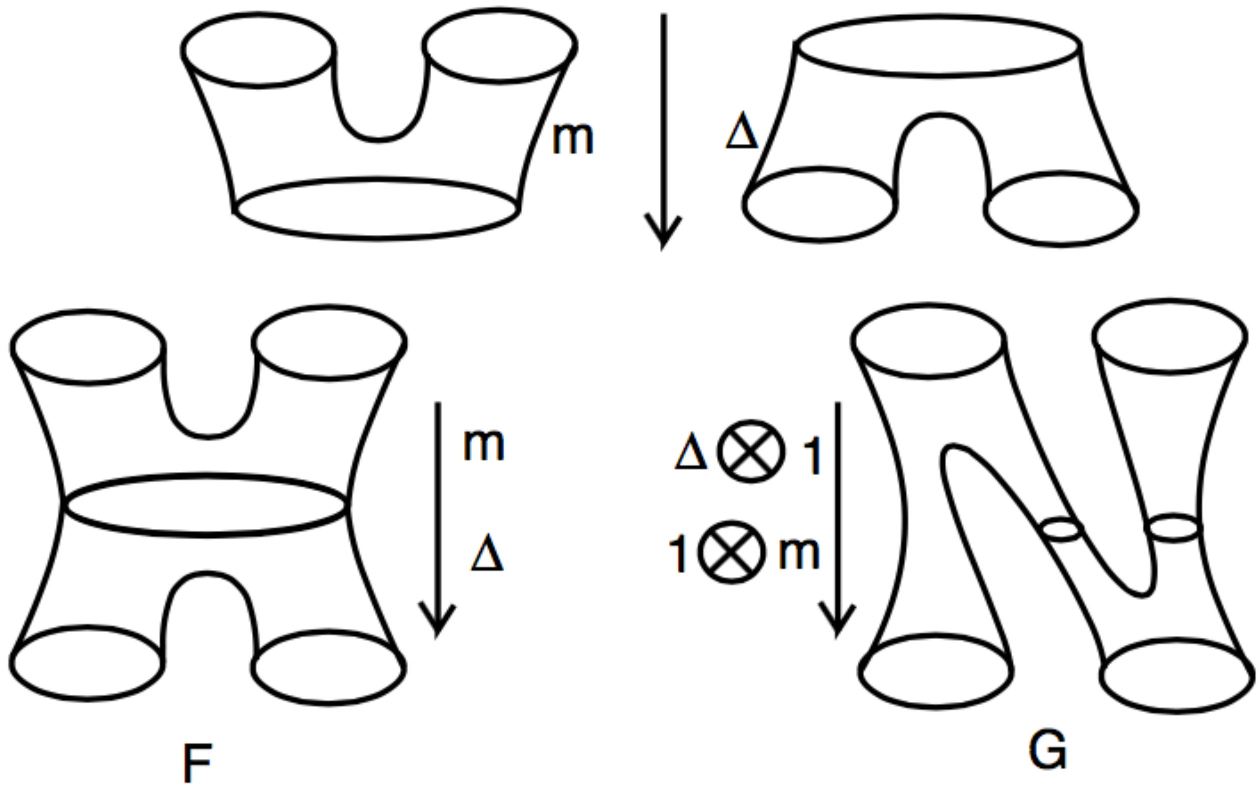


1
x
x
x ⊗ x
1
x
(1⊗x + x⊗1)⊗x
=
1⊗(x⊗x) + x⊗(1⊗x)
∅ + x⊗x
=
x⊗x

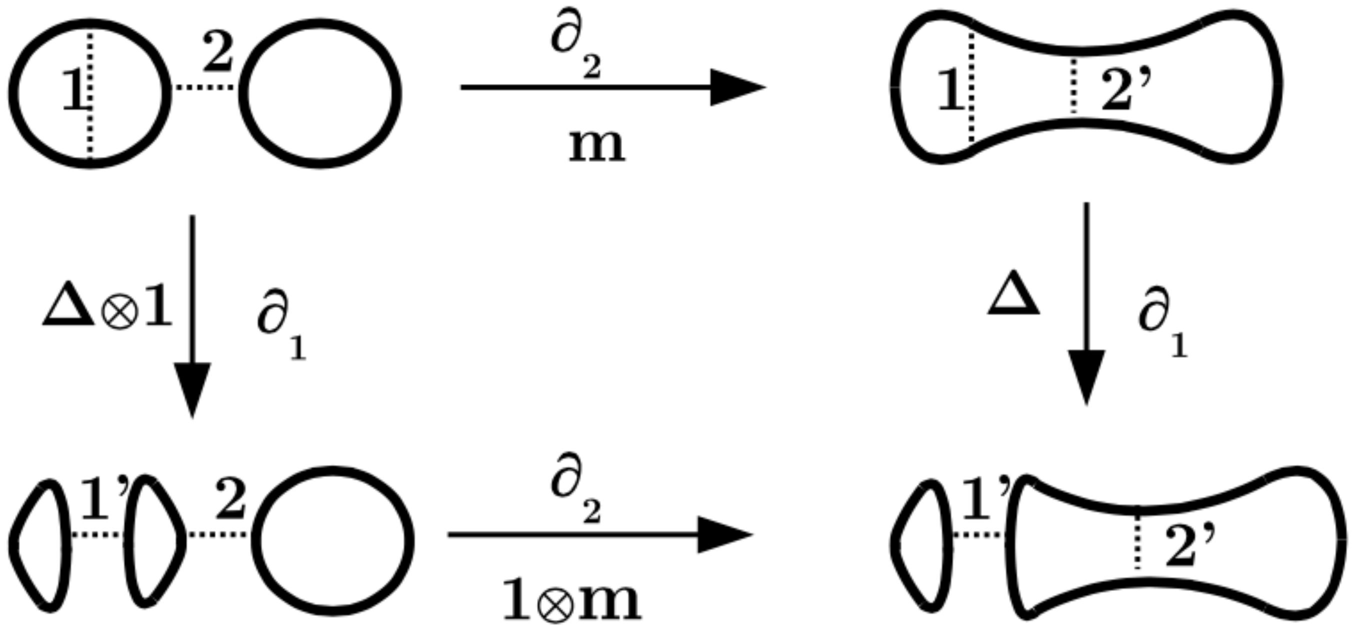

x ⊗ φ = φ



A
∂
A-1
A
Δ
A-1
A
m
A-1



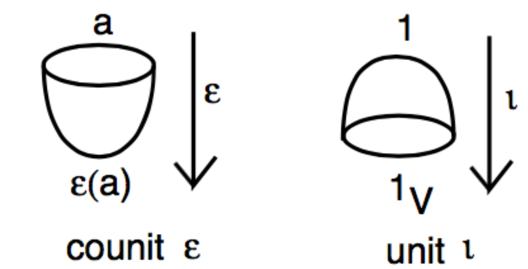
m
Δ
m
Δ
Δ ⊗ 1
1 ⊗ m
F
G



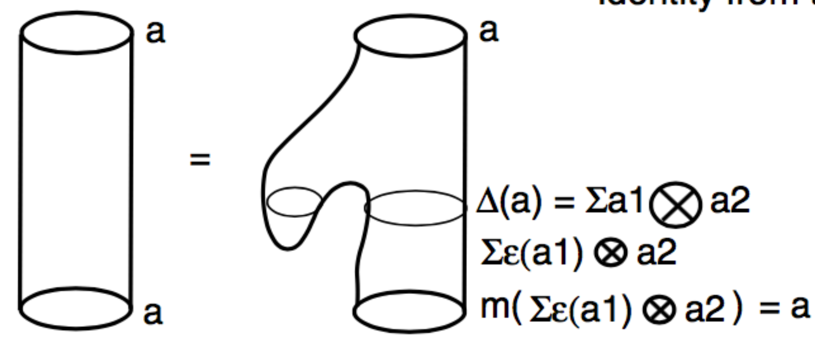
$$\partial_2\partial_1 = (1\otimes m)(\Delta\otimes 1)$$

$$\partial_1\partial_2 = (\Delta)(m)$$

$$\partial_1\partial_2 = \partial_2\partial_1$$



Evaluations at successive levels.
Identity from topology.

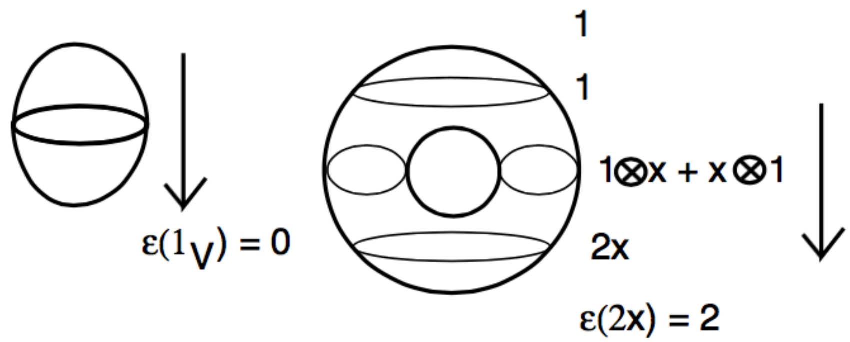


Using special case of a=1, we obtain:

$m(\varepsilon(1)\otimes x + \varepsilon(x)\otimes 1) = 1$

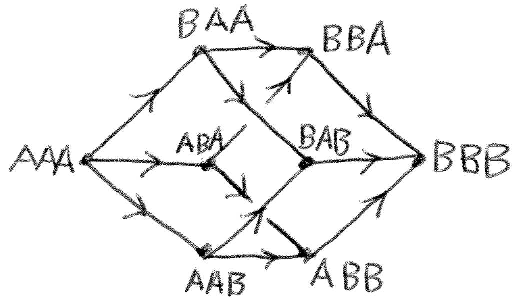
$\Longrightarrow \varepsilon(1)x + \varepsilon(x)1 = 1$

$\Longrightarrow \varepsilon(1) = 0$
$\varepsilon(x) = 1$

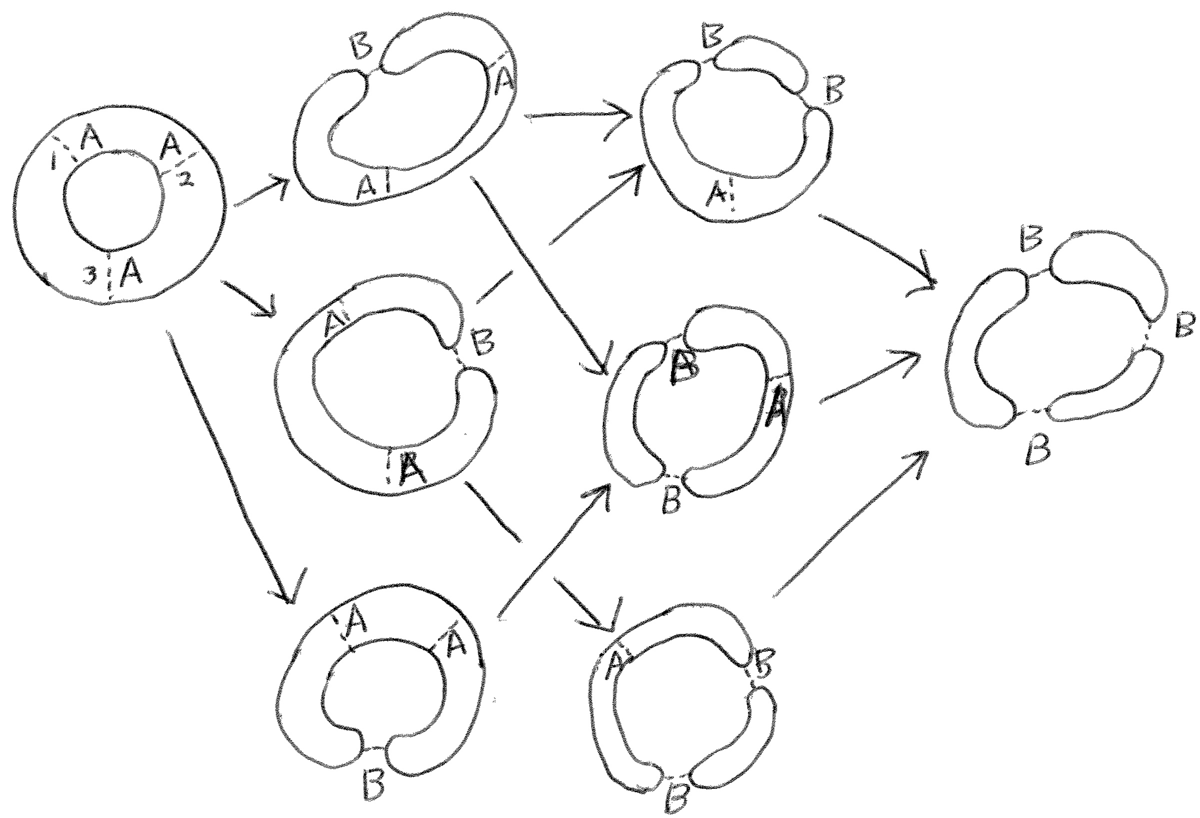


We have arrived at the Frobenius algebra, but there is still work to be done to see the invariance under ambient isotopy of knots and links.

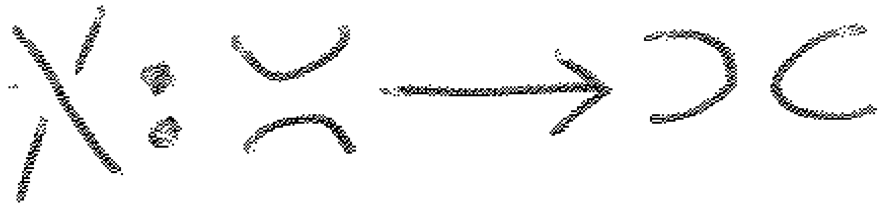
Cubism Again



$\Longleftrightarrow (A \to B)^3$



Categorification and the Morse Dream

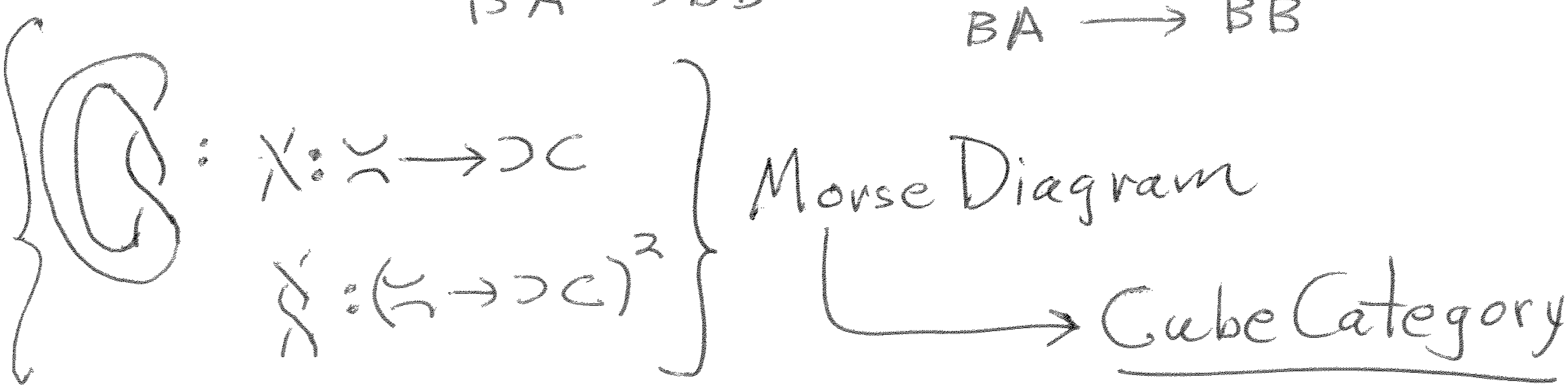


(flattening a higher category)

$$(A\to B)^2 = (A\to B)(A\longrightarrow B) = A(A\to B)\longrightarrow B(A\to B)$$

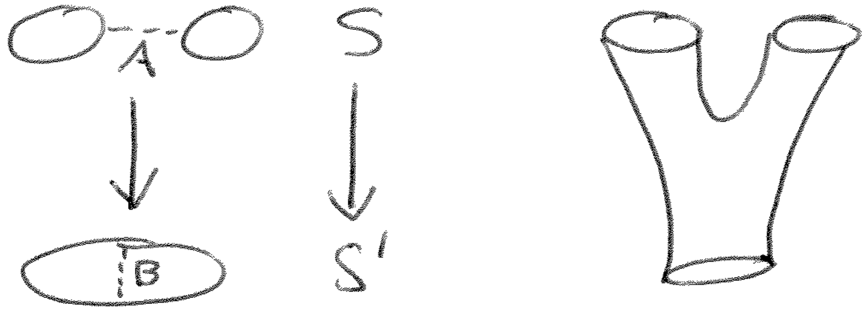
$$= (AA\to AB)\longrightarrow(BA\to BB)$$

$$= \begin{array}{ccc} & AA \to AB & \\ & \downarrow & \\ & BA \to BB & \end{array} = \begin{array}{ccc} AA & \longrightarrow & AB \\ \downarrow & & \downarrow \\ BA & \longrightarrow & BB \end{array}$$

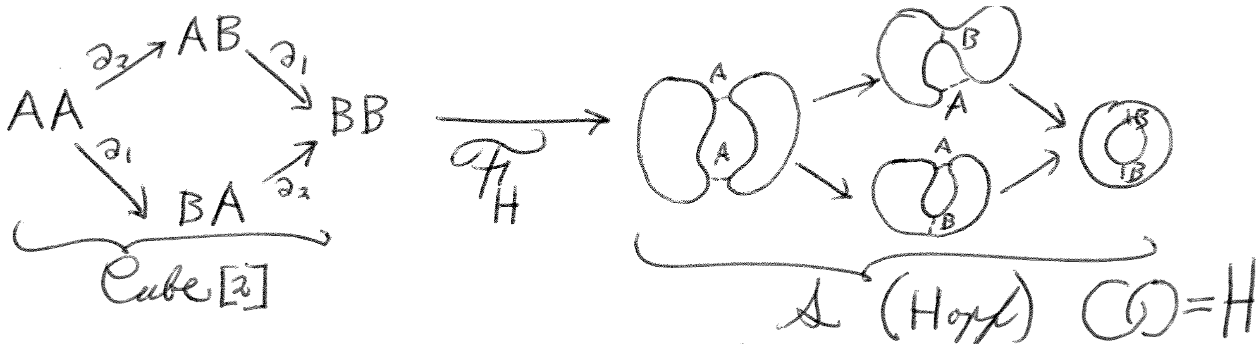


We have category $\mathcal{S}(K)$ whose objects are states S of K + morphisms gen by arrows $S \longrightarrow S'$, $b(S)+1 = b(S')$.

Regard the arrow as a surface cobordism.



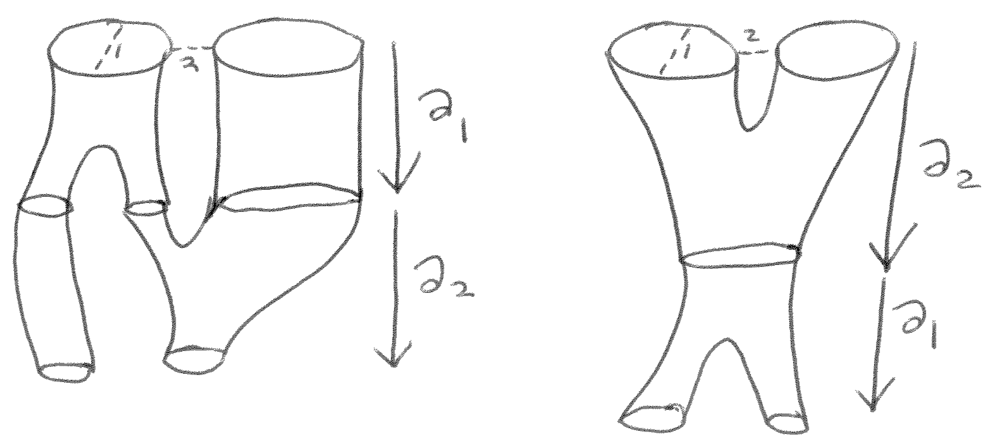
Two surface cobs are = as morphisms iff the corresponding surfaces are homeomorphic.



$$\mathcal{S}(K) = \mathcal{F}_K\left(\text{Cube}[c(K)]\right)$$

and $\mathcal{F}$ extends to a *functor* from the cube category to the category $\mathcal{S}(K)$.

This means that all relevant squares commute. e.g.



We make an abstract analogue of a chain complex from $\mathcal{S}(K)$ by extending to an additive category with dir sums.

$$A_1, \ldots, A_n \rightsquigarrow A = \bigoplus_{i=1}^{n} A_i$$

$$f : A \longrightarrow B, \quad B = \bigoplus_{j=1}^{m} B_j$$

$$f = (f_{ij}), \quad f_{ij} : A_i \longrightarrow B_j .$$

$$g : B \longrightarrow C$$

$$A_i \xrightarrow{f_{ik}} B_k \xrightarrow{g_{kj}} C_j$$

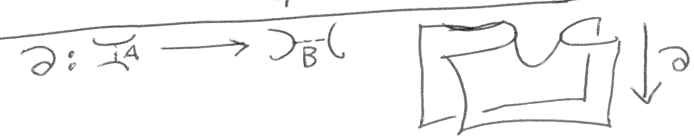
$$(g \circ f)_{ij} = \sum_k g_{kj} \circ f_{ik}$$

$$S \in \mathcal{O}bj(\mathcal{S}(K))$$

$$\mathcal{C}^i(K) = \bigoplus_{\substack{S \in \mathcal{O}bj(\mathcal{S}(K)) \\ b(S) = i}} S$$

$$\partial : \mathcal{C}^i(K) \longrightarrow \mathcal{C}^{i+1}(K)$$

$$\partial = \sum_{k=1}^{c(K)} \pm \partial_k$$



Dror's Canopoly

An abstract categorical analog of a chain complex. That can be taken up to chain homotopy. The maps are additive combinations of surface cobordisms.

We say $f \sim g$ iff $\exists H : \mathcal{C} \to \mathcal{C}'$ s.t. $\partial H + H\partial = f - g$.

Work Mod 2.

Categorical Chain Homotopy

Question: What is least equiv reln on $\mathcal{CS}(K)$ s.t. $[\mathcal{CS}(K)]$ (= chain homotopy equiv class of ↑) is invar under RM's ?

We examine this question as though we had not seen the Frobenius algebra.

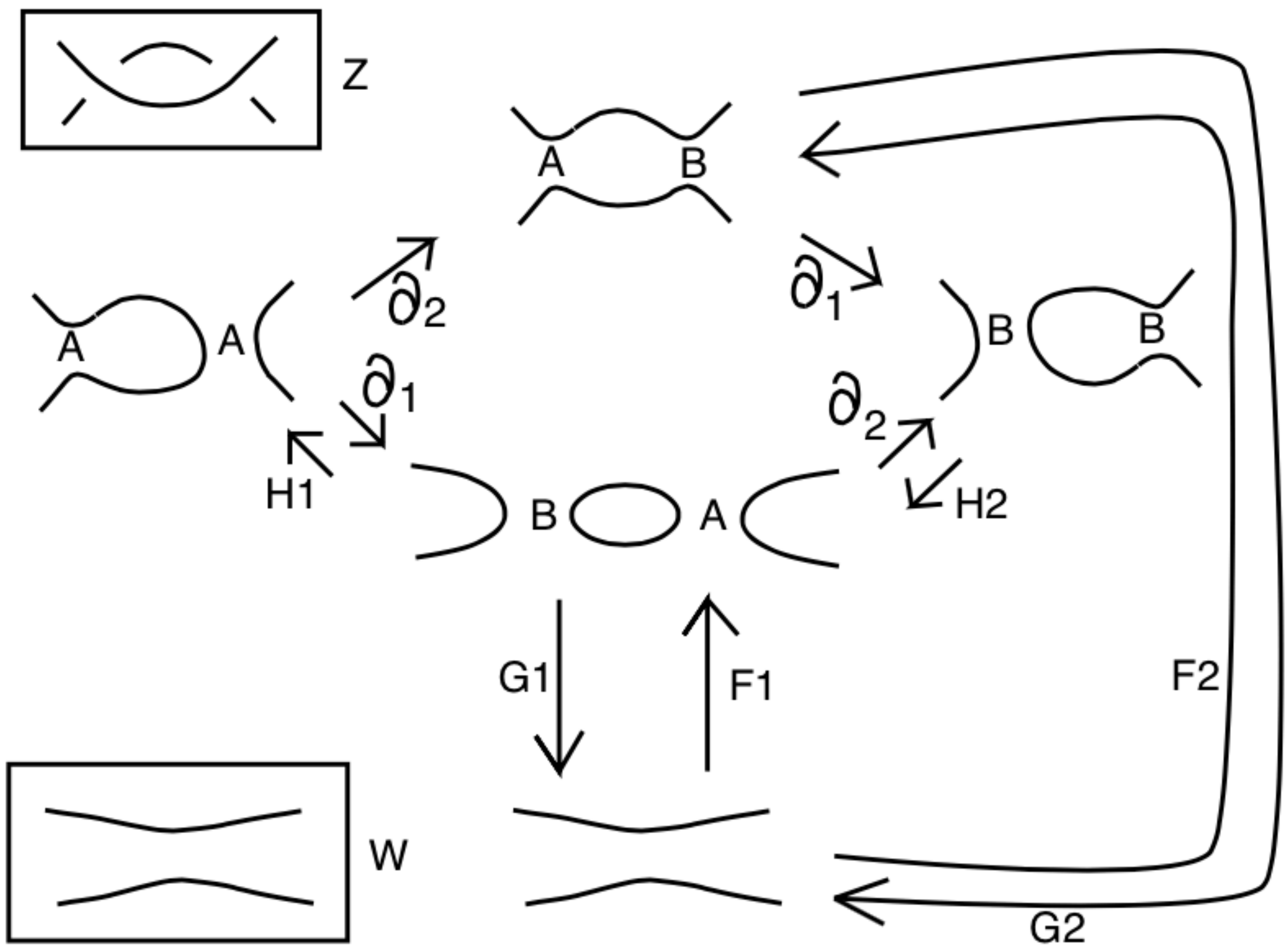


Figure 11: **Complexes for Second Reidemeister Move**

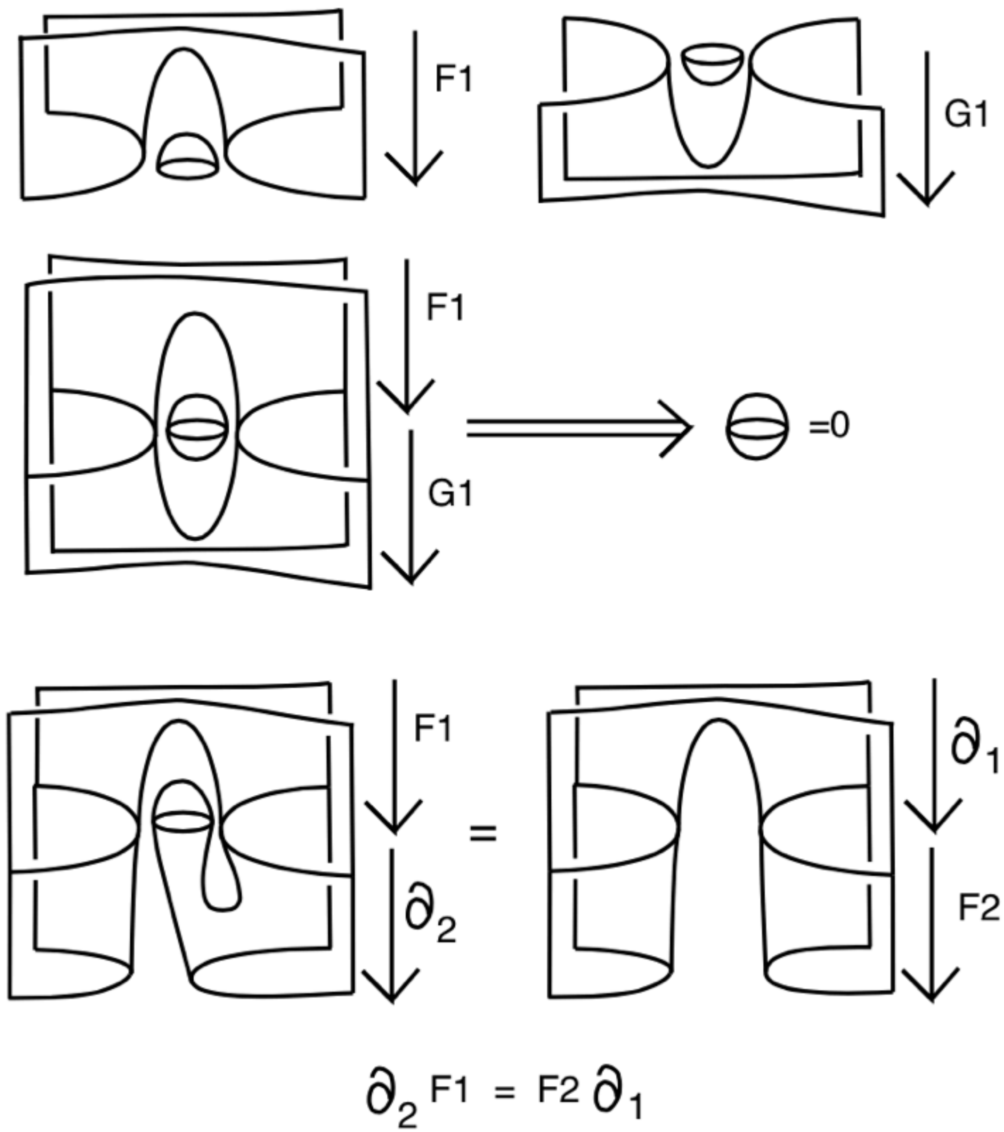


Figure 12: **Cobordism Compositions for Second Reidemeister Move**

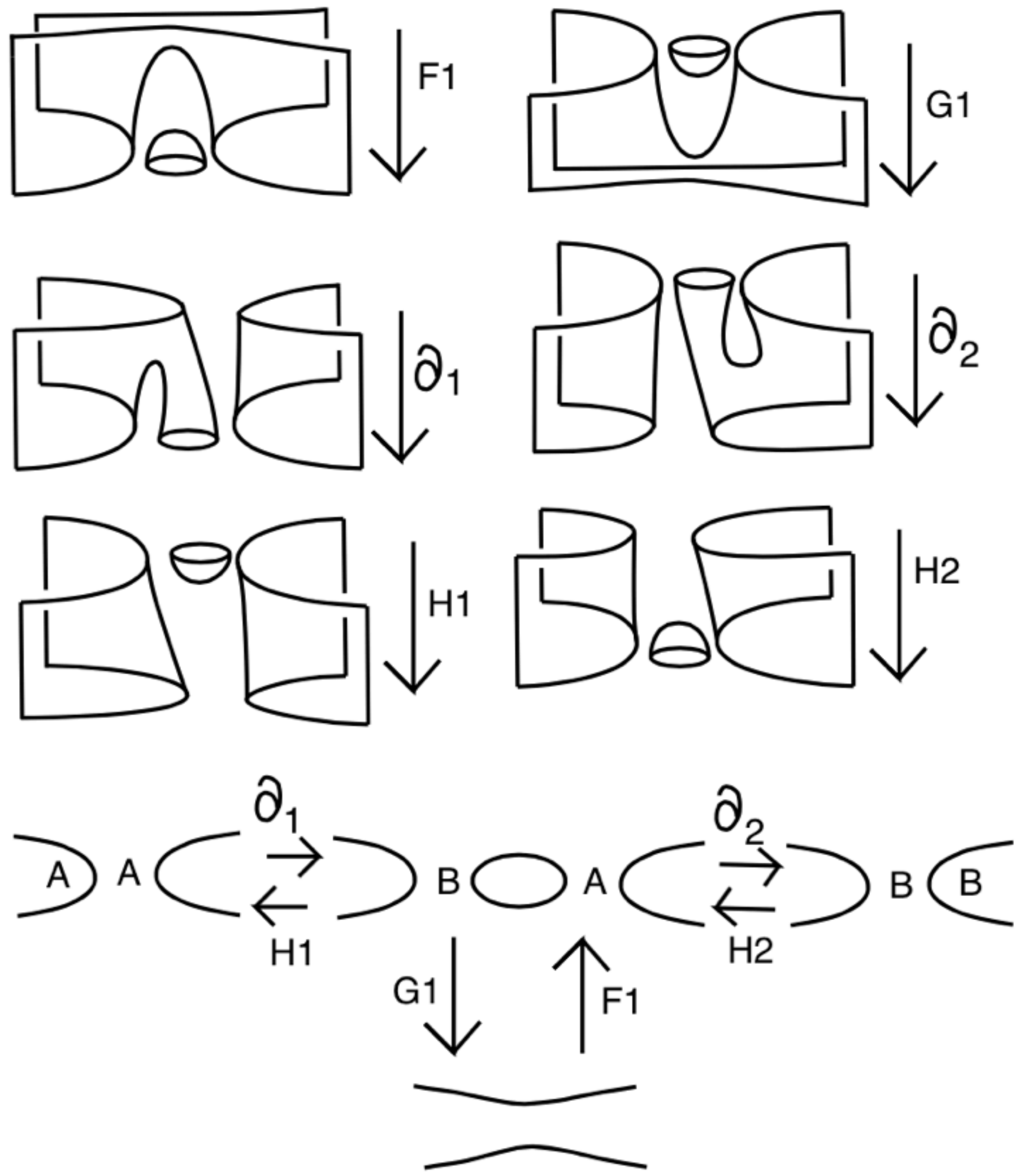


Figure 13: **Preparation for Homotopy for Second Reidemeister Move**

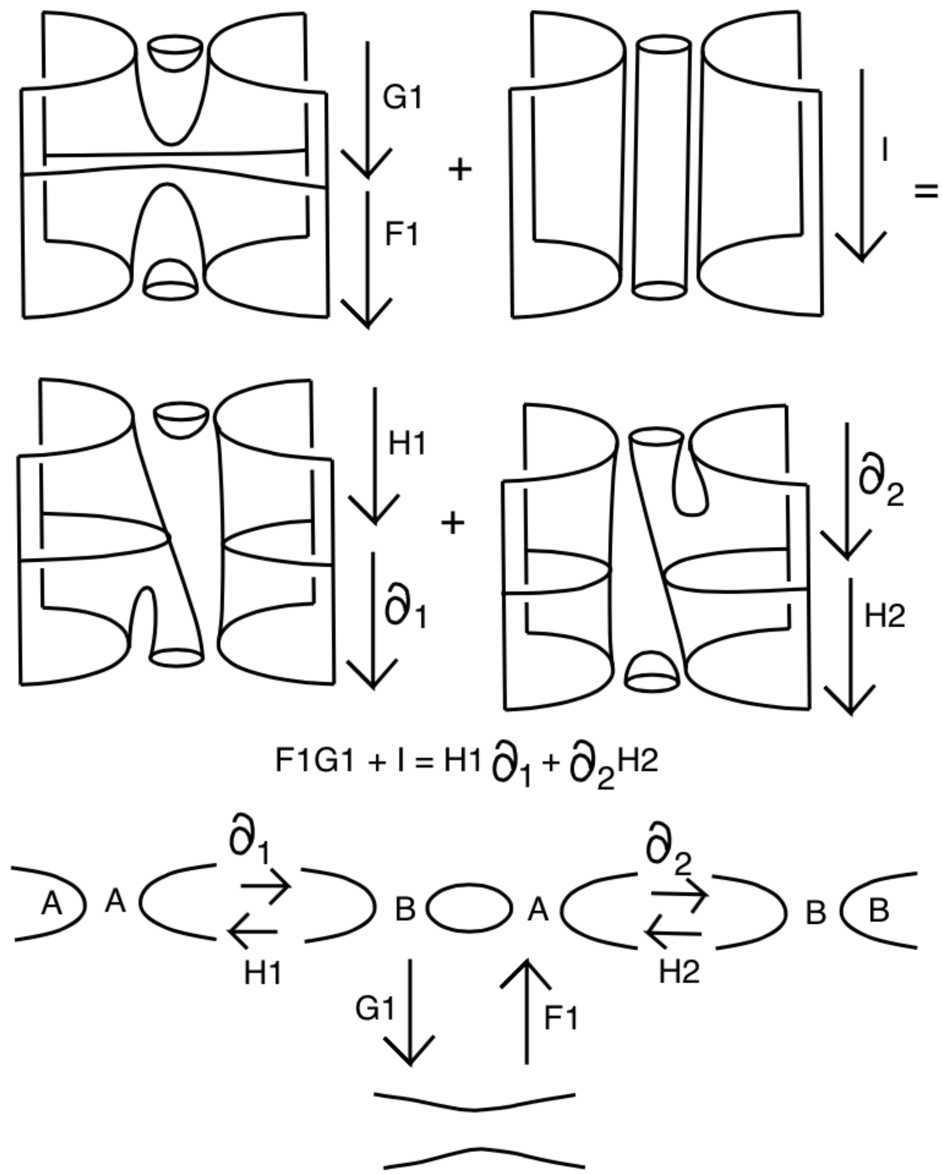


Figure 14: **Homotopy for Second Reidemeister Move**

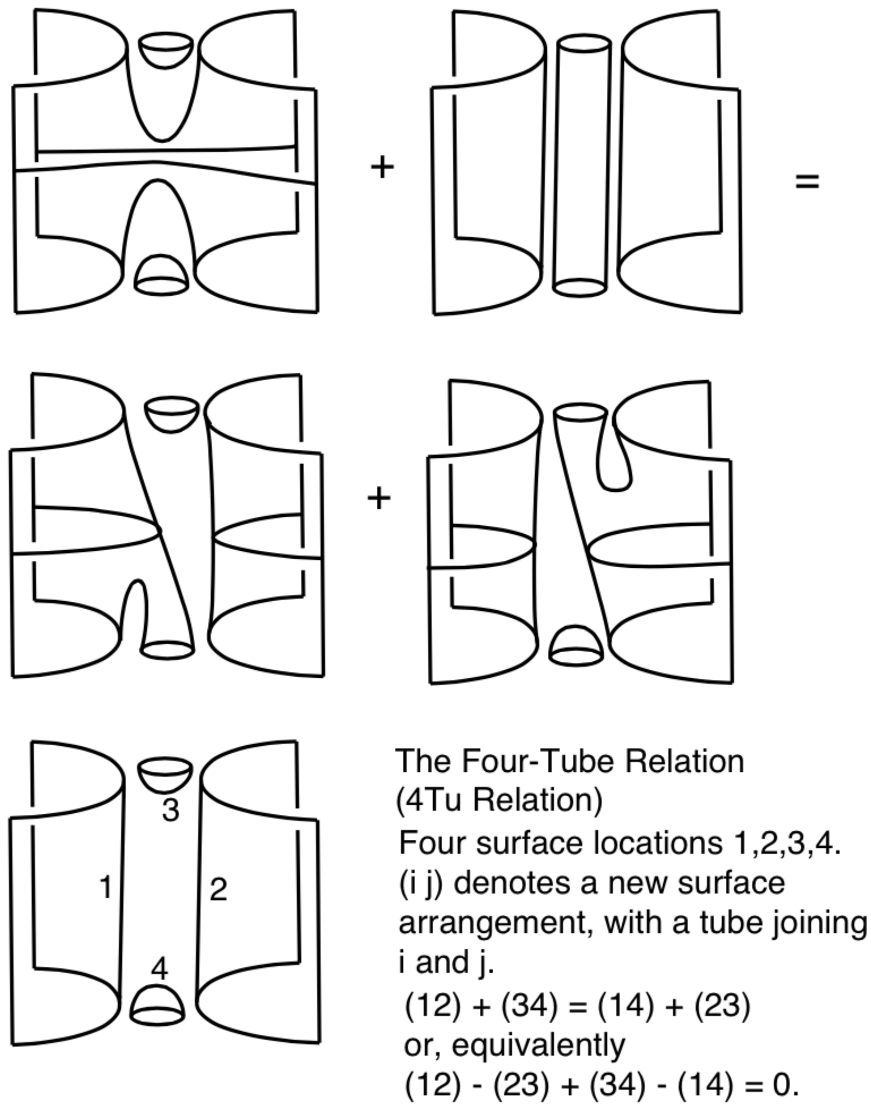
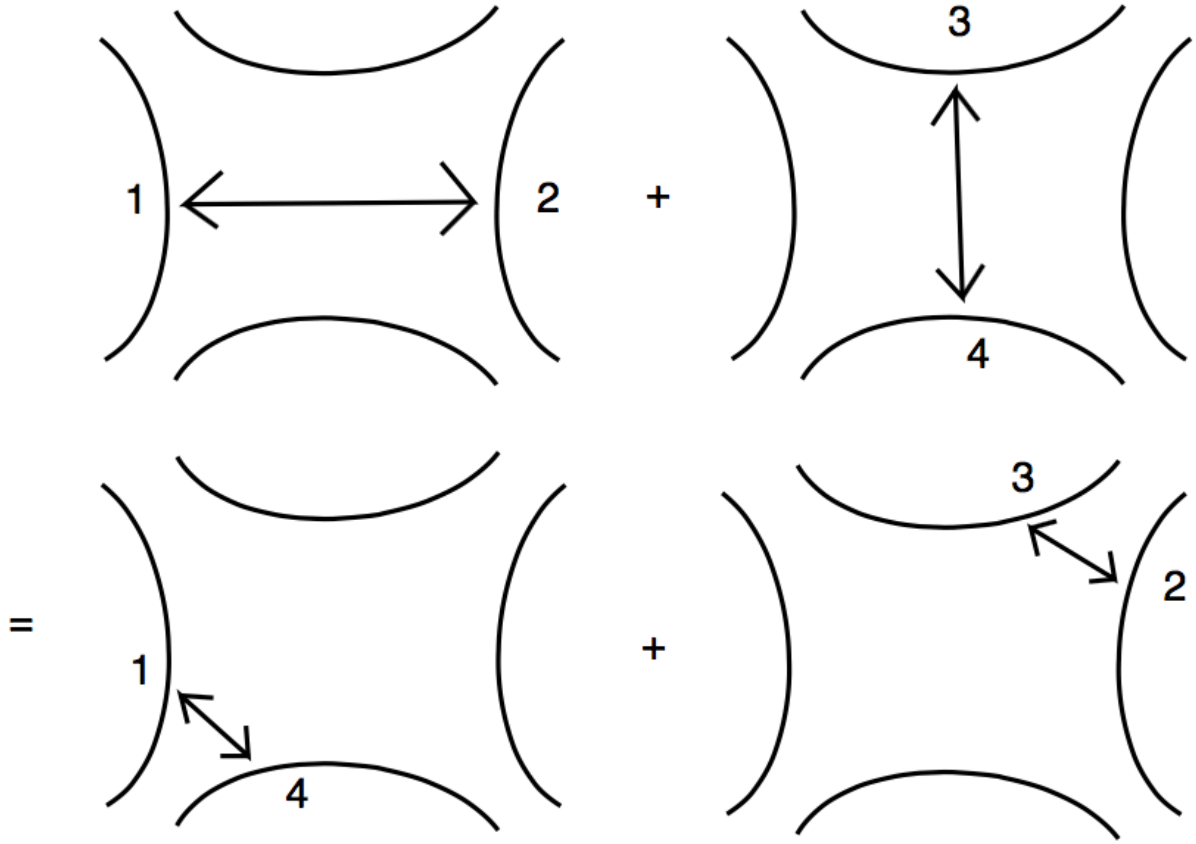
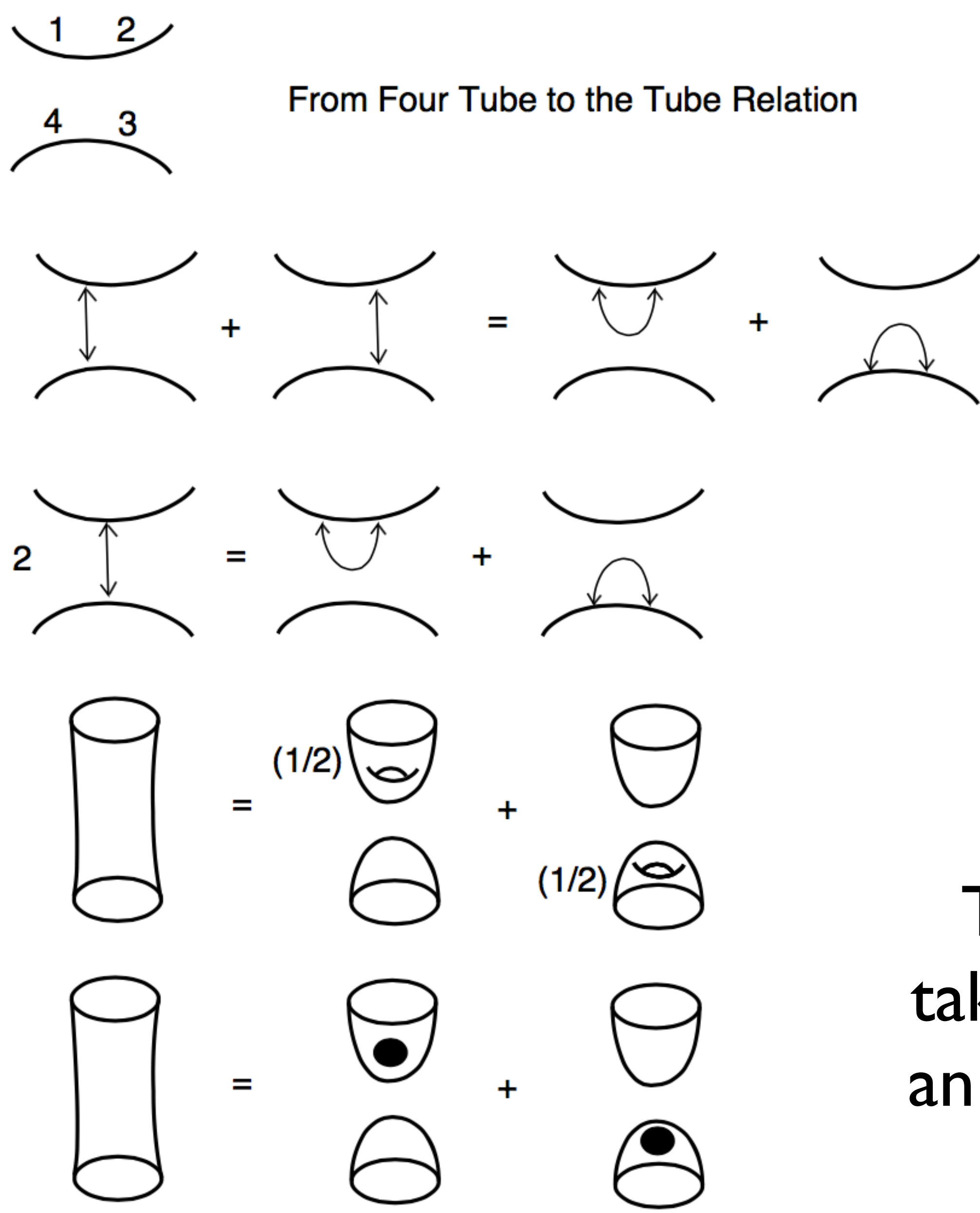


Figure 15: **Four-Tube Relation From Homotopy**

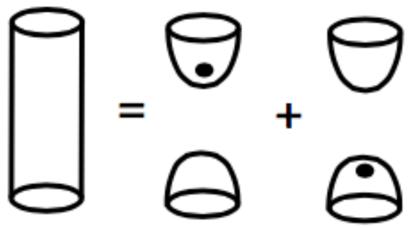


1234 - 1234 +1234 - 1234 = 0

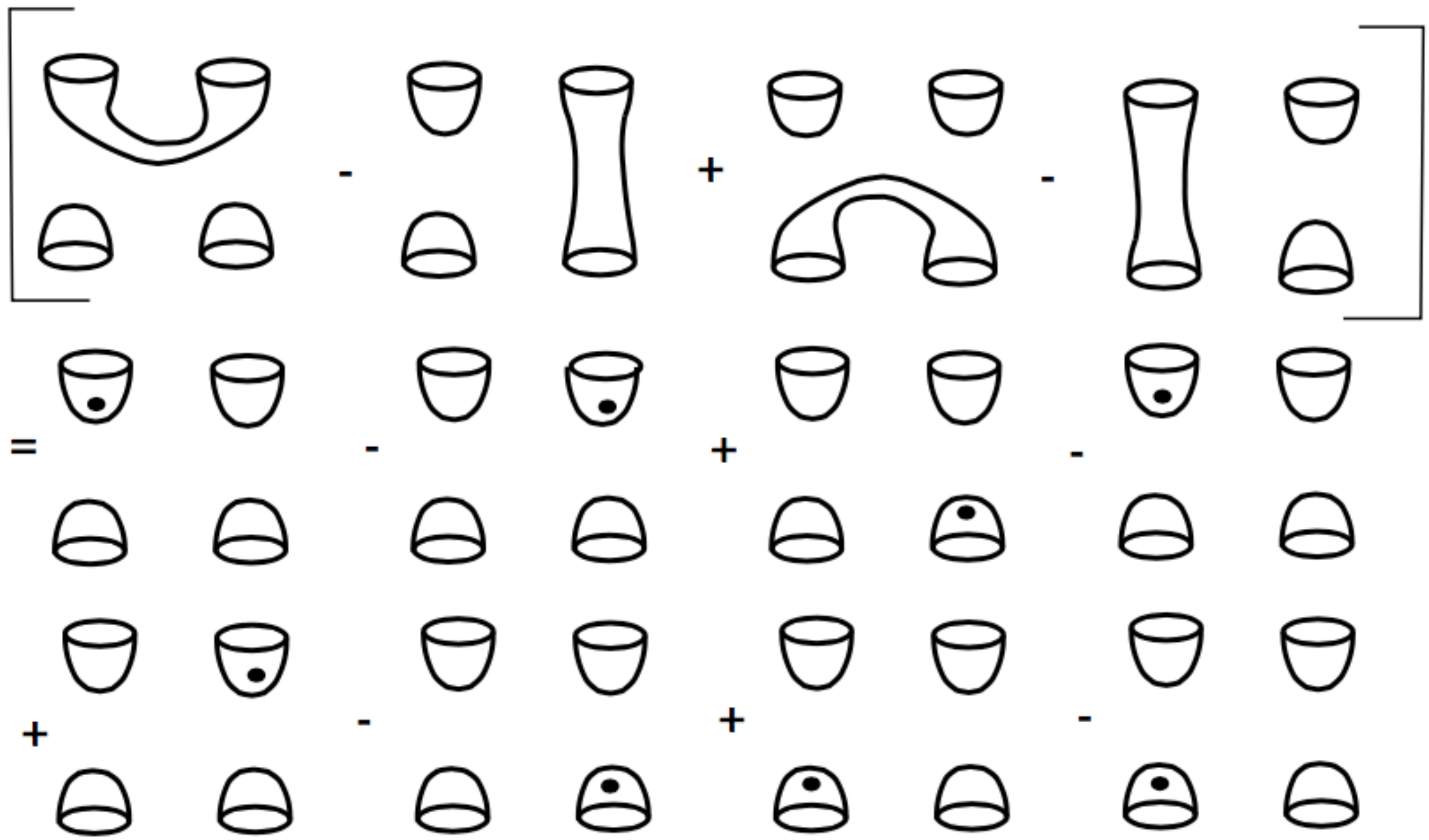
Schematic Four-Tube Relation



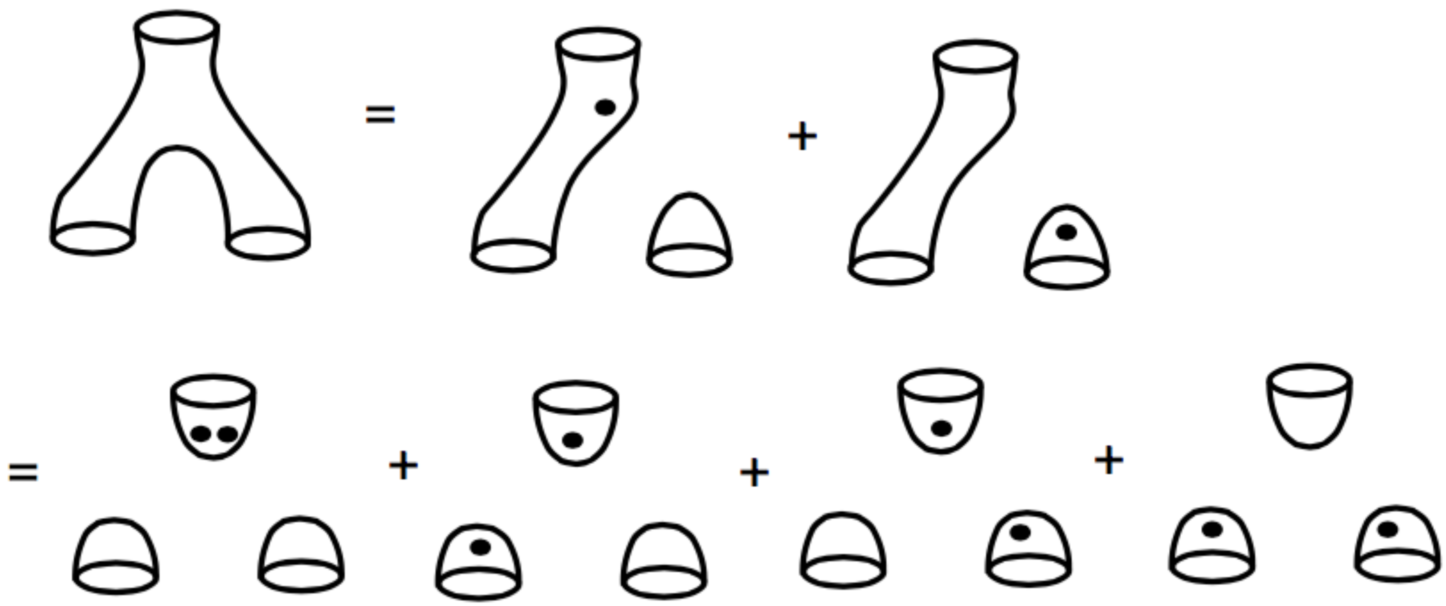
The dot can be taken to represent an algebra element x.



The Tube Relation implies the
Four Tube Relation.

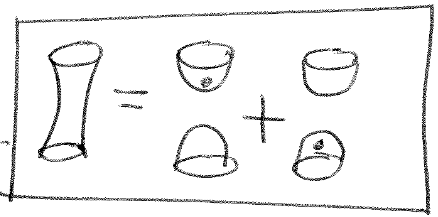


= 0.

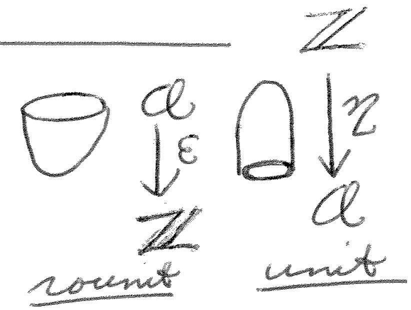


Coproduct via the Tube-Relation

From 4Tu to Frobenius Algebra



Tube Relation



Now find algebra $\mathcal{A}$

$\bullet \equiv x \in \mathcal{A}$, $\alpha \in \mathcal{A}$

$$\alpha = \varepsilon(\alpha x)1 + \varepsilon(\alpha)x$$

$$\Rightarrow \quad \begin{cases} x = \varepsilon(x^2)1 + \varepsilon(x)x \\ 1 = \varepsilon(x)1 + \varepsilon(1)x \end{cases}$$

$$\varepsilon(x) = 1, \quad \varepsilon(x^2) = 0, \quad \varepsilon(1) = 0$$

$$x^2 = \varepsilon(x^3)1 + \varepsilon(x^2)x$$

$$\Rightarrow x^2 = k1, \quad k \in \mathbb{Z}$$

$$x^2 = k$$

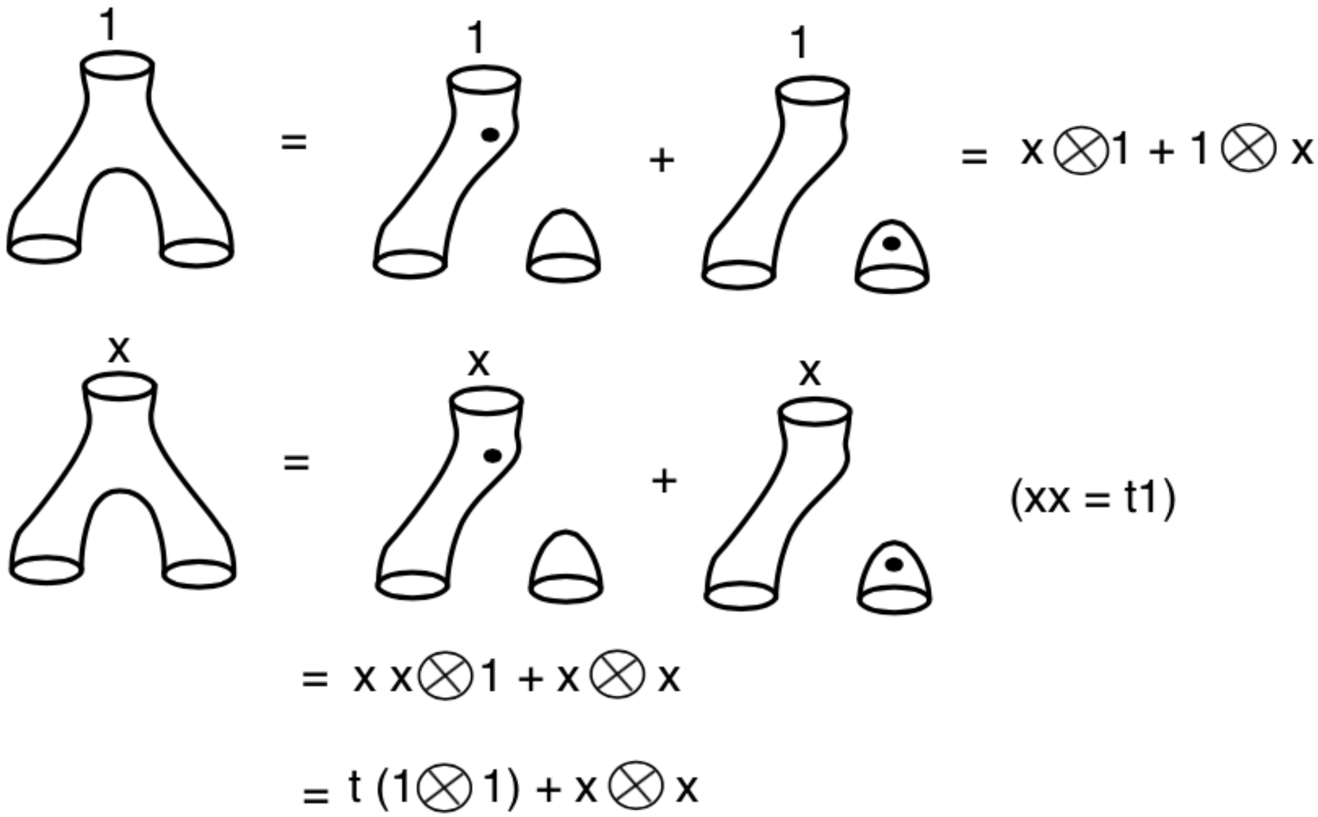


Figure 20: **Coproducts of** 1 **and** x **Via Tube-Cutting Relation**

Algebra from 4Tu - Guaranteed to Produce Link Homology

$\mathcal{A} = \mathbb{Z}[x]/(x^2 - k)$

$\varepsilon(x) = 1, \quad \varepsilon(1) = 0$

$\Delta(1) = 1 \otimes x + x \otimes 1$

$\Delta(x) = k(1 \otimes 1) + x \otimes x$

$k = 0$: Khovanov

$k = 1$: Lee

Lee's Algebra

$$x^2 = 1,$$

$$\Delta(1) = 1 \otimes x + x \otimes 1,$$

$$\Delta(x) = x \otimes x + 1 \otimes 1,$$

$$\epsilon(x) = 1,$$

$$\epsilon(1) = 0.$$

This gives a link homology theory that is distinct from Khovanov homology. In this theory, the quantum grading j is not preseved, but we do have that

$$j(\partial(\alpha)) \geq j(\alpha)$$

for each chain α in the complex. This means that *one can use j to filter the chain complex for the Lee homology*. The result is a spectral sequence that starts from Khovanov homology and converges to Lee homology.

Lee's Algebra

$$\mathcal{A} = \mathbb{Q}[x]/(x^2-1)$$

$$\varepsilon(x) = 1, \quad \varepsilon(1) = 0$$

$$\Delta(1) = 1 \otimes x + x \otimes 1$$

$$\Delta(x) = (1 \otimes 1) + x \otimes x$$

Let $r = \frac{1+x}{2}$, $g = \frac{1-x}{2}$

$$\varepsilon(r) = 1/2, \quad \varepsilon(g) = -1/2$$

$$r + g = 1$$

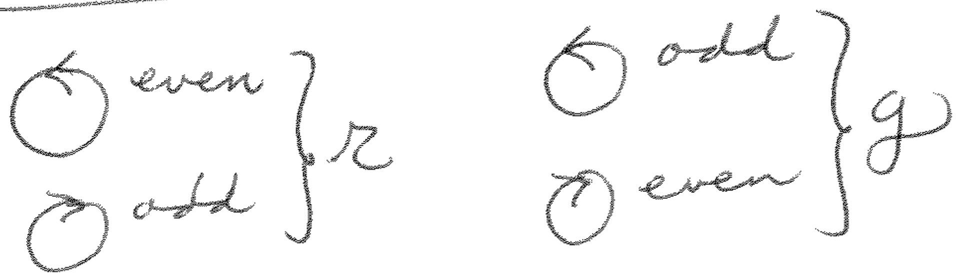
$$r^2 = r, \quad g^2 = g$$

$$rg = 0$$

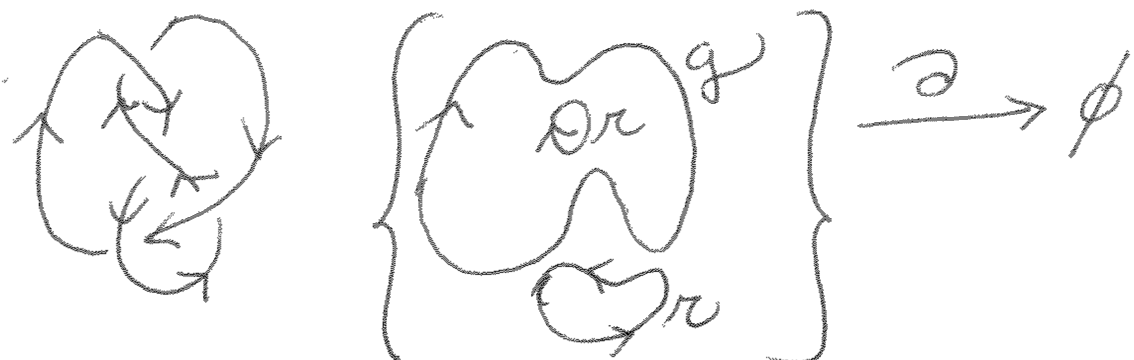
$$\Delta(r) = 2\, r \otimes r$$

$$\Delta(g) = -2\, g \otimes g$$

Lee Homology is gen by Seifert smoothing states for all choices of orientation of link.



e.g.



Lee homology is simple. One has that the dimension of the Lee homology is equal to $2^{comp(L)}$ where $comp(L)$ denotes the number of components of the link L. Up to homotopy, Lee's homology has a vanishing differential, and the complex behaves well under link concondance. In his paper [4] Dror BarNatan remarks "In a beautiful article Eun Soo Lee introduced a second differential on the Khovanov complex of a knot (or link) and showed that the resulting (double) complex has non-interesting homology. This is a very interesting result." Rasmussen [49] uses Lee's result to define invariants of links that give lower bounds for the four-ball genus, and determine it for torus knots. This gives an (elementary) proof of a conjecture of Milnor that had been previously shown using gauge theory by Kronheimer and Mrowka [29].

Rasmussen's result uses the Lee spectral sequence. We have the quantum (j) grading for a diagram K and the fact that for Lee's algebra $j(\partial(s)) \geq j(s)$. Rasmussen uses a normalized version of this grading denoted by $g(s)$. Then one makes a filtration $F^k C^*(K) = \{v \in C^*(K) | g(v) \geq k\}$ and given $\alpha \in Lee^*(K)$ define

$$S(\alpha) := max\{g(v) | [v] = \alpha\}$$

$$s_{min}(K) := min\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$
$$s_{max}(K) := max\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$

and

$$s(K) := (1/2)(s_{min}(K) + s_{max}(K)).$$

This last average of s_{min} and s_{max} is the Rasmussen invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$$\begin{cases} n_+ = \#\text{ of } + \text{ crossings in } K. \\ n_- = \#\text{ of } - \text{ crossings in } K. \end{cases}$$

$$\boxed{j(s) = \#(\text{B smoothings}) + \#(1\text{'s}) - \#(X\text{'s})}$$

We now enter the following sequence of facts:

1. $s(K) \in Z$.

2. $s(K)$ is additive under connected sum.

3. If K^* denotes the mirror image of the diagram K, then

$$s(K^*) = -s(K).$$

4. If K is a positive knot diagram (all positive crossings), then

$$s(K) = -r + n + 1$$

where r denotes the number of loops in the canonical oriented smoothing (this is the same as the number of Seifert circuits in the diagram K) and n denotes the number of crossings in K.

5. For a torus knot $K_{a,b}$ of type (a, b), $s(K_{a,b}) = (a-1)(b-1)$.

6. $|s(K)| \leq 2g^*(K)$ where $g^*(K)$ is the least genus spanning surface for K in the four ball.

7. $g^*(K_{a,b}) = (a-1)(b-1)/2$. This is Milnor's conjecture.

This completes a very skeletal sketch of the construction and use of Rasmussen's invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$n_+ = \#$ of $+$ crossings in K.

$n_- = \#$ of $-$ crossings in K.

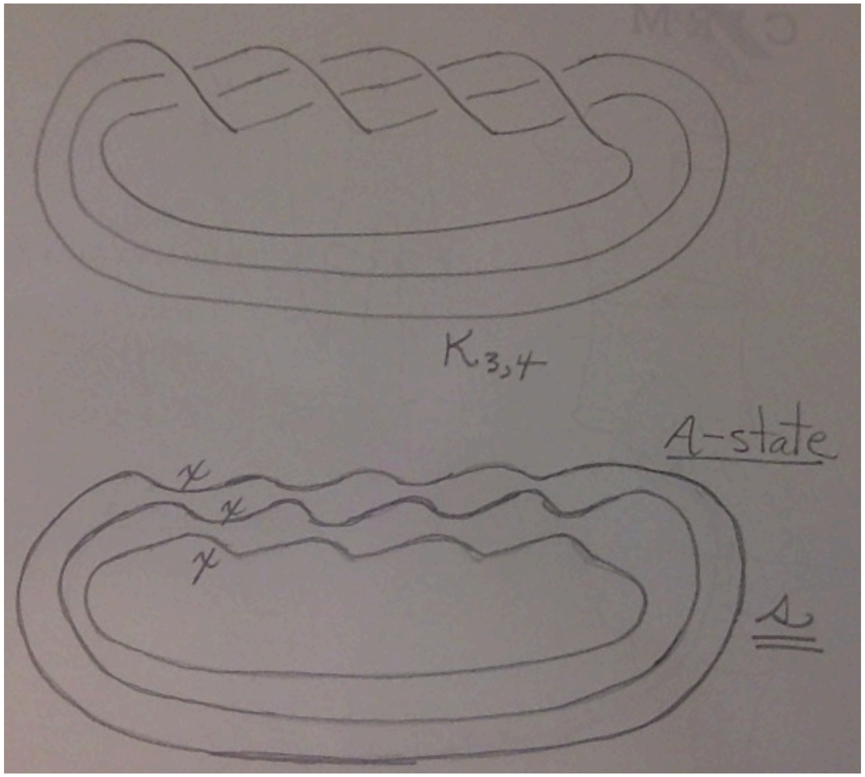
$$j(s) = \#(\text{B smoothings}) + \#(1\text{'s}) - \#(x\text{'s})$$

Facts: $s_{max}(K) = s_{min}(K) + 2$

$s(K) = s_{min}(K) + 1$

A-State: $s(K) = 1 - (\#\text{loops}) + (\#\text{ crossings}) = 2\text{genus}(\text{Seifert}(K))$

For positive knot all loops labelled x.



For A-state of a $K_{p,q}$ torus knot have (with all x's):

- p loops
- $(p-1)q$ crossings

So

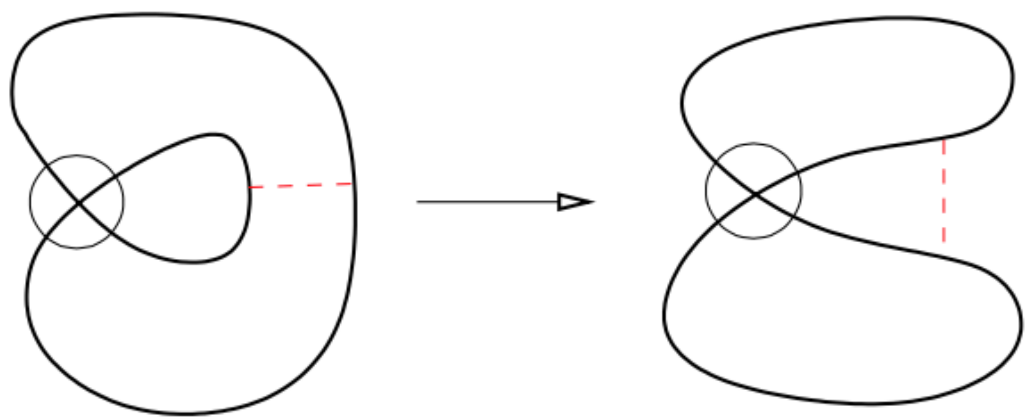
$$g(s) = (0 - p) + (p-1)q$$

$$= pq - q - p$$

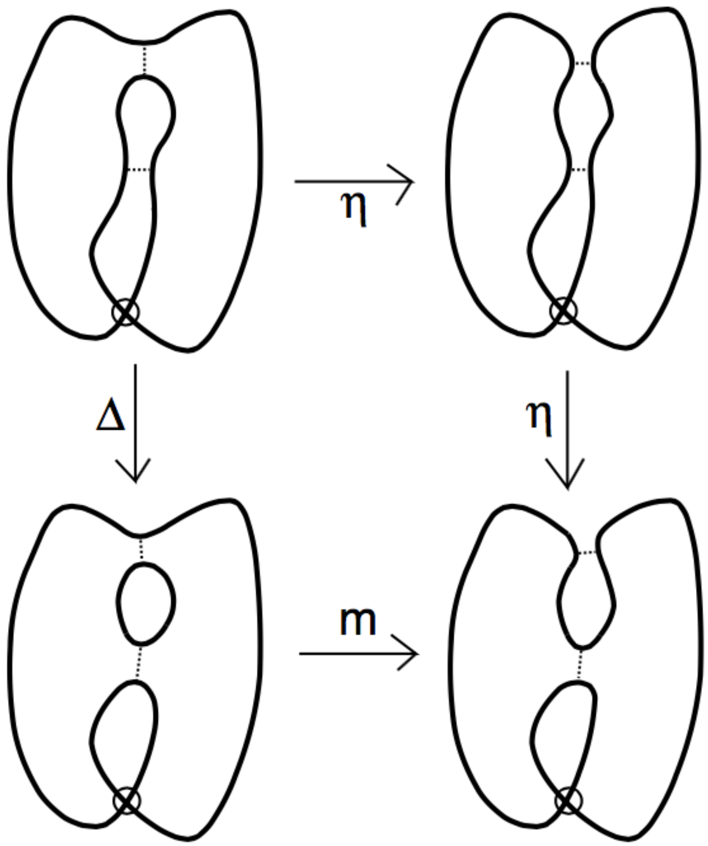
$$g(s) = (p-1)(q-1) - 1$$

$$\Longrightarrow \boxed{s(K_{pq}) = (p-1)(q-1)}$$

Khovanov Homology for Virtual Links
(our latest evolution of Manturov's approach)
(with Dye, Kaestner, Ogasa, Baldridge, McCarty)



Problem of Single Cycle
Smoothing

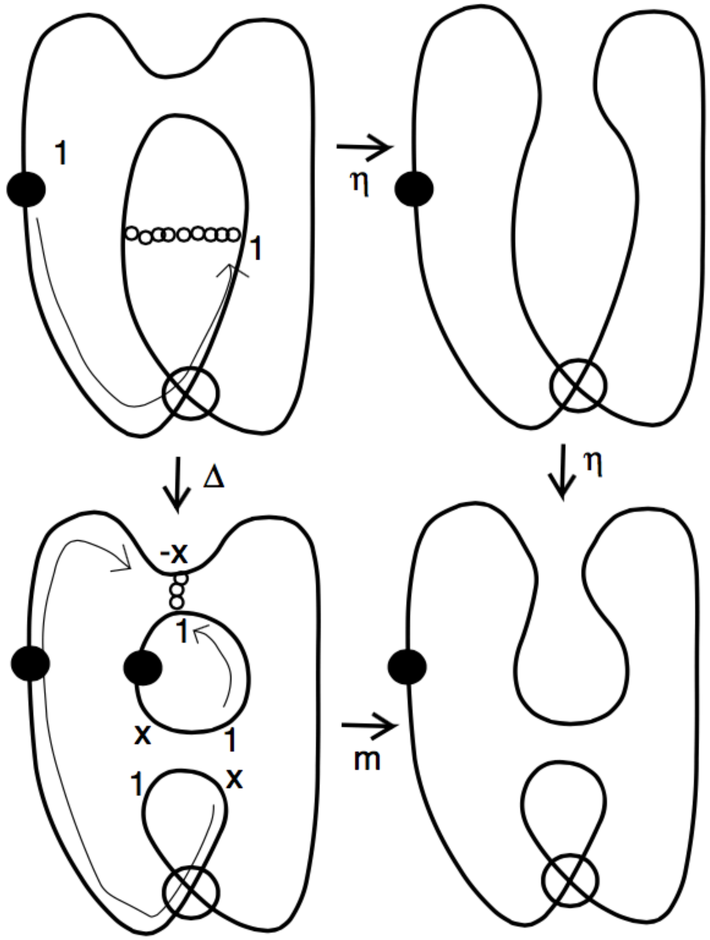


$\eta(a) = 0$ for all a

$$\eta \circ \eta = 0.$$

But composing along the opposite sides we see

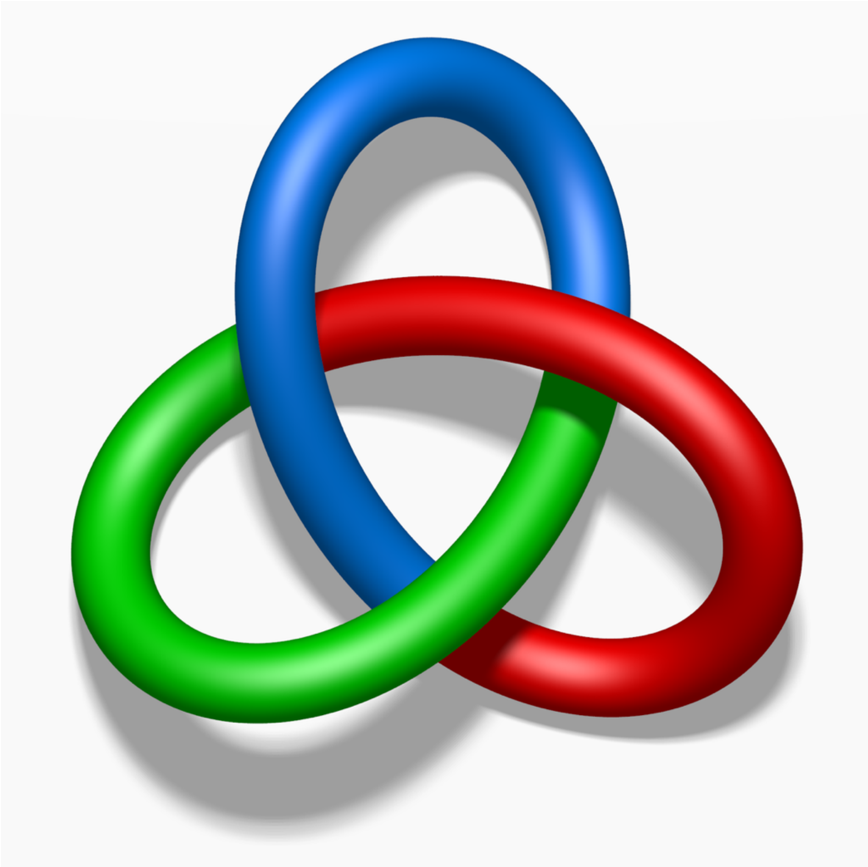
$$m \circ \Delta(1) = m(1 \otimes X + X \otimes 1) = X + X = 2X.$$



Thus 1⊗x+x⊗1 is transported to 1 ⊗ x + (–x) ⊗ 1 at the multiplication site. Upon multiplying we have m(1⊗x+(–x)⊗1) = x–x = 0.

Solution over the integers by counting transport through virtual crossings.
Change x to -x for each virtual passing.
Local coefficient system.
Use the unoriented complex.
Extends to Lee homology.
Has Rasmussen Invariant.

Virtual Knot Cobordism



Virtual Knot Cobordism

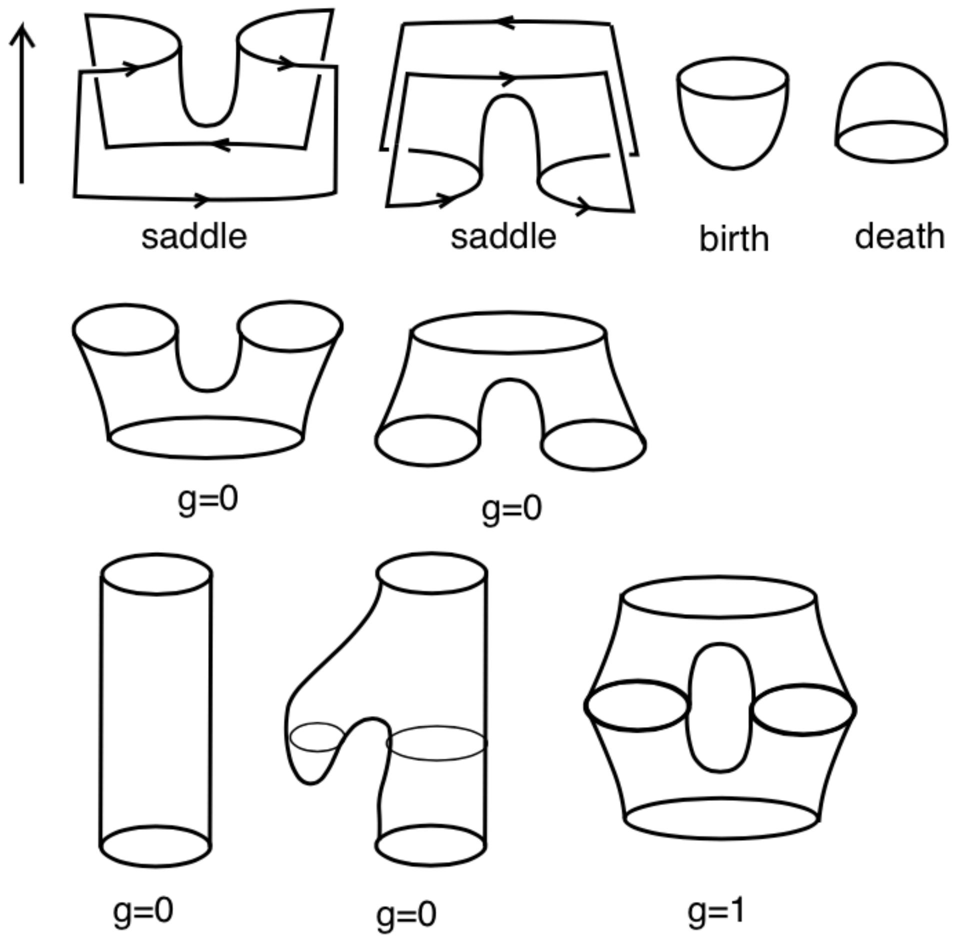


Figure 16: **Saddles, Births and Deaths**

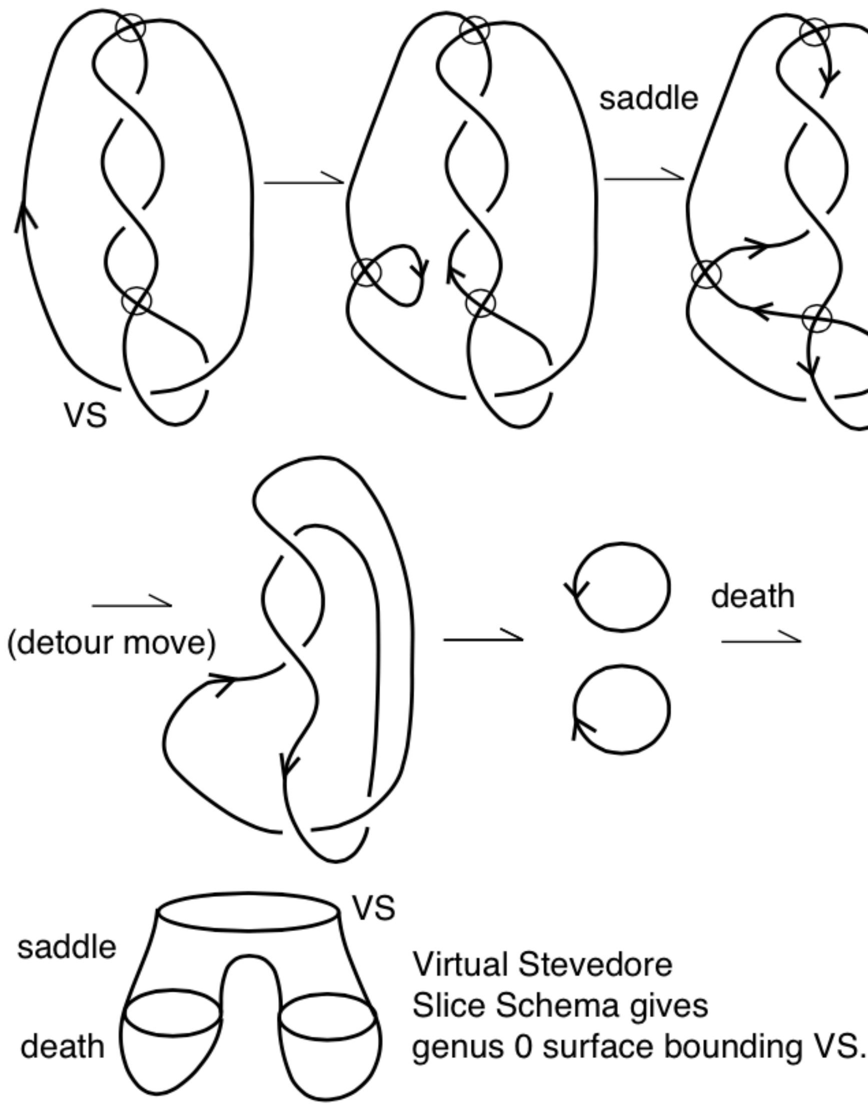
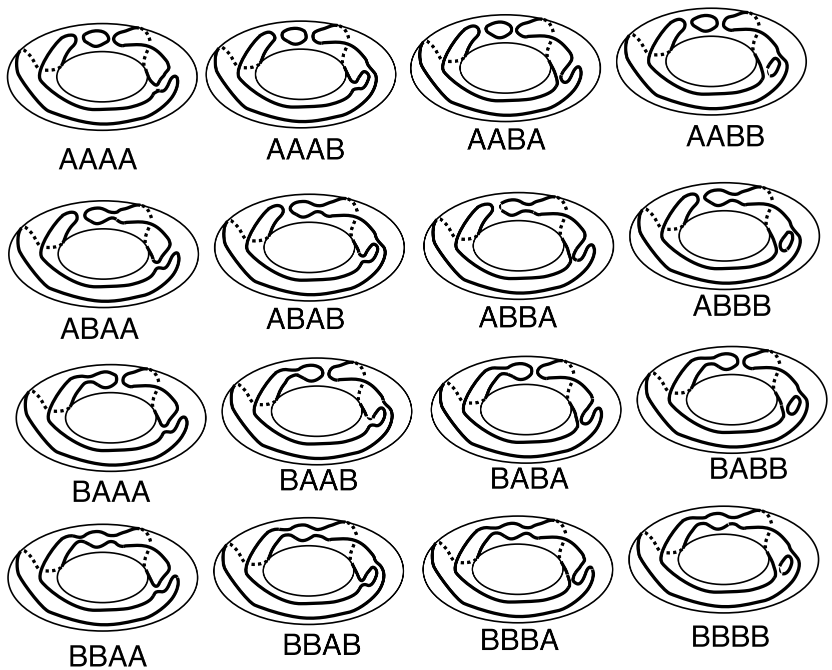
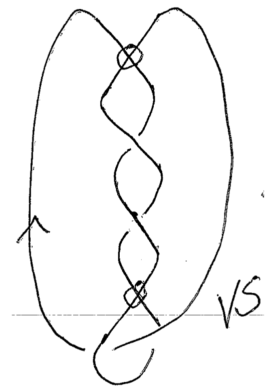
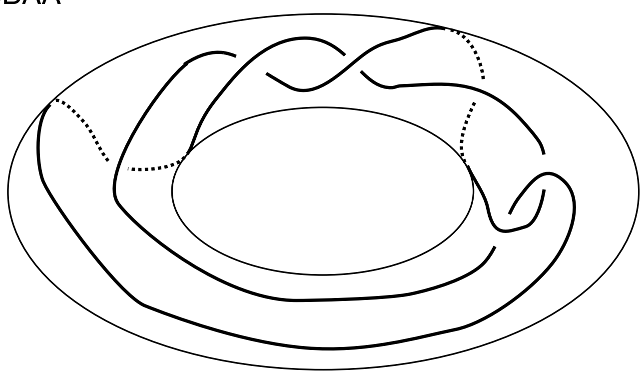


Figure 17: **Virtual Stevedore is Slice**

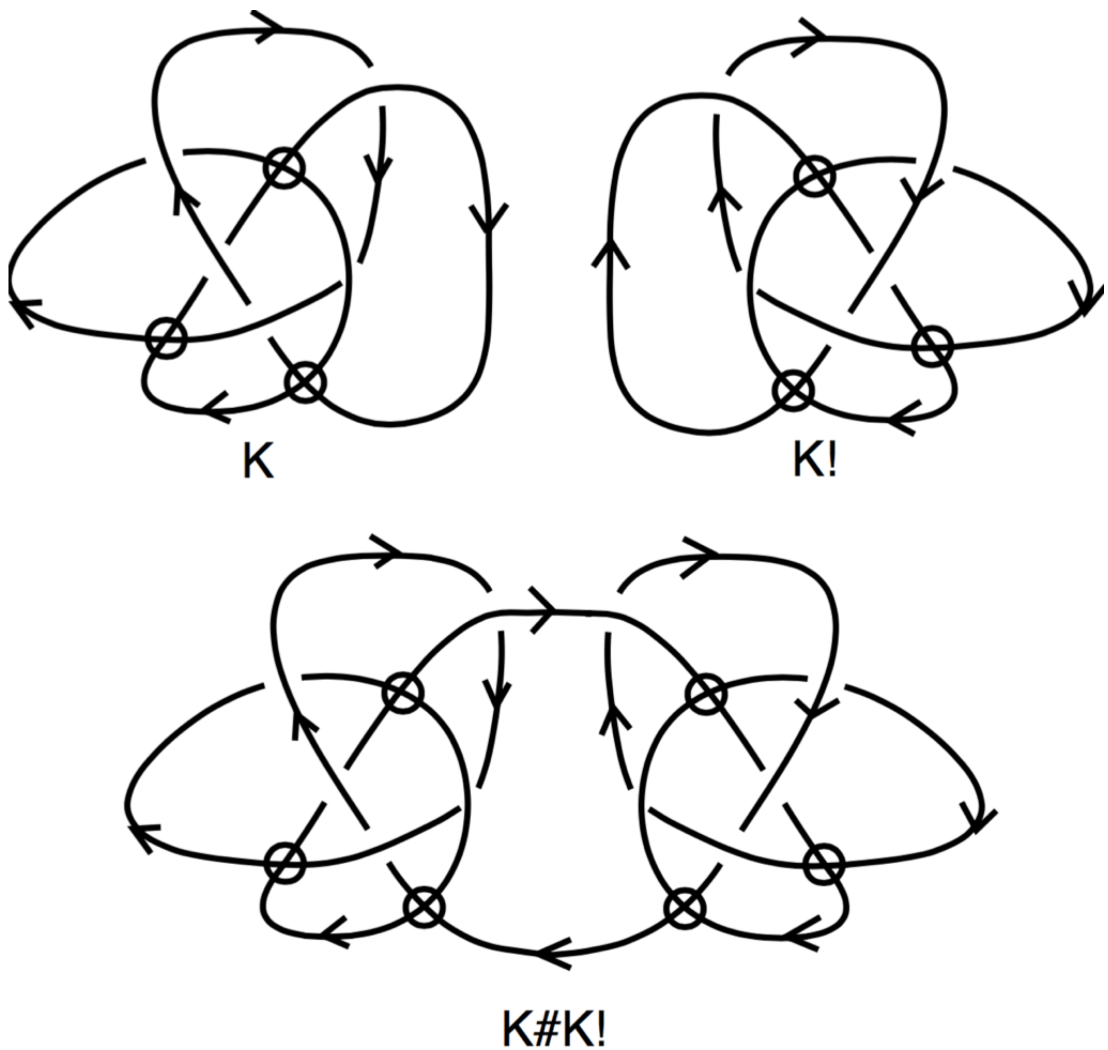
Virtual Stevedore is not classical.

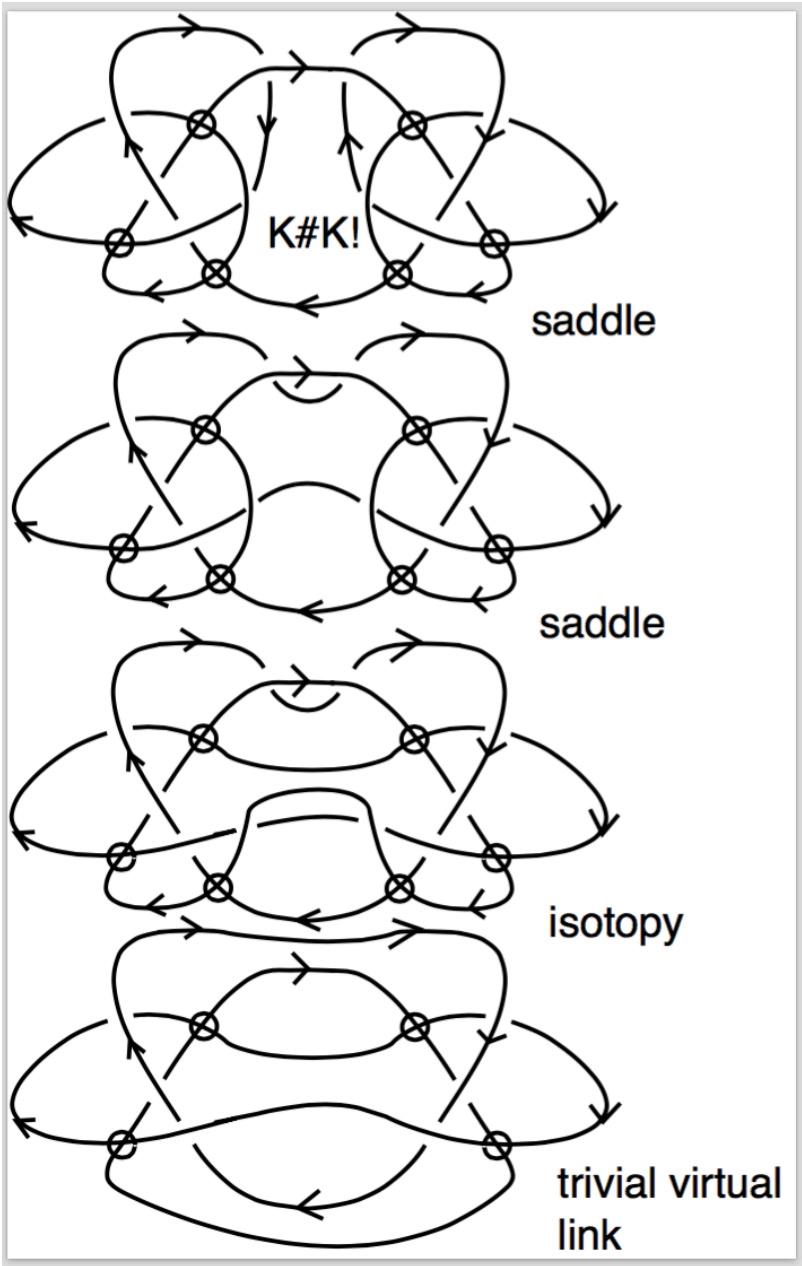






Vertical Mirror Image





Connected Sum
with the
Vertical Mirror Image
is
Slice.

We say that K is concordant to K`

$K \sim_c K`$

if there exists a cobordism from K to K` of genus 0.

A virtual knot is said to be slice
if it is concordant to the unknot.

Spanning Surfaces for Knots and Virtual Knots.

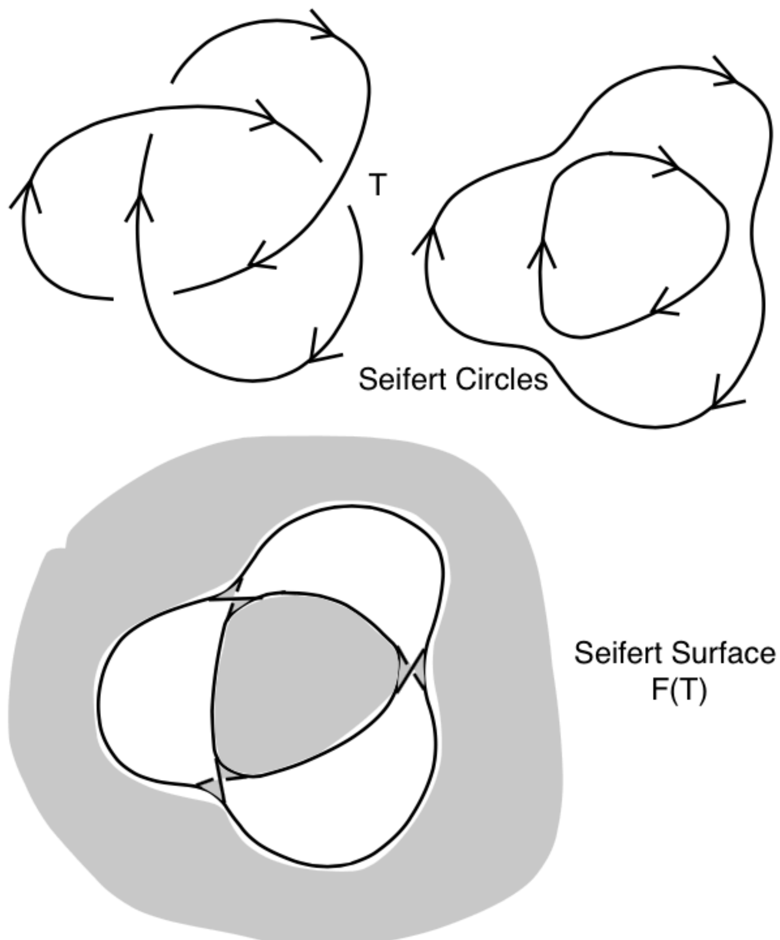
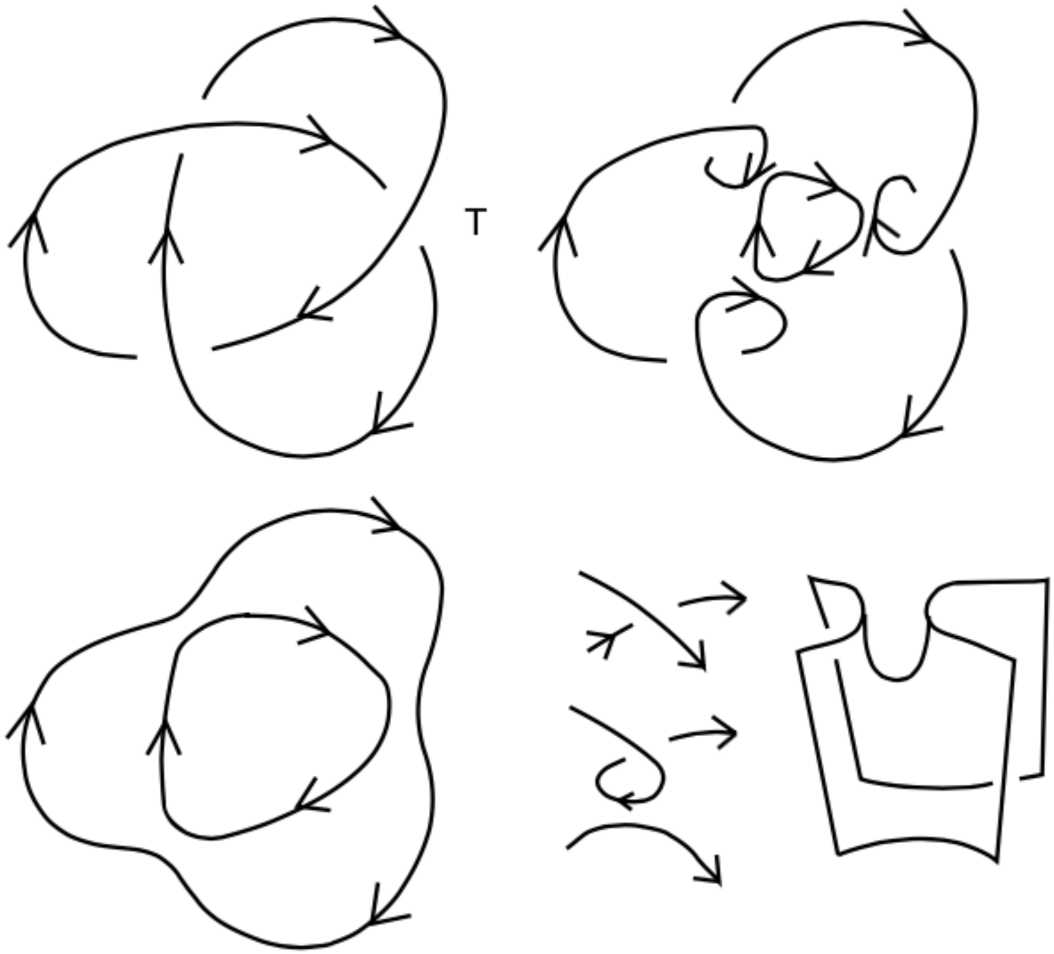
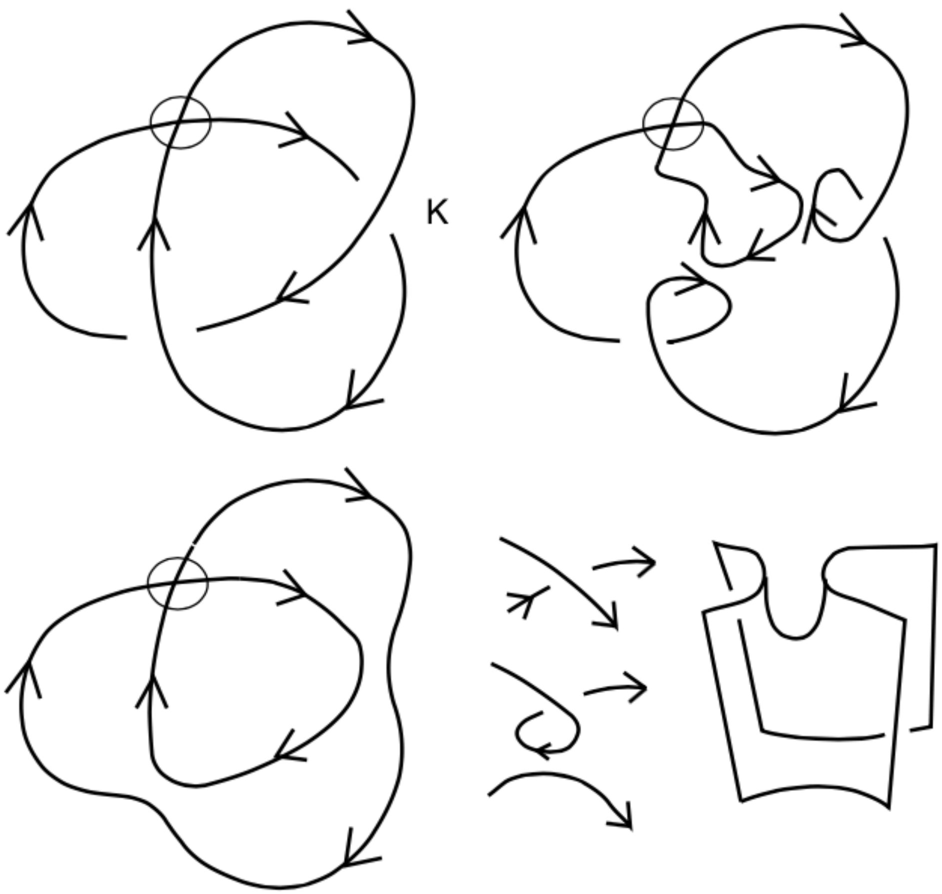


Figure 18: **Classical Seifert Surface**



Every classical knot diagram bounds a surface in the four-ball whose genus is equal to the genus of its Seifert Surface.

Figure 19: **Classical Cobordism Surface**

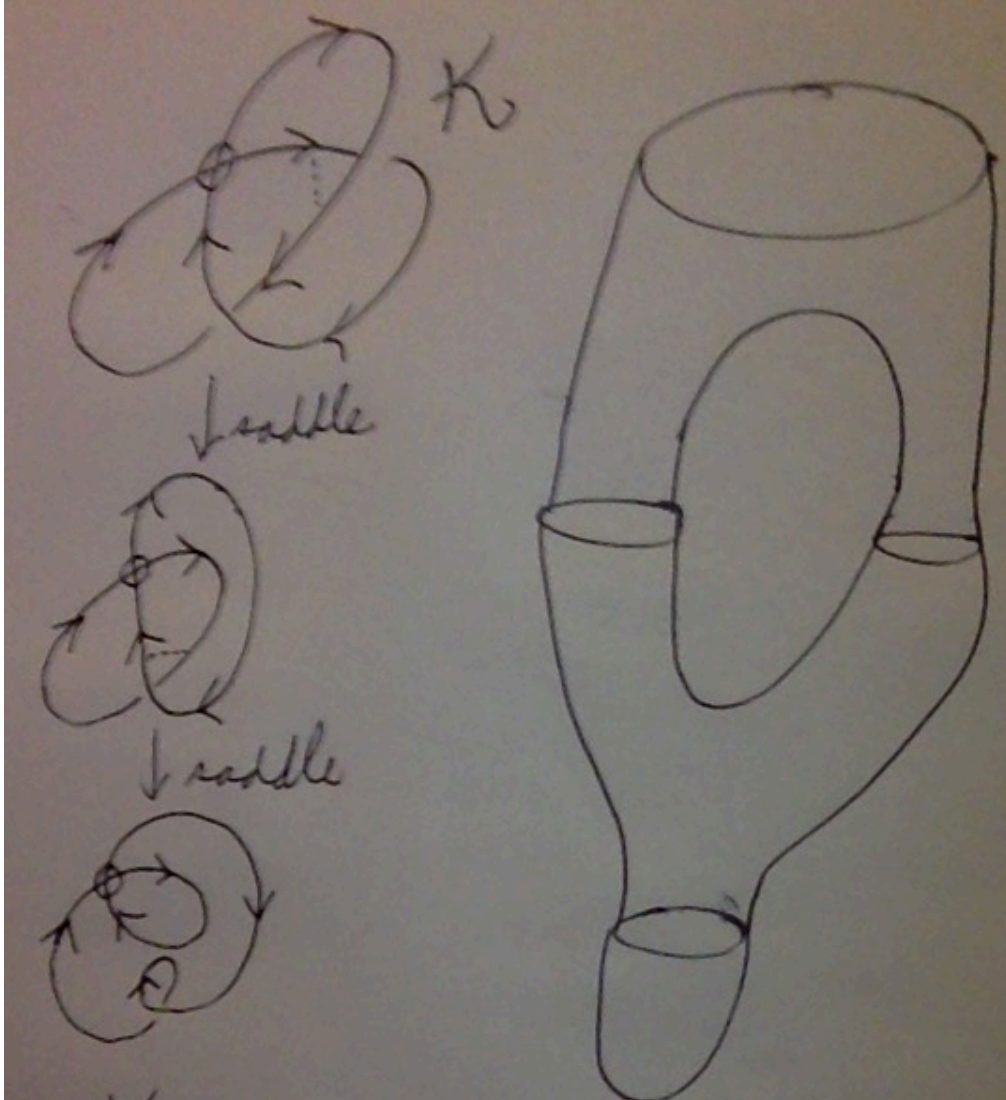


Seifert Circle(s) for K

Every virtual diagram K bounds a virtual orientable surface of genus $g = (1/2)(-r + n + 1)$ where r is the number of Seifert circles, and n is the number of classical crossings in K.
This virtual surface is the cobordism Seifert surface when K is classical.

Figure 20: **Virtual Cobordism Seifert Surface**

$r = 1$,
$n = 2$,
$g = (1/2)(-1 + 2 + 1) = 1$



K bounds a surface of genus 1 in the virtual 4-ball.

Heather Dye, Aaron Kaestner and LK, prove the following generalization of Rasmussen's Theorem, giving the four-ball genus of a positive virtual knot.

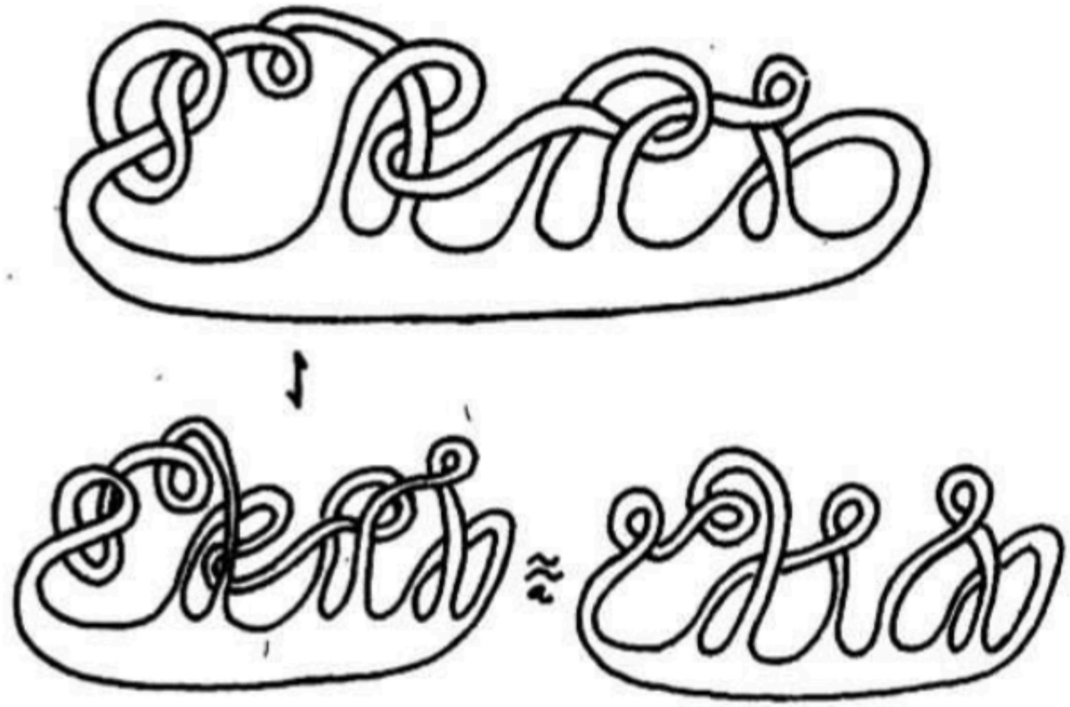
Theorem [2]. Let K be a positive virtual knot (all classcial crossings in K are positive), then the four-ball genus $g_4(K)$ is given by the formula

$$g_4(K) = (1/2)(-r + n + 1) = g(S(K))$$

where r is the number of virtual Seifert circles in the diagram K and n is the number of classcial crossings in this diagram. In other words, that virtual Seifert surface for K represents its minimal four-ball genus.

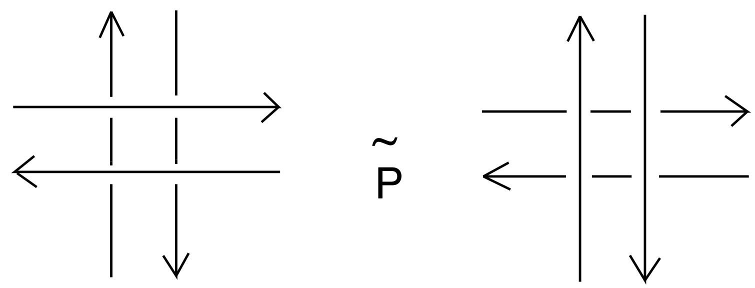
The virtual Seifert surface for positive virtual K represents the minimal four-ball genus of K.

The Theorem is proved by generalizing both Khovanov and Lee homology to virtual knots and generalizing the Rasmussen invariant to virtual knots.



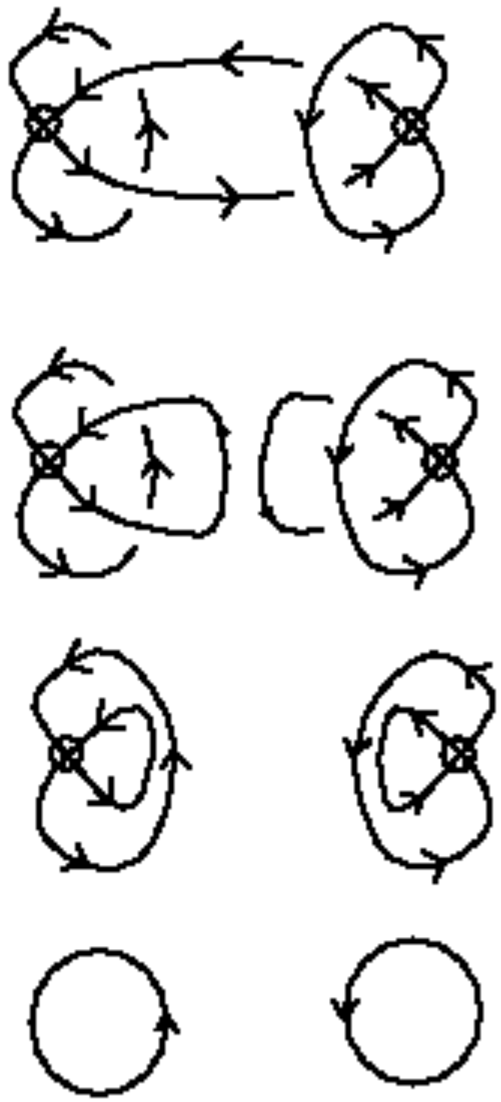
Classical Spanning Surfaces simplify by passing bands. Every classical knot is pass equivalent to either a trefoil or an unknot. Trefoil and unknot are distinguished by the Arf invariant.

Virtual Band Passing
VKT +



Classically there are two pass classes for knots: Trefoil and Unknot.

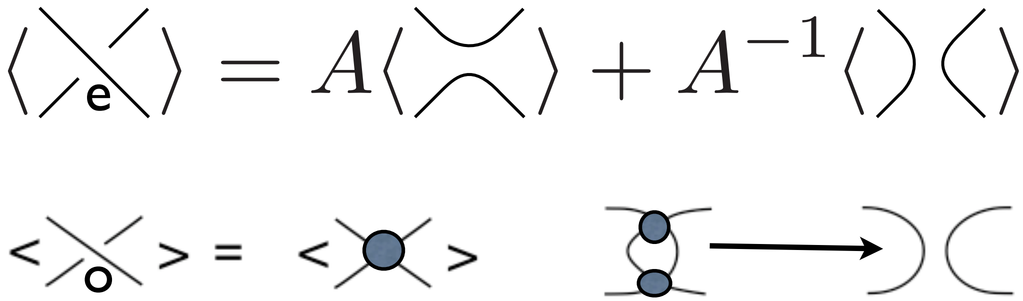
What are the pass classes for virtual knots and links?



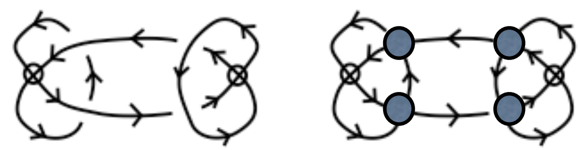
The Kishino
diagram gives a
virtual knot
that is slice but it
is not
PASS trivial.

Kishino is not pass trivial
since it is a non-trivial flat virtual knot. And its flat class IS its pass class since passing does not affect it as a flat.

Manturov Parity Bracket

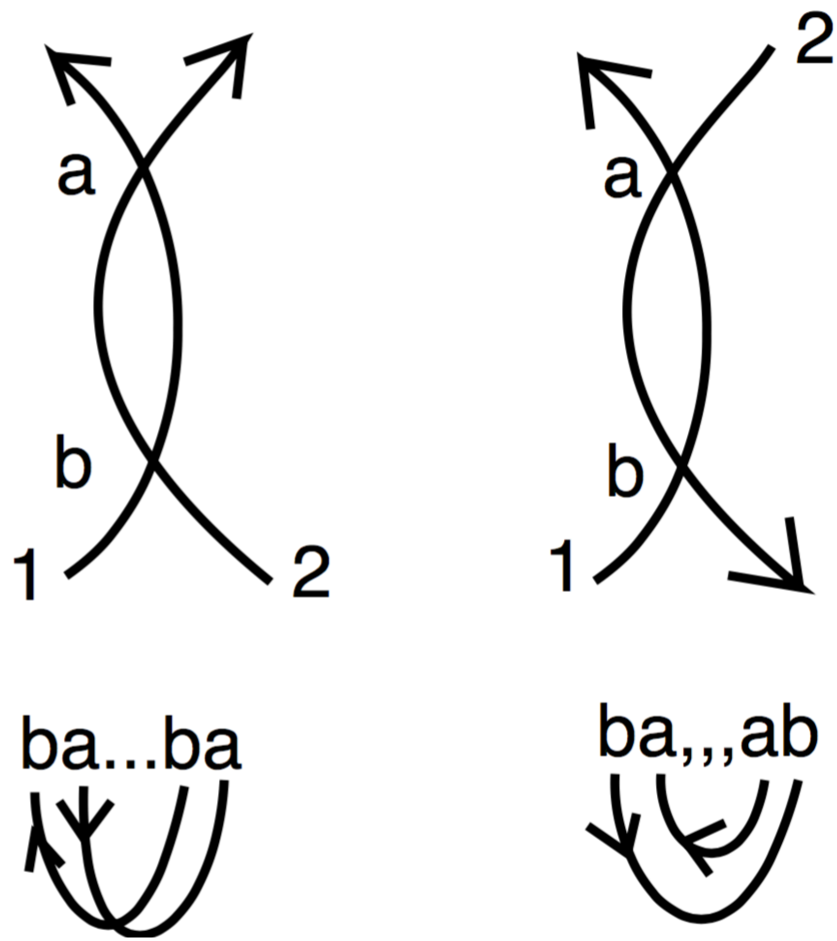


Kishino
Knot

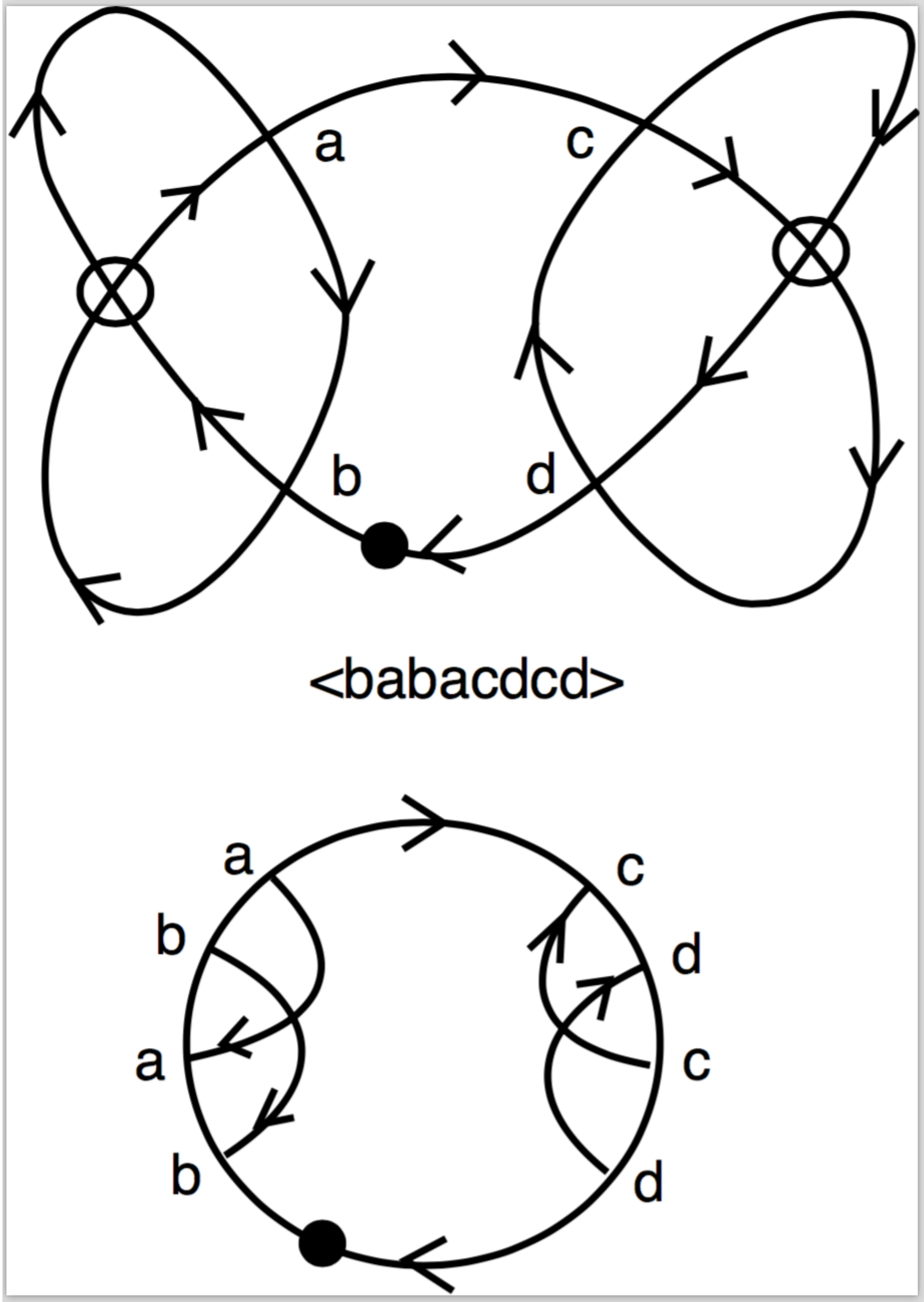


The Parity Bracket provides the simplest proof that the Kishino diagram is non-trivial.

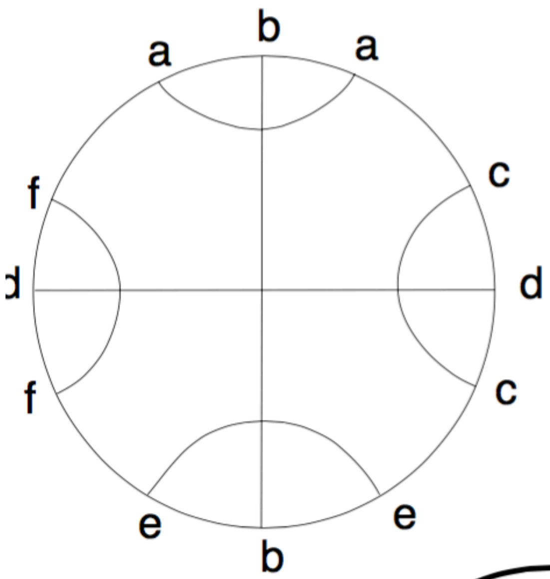
Parity bracket is calculated for virtuals and flat virtuals by replacing all odd crossings (odd interstice in Gauss code) with nodes. Then apply state sum with graphs (up to type two reducion) and polynomial coefficients. Kishino invariant is a single reduced diagram.



In flat Gauss code, two-moves require oppositely oriented parallel or crossed chords.

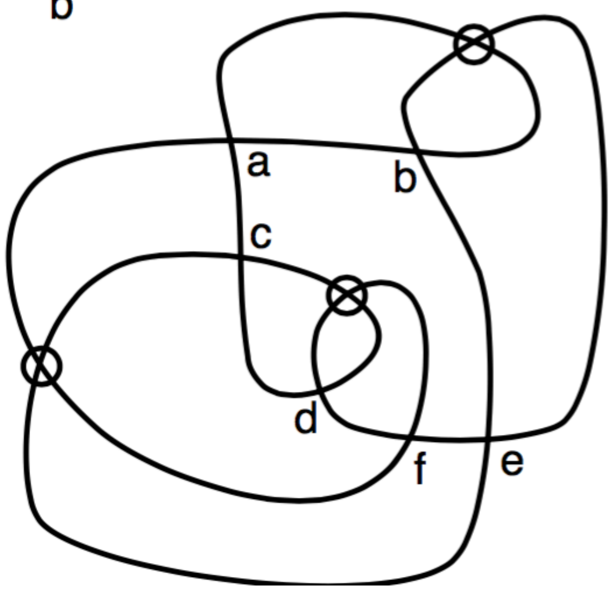


Reducing two-moves
are not available
on the flat
Kishino diagram.

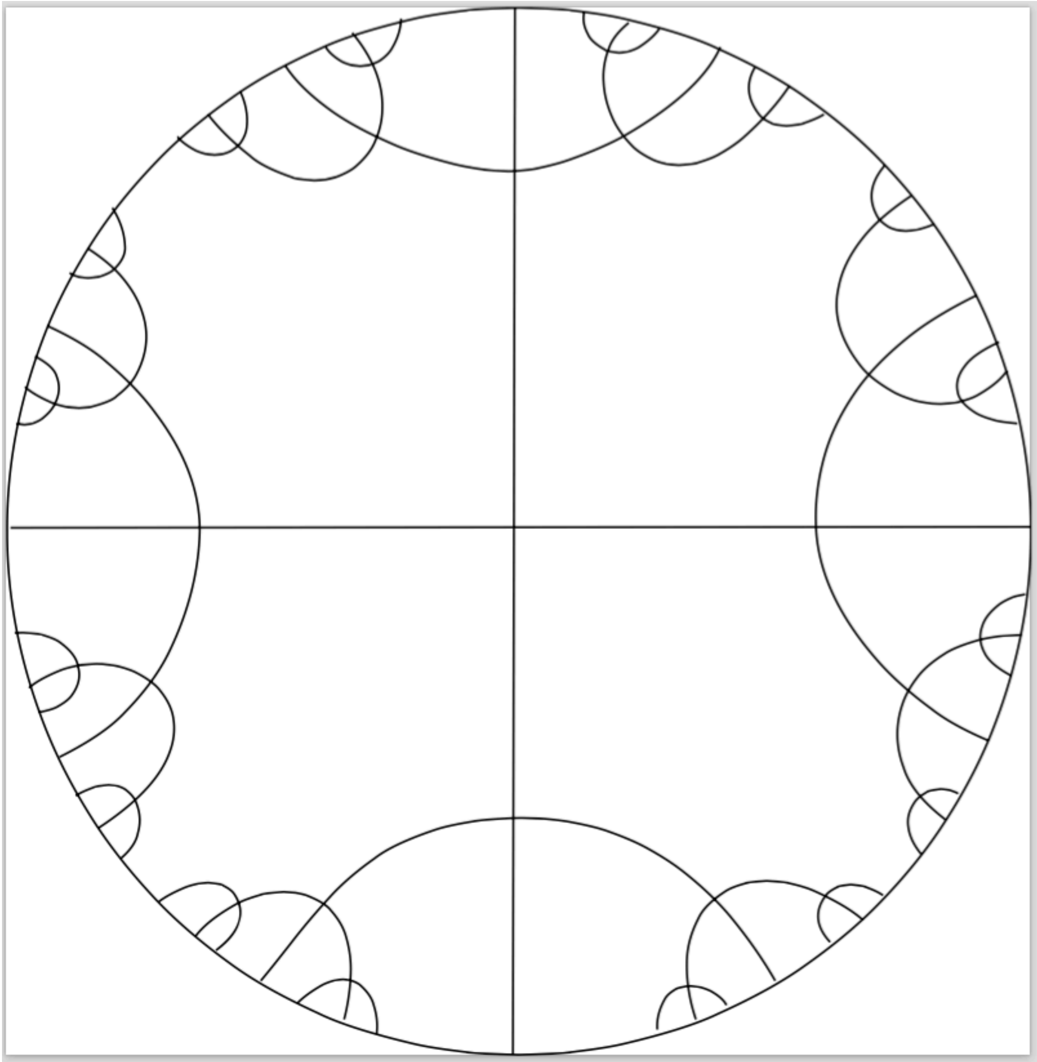


All odd crossings
and irreducible
as flat virtual diagram.

<abacdcebefdf>



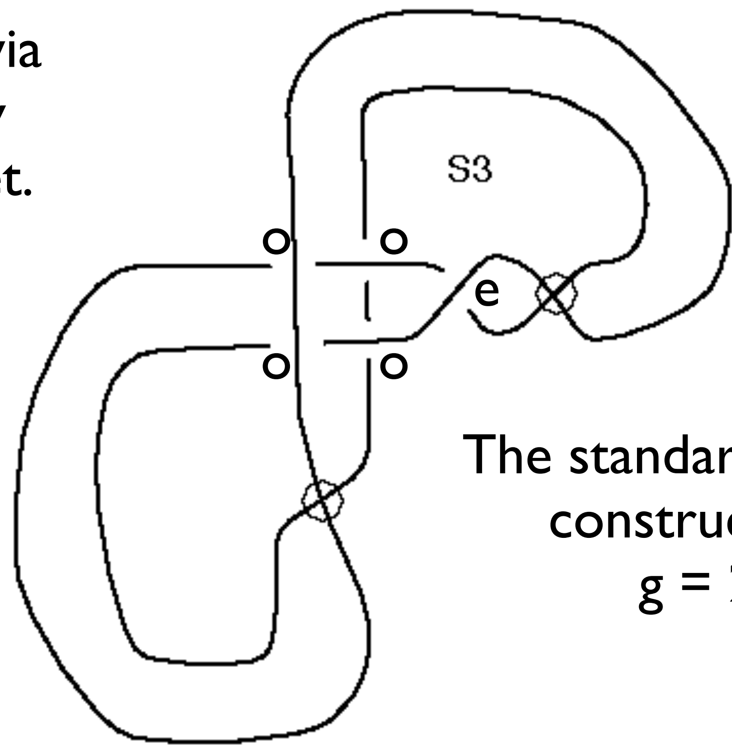
Here is another example of a flat with all odd crossings. It is non trivial by parity bracket and it is its own pass class.



This Gauss code schema shows how to produce infinitely many distinct flat virtuals, each their own pass class. Thus there are infinitely many distinct pass classes for virtual knots.

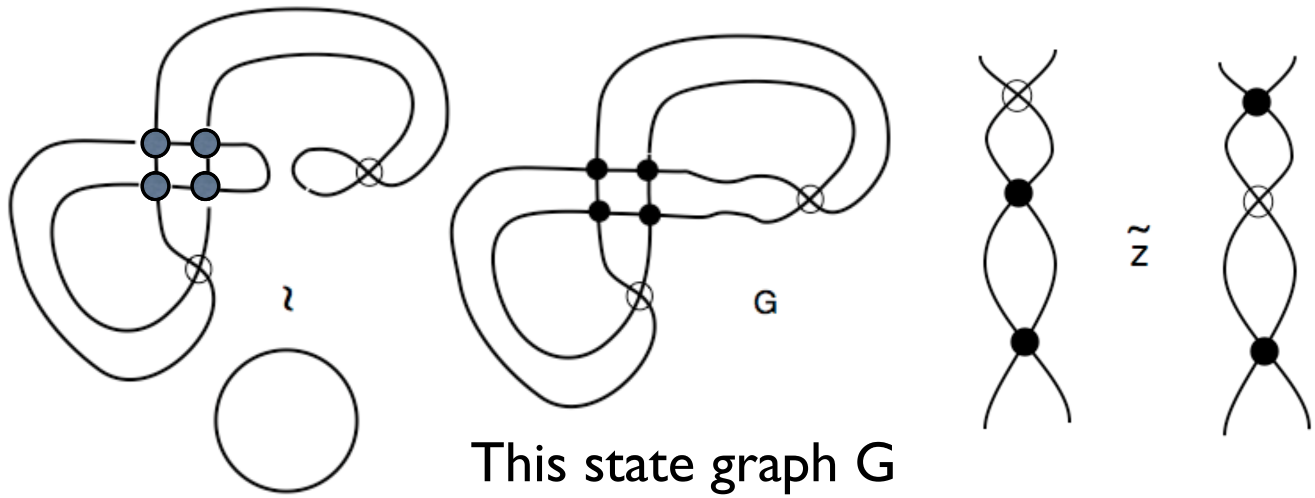
The Knot S3 (found with Slavik Jablan) has unit Jones polynomial. It is not Z-equivalent to a classical knot.

Proof via Parity Bracket.



The standard surface construct has $g = 2$.

A[S3] = -2K1^2 + K2 + A^4 (1 - 2K1^2 + K2)

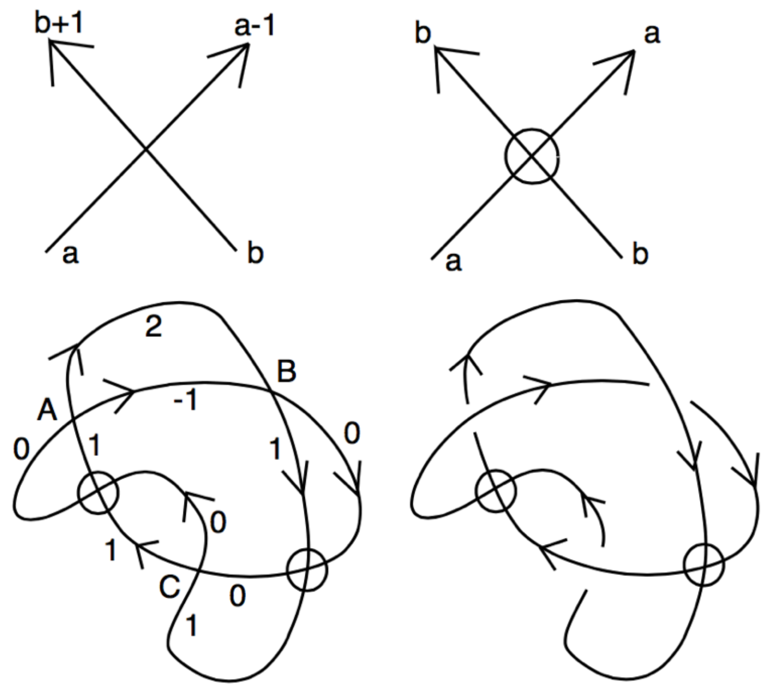


This state graph G
has g = 2 and does not reduce under
graphical Z move.

The Parity bracket of S3 has only two terms and includes the graph G. The virtual graph G cannot be reduced by Reidemeister Two moves on its nodes.

Conclusion: The knot S3 has surface genus g = 2.

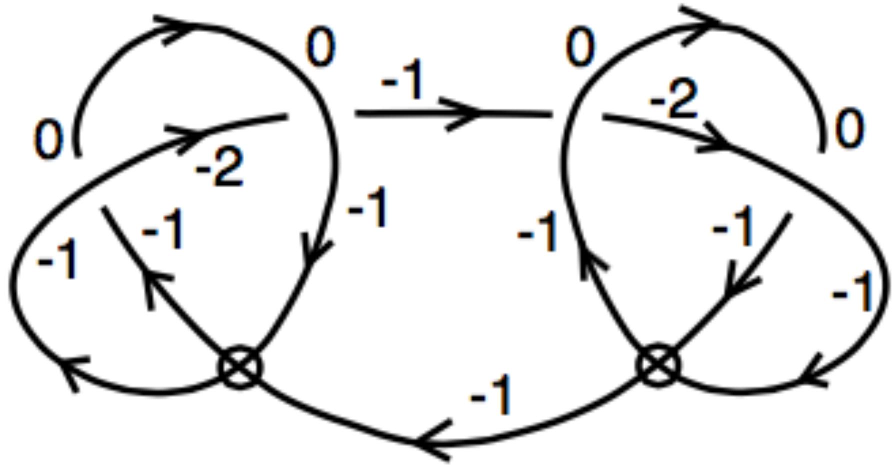
Affine Index Polynomial
(See LK and Folwazcny and variants from Henrich,Cheng,Dye,...)



	W_-	W_+
A	-2	+2
B	+2	-2
C	0	0

sgn(A) = sgn(B) = +1
sgn(C) = -1

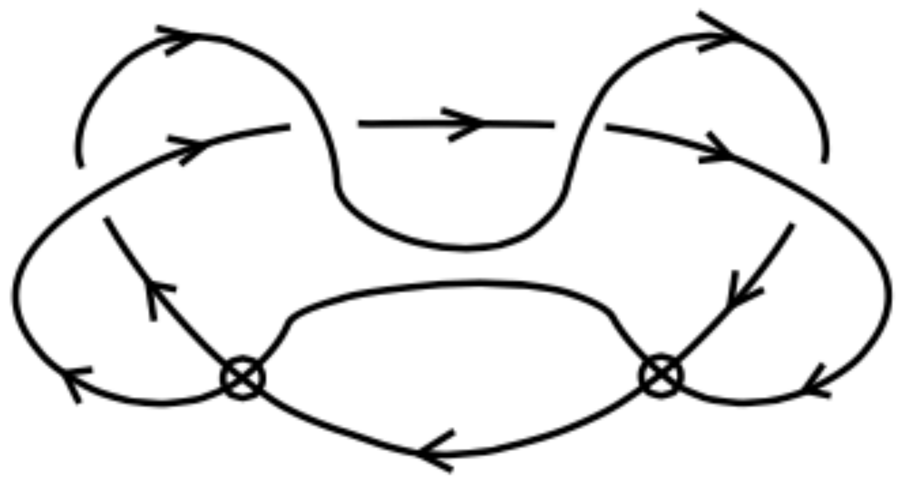
$P_K(t) = t^{-2} + t^{2} - 2$



K#K!

$$P_{K\#K!} = t^{-1} + t - t^{-1} - t = 0.$$

A slice knot with non-zero but cancelling weights.



$$P_K = \sum_c sgn(c)(t^{W_K(c)} - 1) = \sum_c sgn(c)t^{W_K(c)} - wr(K)$$

$$P_K = \sum_{n=1}^{\infty} wr_n(K)(t^n - 1)$$

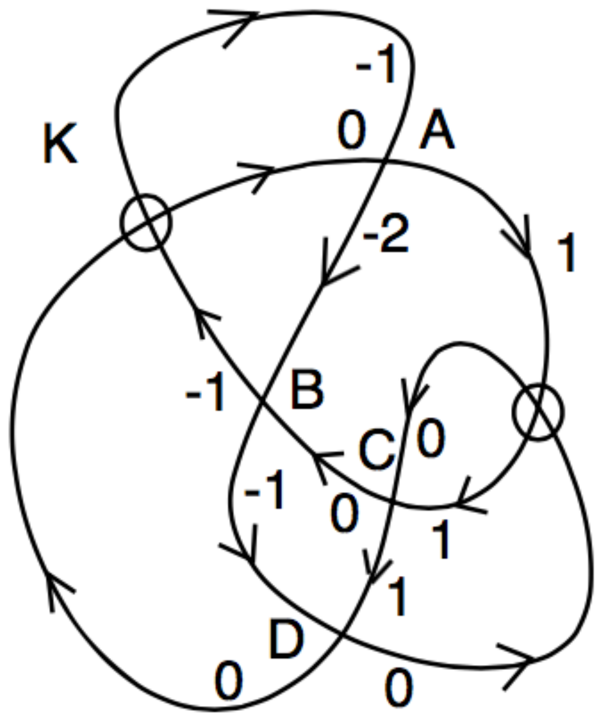
$$wr_n(K) = \sum_{c:W_K(c)=n} sgn(c).$$

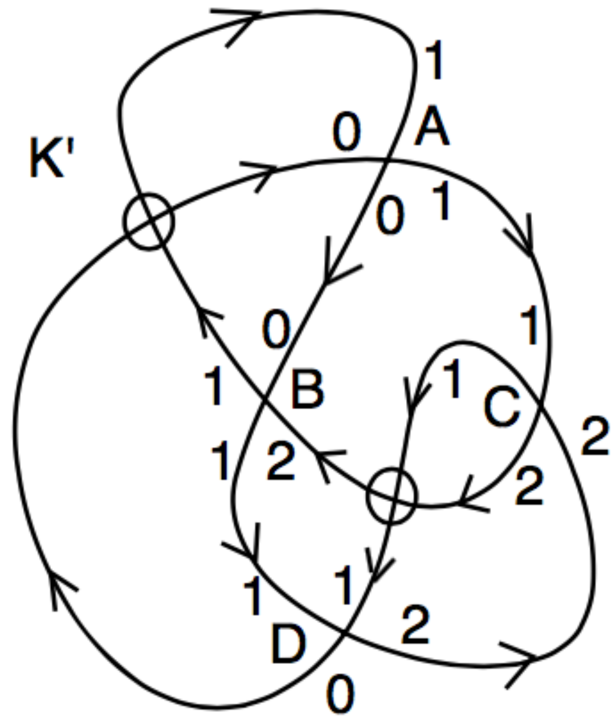
Remark. We define the *flat affine index polynomial*, PF_K, for a flat virtual knot K (in a flat virtual link the classical crossings are immersion crossings, neither over not under, Reidemeister moves are allowed independent of over and under, but virtual crossings still take detour precedence over classical crossings [14]) by the formula

$$PF_K(t) = \sum_c (t^{|W_K(c)|} + 1)$$

where the polynomial is taken over the integers modulo two, but the exponents (the absolute values of the weights at the crossings) are integral. It is not hard to see that $PF_K(t)$ is an invariant of flat virtual knots, and that the concordance results of the present paper hold in the flat category for this invariant. These results will

The Flat Index Polynomial is of particular interest here because we will prove below that it is a flat concordance invariant.



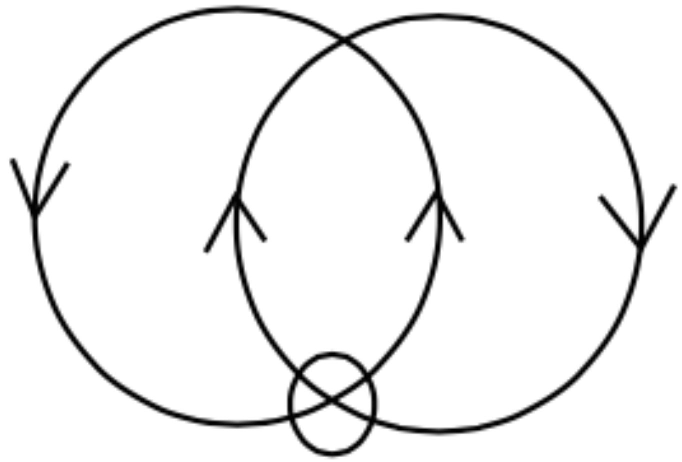


	w+	w-
A	-2	2
B	1	-1
C	0	0
D	1	-1

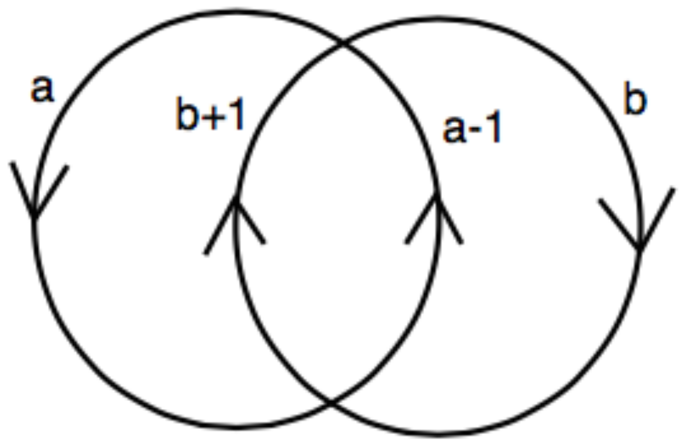
PF(K) = t^2 + 1 (mod 2)

	w+	w-
A	0	0
B	1	-1
C	0	0
D	-1	1

PF(K') = 0 (mod 2)

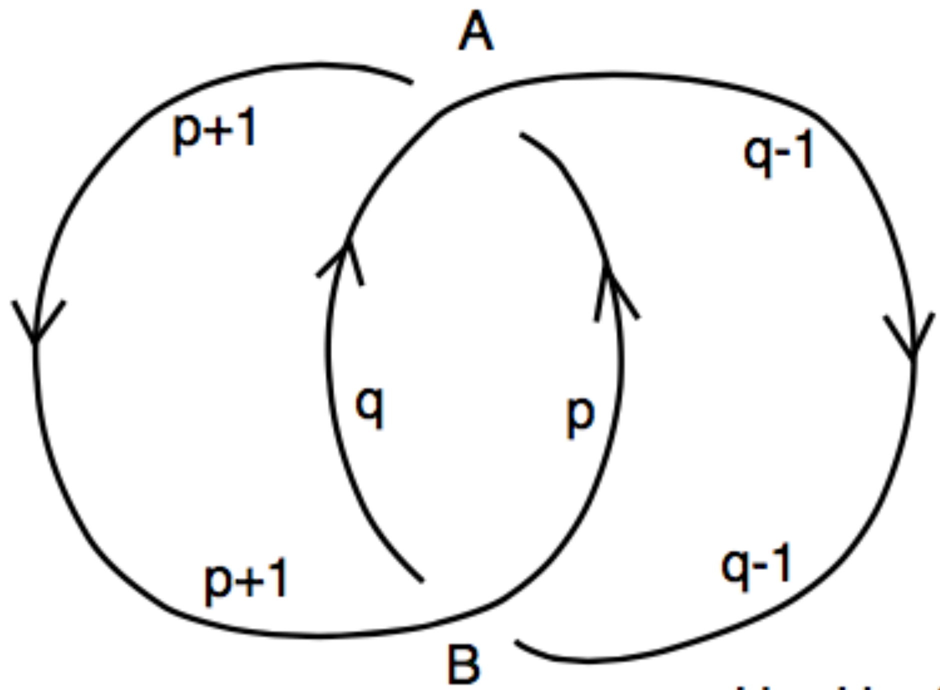


impossible to label



can be labeled

Index Invariant for Links



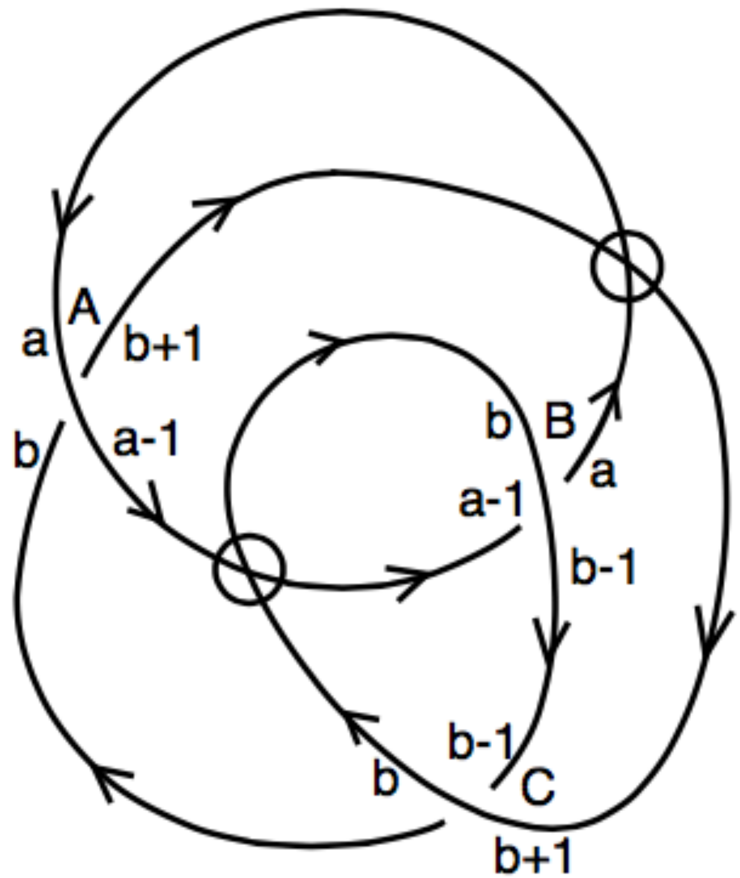
H = Hopf Link

N = p-q

W(A) = q-p-1 = -N-1

W(B) = p-q +1 = N+1

$$P_H(t) = t^{-N-1} + t^{N+1} - 2$$



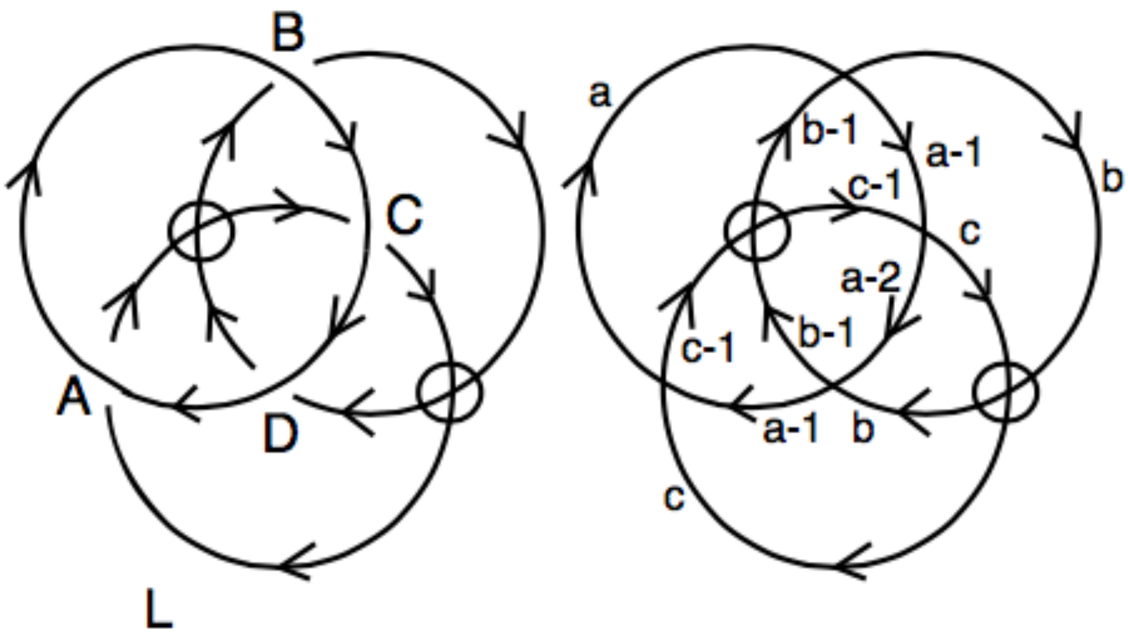
N = a-b

	w+	w-
A	N-1	1-N
B	-N	N
C	1	-1

Virtual Link L.

$$PL = t^{N-1} + t^{-N} + t - 3$$

Virtual Borromean Rings

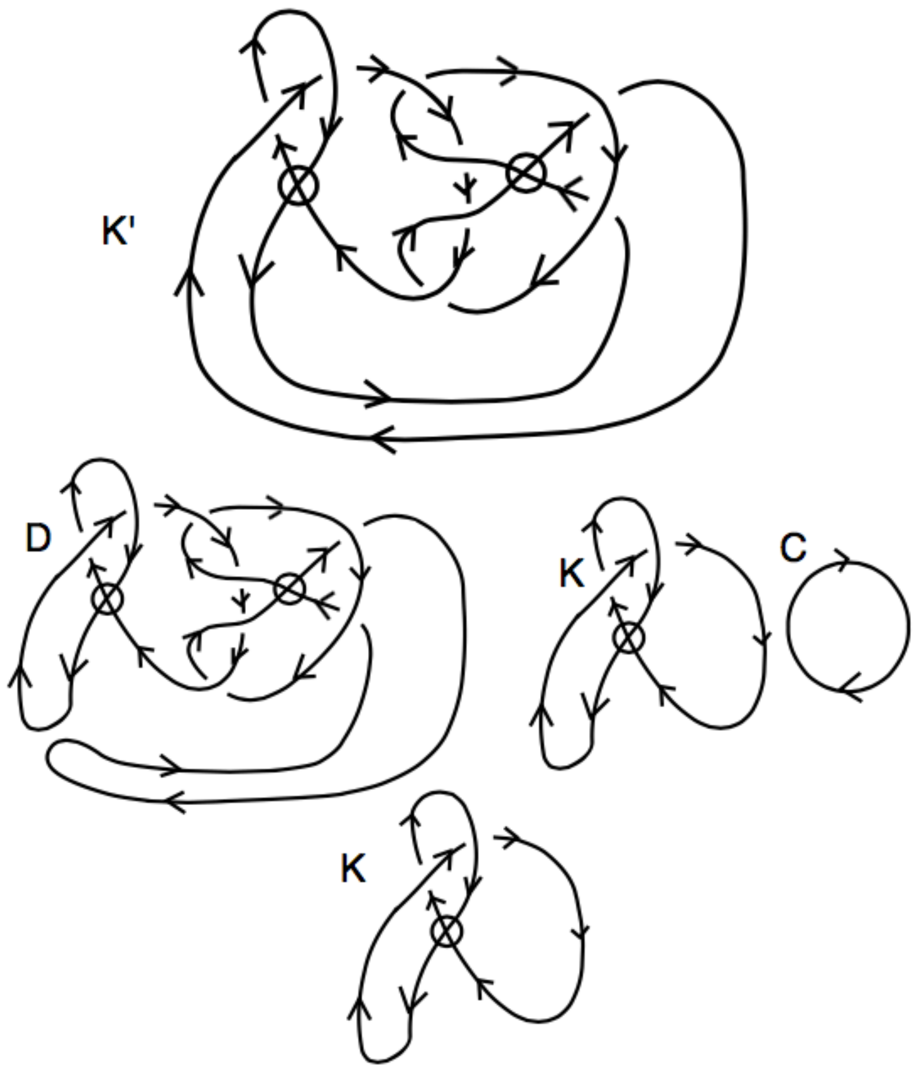


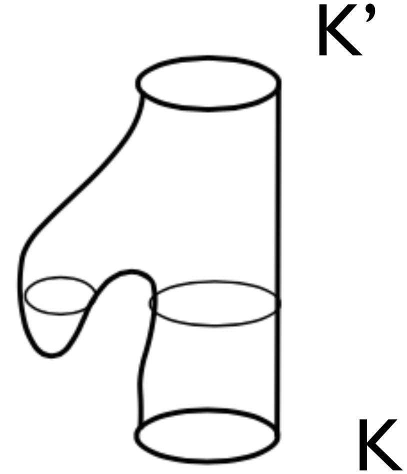
$PL = -t^{M} + t^{N} + t^{M-1} - t^{N-1}$

N = a-b, M = a-c

	w+	w-
A	-M	M
B	N	-N
C	M-1	-M+1
D	-N+1	N-1

Concordances are Composed of Elementary Concordances
(Cancellation of Saddle and Max or Min)

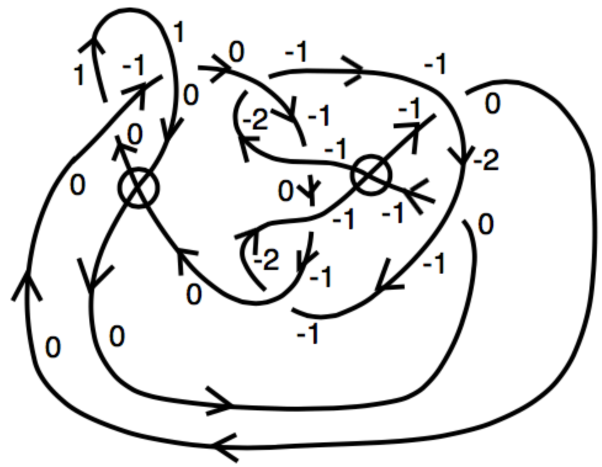




Theorem. P_K and PF_K are concordance invariants.

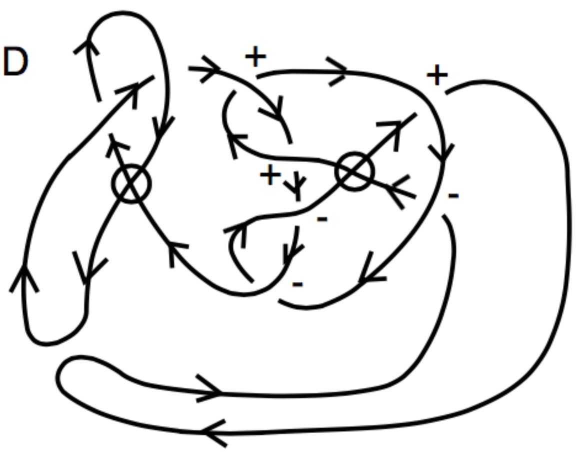
Proof. Concordances are compositions of elementary concordances.//

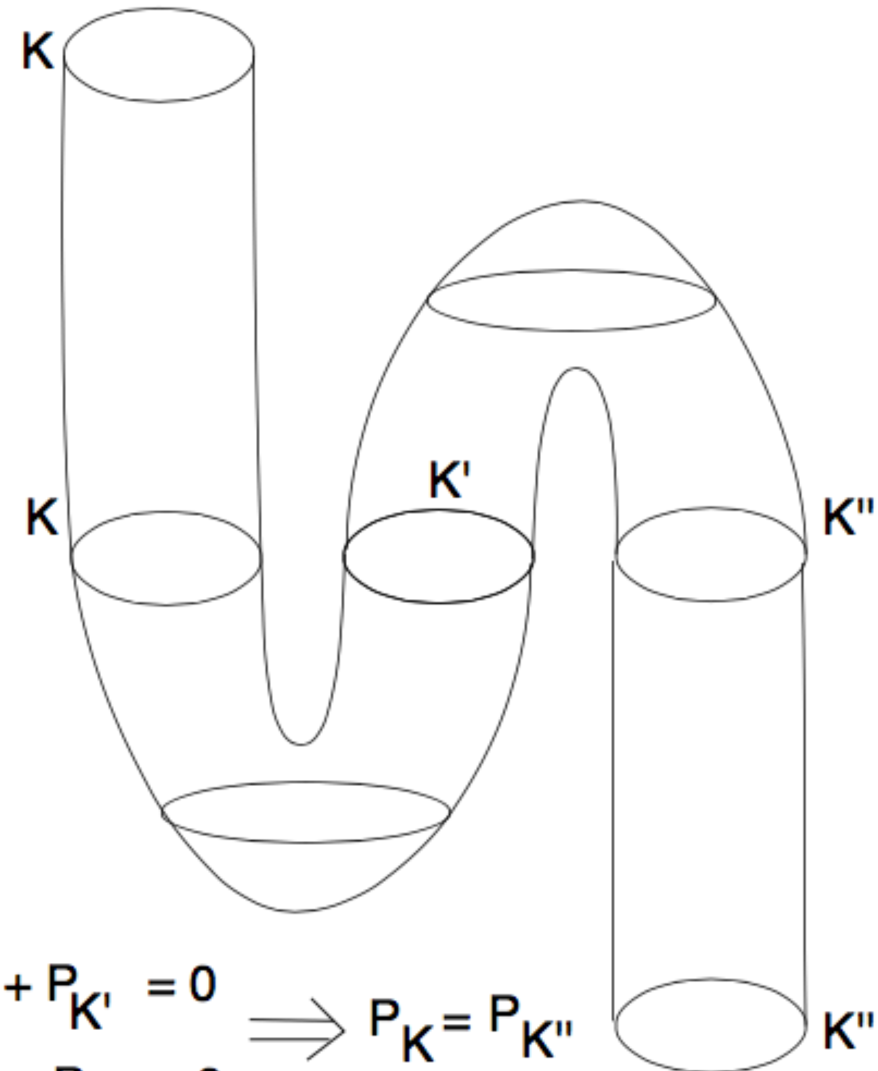
K'



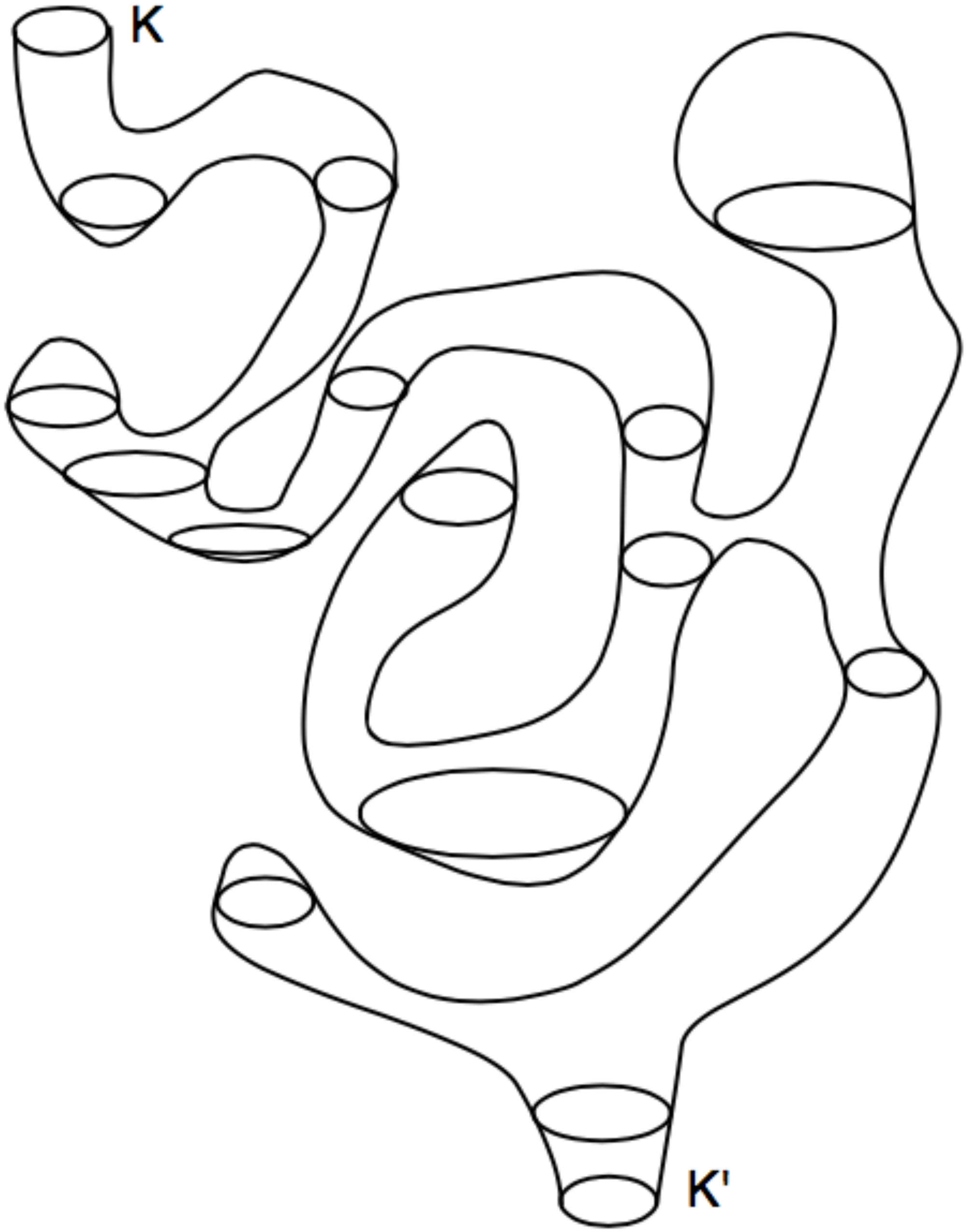
PK' = $t^{-1} + t + 2$

D





$$\left.\begin{aligned} P_K + P_{K'} &= 0 \\ P_{K'} + P_{K''} &= 0 \end{aligned}\right\} \Longrightarrow P_K = P_{K''}$$

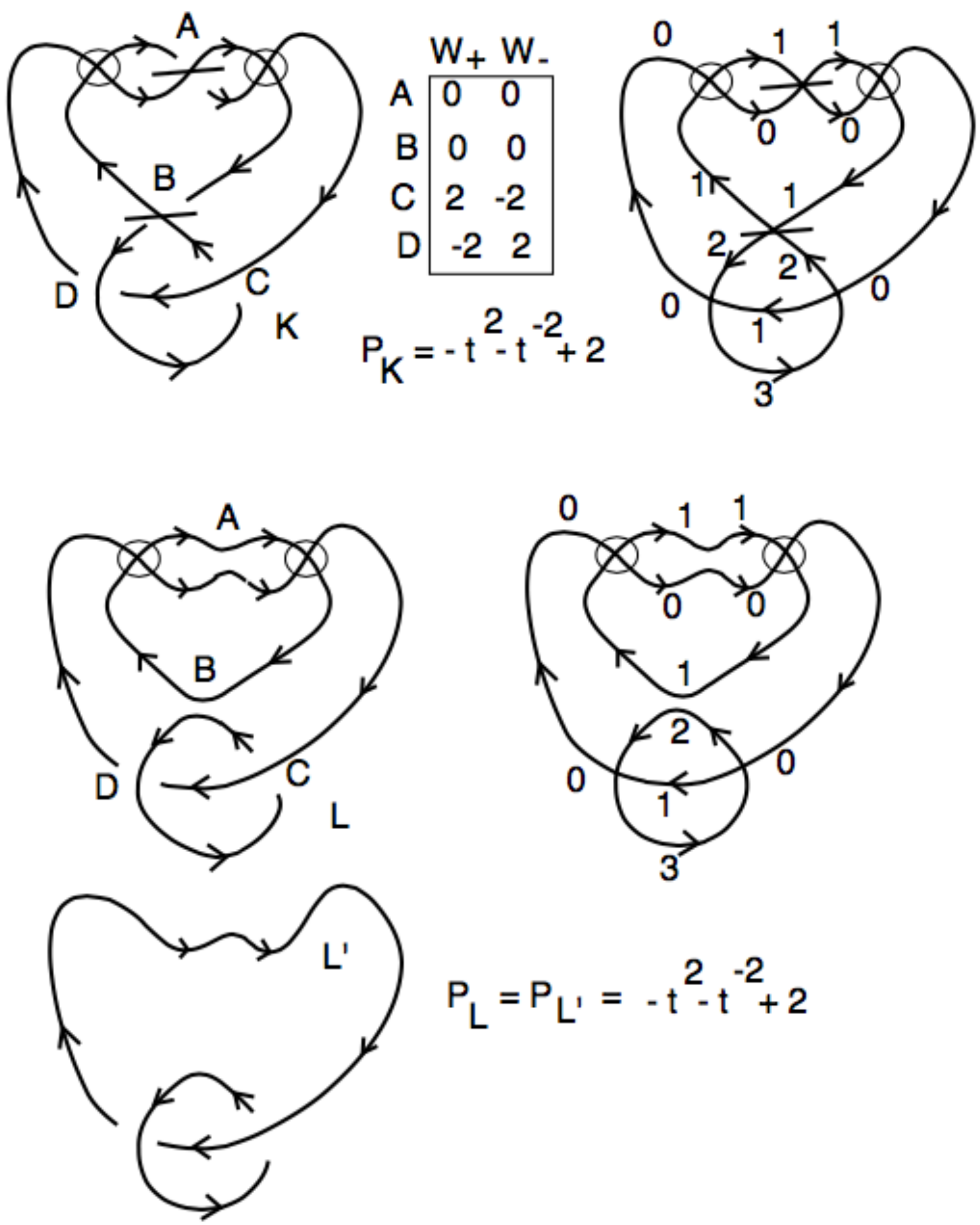

K
K'

Theorem. P_K and PF_K are concordance invariants.

Proof. Concordances are compositions of elementary concordances.//

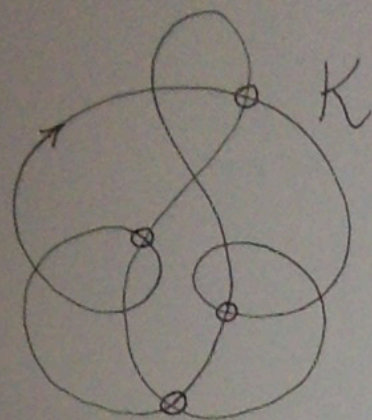
A special concordance of links is DEFINED to be a composition of elementary concordances.

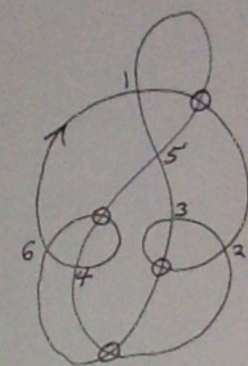
P_K and PF_K are invariants of special concordance for links that have an affine labeling.

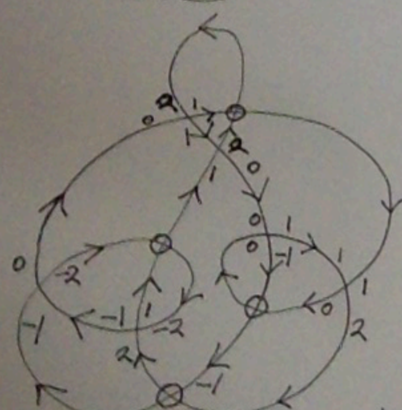


A labeled cobordism of a knot to a link.

K = Allison Henrich Buddah Knot







	w_+	w_-
1	1	−1
2	−1	1
3	−1	1
4	3	−3
5	−1	1
6	−1	1

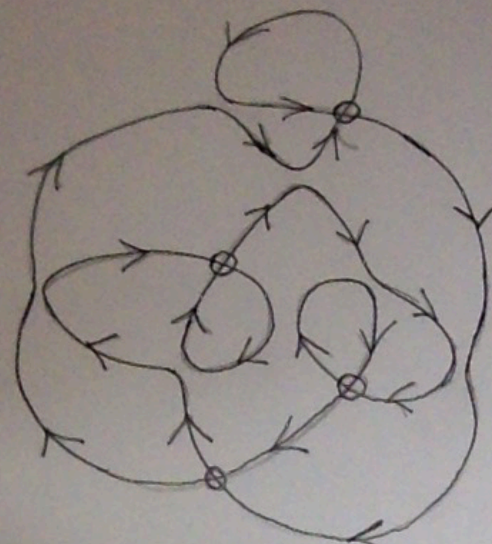
$$P_{K+} = t^3 + t^1 + 4t^{-1} - 6$$

$$PF_K \equiv t^3 + t \pmod 2$$

Flat Gauss Code

1 2 3 3 4 5 1 5 3 6 4 6 } all odd crossings

$\Rightarrow$ Non-Trivial Parity Bracket



K has <u>one</u> virtual Seifert Circuit.

$r = 1$.

$$\text{Virtual Seifert Genus} = \frac{1}{2}(-r + v + 1)$$

$$= \frac{1}{2}(-1 + 6 + 1) = 3.$$

$\Rightarrow$ K_+ has virtual 4-ball genus 3.

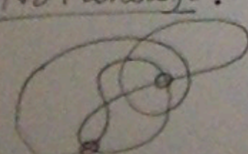
<u>Nota Bene</u>: $\overbrace{\frac{1}{2}(-R + v + 1)}^{\text{Seifert Genus}} = \overbrace{1 + \frac{v - \lambda}{2}}^{\text{Suface Genus}}$

$$\Leftrightarrow \quad -R + v + 1 = 2 + v - \lambda$$

$$\Leftrightarrow \quad -R + 1 = 2 - \lambda$$

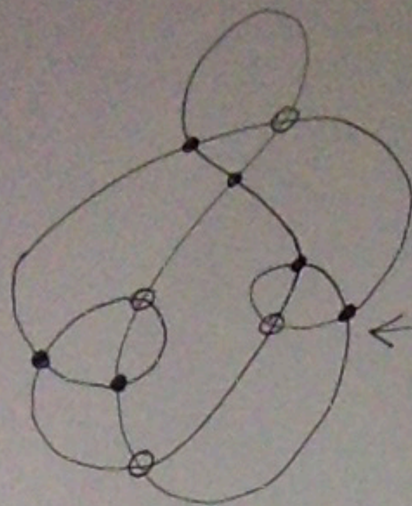
$$\Leftrightarrow \quad \boxed{\lambda = R + 1}$$

<u>Not always!</u>

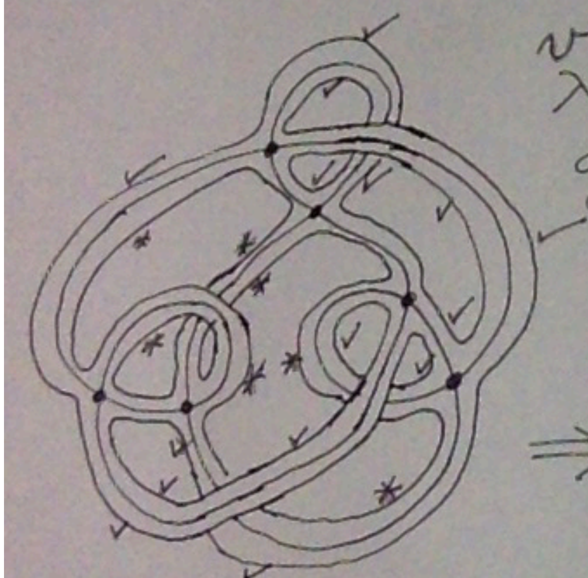


$\lambda = 4$

$R = 1$



Parity Bracket $\Rightarrow$ Surface genus of K = genus of surface for Ⓑ.



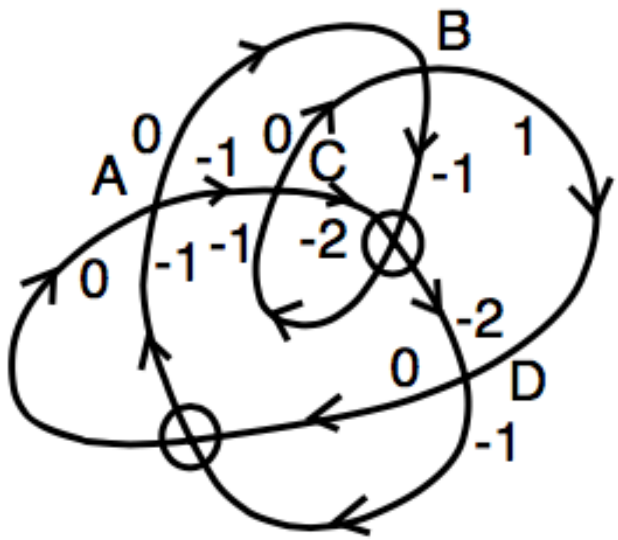
$\nu = 6$

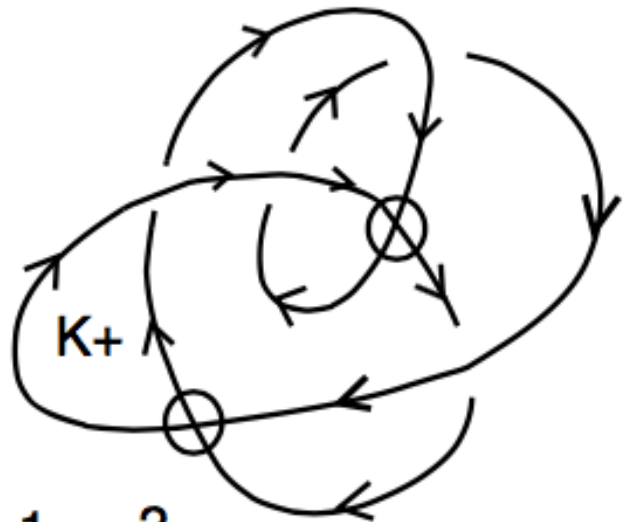
$\lambda = |\{\checkmark, *\}| = 2$

$$g = 1 + \frac{\nu - \lambda}{2} = 1 + \frac{4}{2}$$

$g = 3$

$\Rightarrow$ K has surface genus 3.

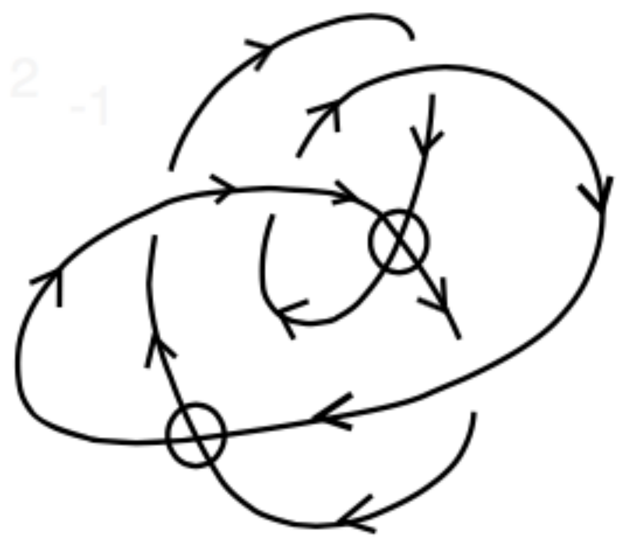




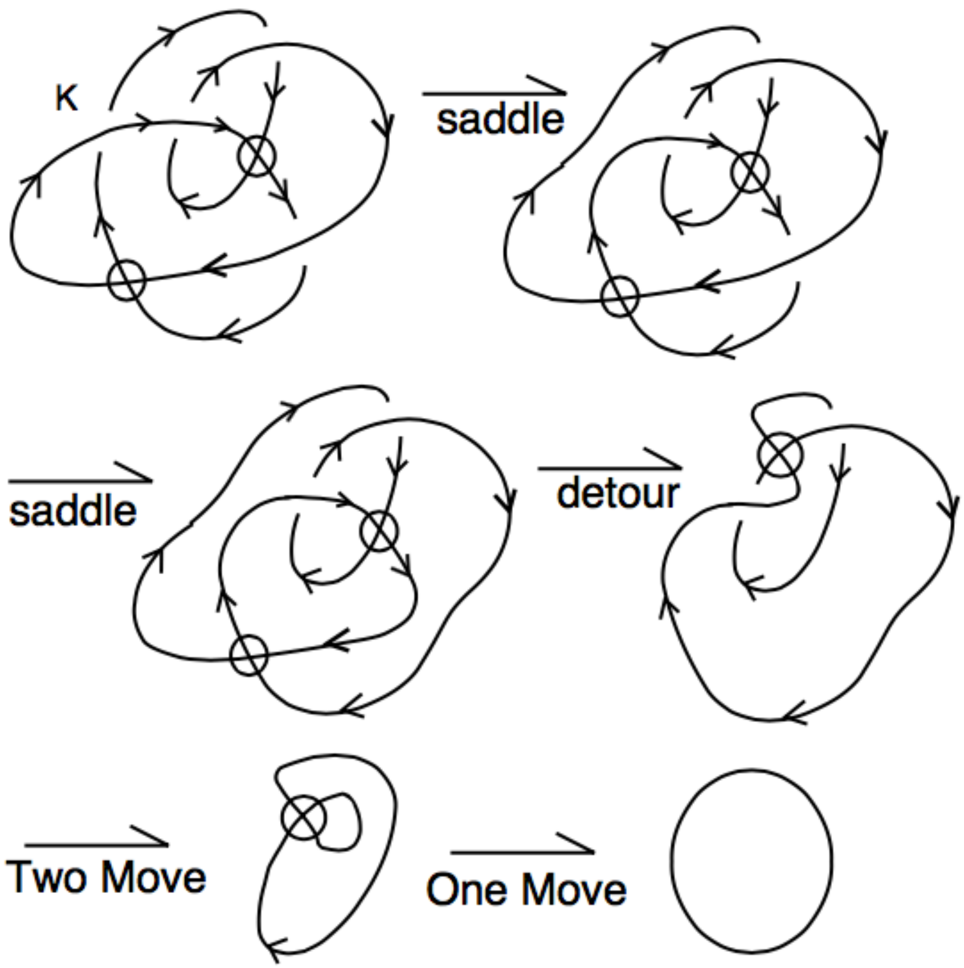
$PK_+ = 2t^{-1} + t^2 - 3$

$PK = t^2 - 1$

	w+	w-
A	0	0
B	-1	1
C	-1	1
D	2	-2



$PK = t^2 + t - t^{-1} - 1$



K bounds a virtual surface of genus one.

Hence, via P_K, K has genus one.

We can often use the concordance invariance of Index polynomials to determine genus one, when it occurs.

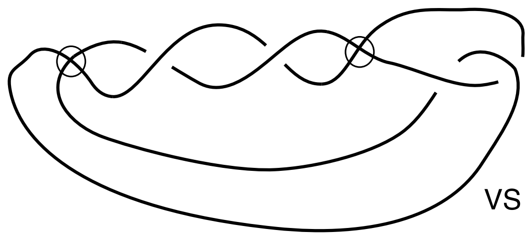
Is there a deeper relationship between virtual genus and the Index polynomials?

Understand concordance of virtual knots.

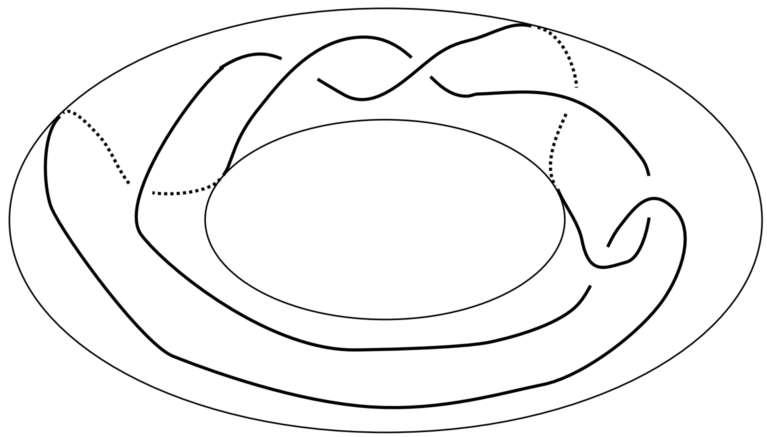
Fully understand concordance of flat virtual knots and links.

Thank you for your attention!

More about the Virtual Stevedore's Knot



VS



VS on a torus.