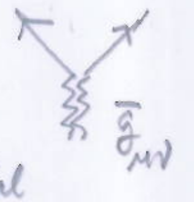
Lecture 2

Recap of lecture 1

- Different choices of vacua correspond to different notions of 'particles'
- particle creation in time-dependent cosmology



$|0_{in}\rangle \doteq$ bath of out-particles

- Non-inertial Obervers in Minkowski space can detect particles in the Minkowski vacuum.

$$\underset{|pure\rangle}{|0_M\rangle} = \frac{1}{\sqrt{Z_1}} \sum_i e^{-\beta E_i/2} \underset{|pure\rangle}{|i,i\rangle} \qquad Z_1 = \sum_i e^{-\beta E_i}$$

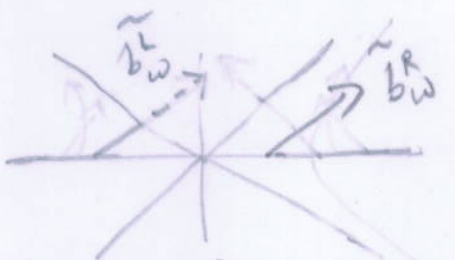
$\longrightarrow$ entangled state

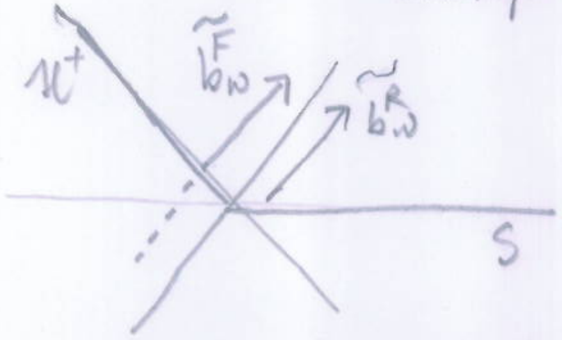
$\longrightarrow$ for the R observables

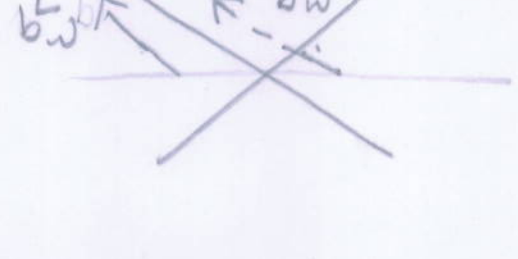
$$\rho_R = Tr_L |0_M\rangle\langle 0_M| = \frac{1}{Z_1}\sum_i e^{-\beta E_i} |i\rangle\langle i|$$

$\longrightarrow$ $|0_M\rangle$ provides a 'purification' of ρ_R by coupling the R-system to a 'bath' of L-modes

$$\rightarrow \prod_\omega e^{-\gamma_\omega \underbrace{b^{R+}_\omega b^{L+}_\omega}_{\substack{\uparrow \\ \text{Entangled pairs}}}} |0_R\rangle$$





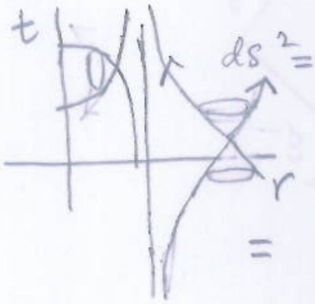


$\longrightarrow$ $S^{[A]}_{ent}$ can be computed in an AdS/CFT context,
Maldacena-Susskind EPR = ER

⑦

§4. Schwarzschild Black holes

- Coordinate systems & Penrose diagrams



$$ds^2 = -dt^2\left(1-\frac{2m}{r}\right) + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2$$ ✓ Schwarzschild

I: $r \in (2m, \infty)$

II: $r \in (0, 2m)$

$$= \left(1-\frac{2m}{r}\right)\left[-dt^2 + dr_*^2\right] + \text{''}$$

$r_* = r + 2m \ln\left|\frac{r}{2m} - 1\right|$

$r_* \to -\infty$ at horizon ✓

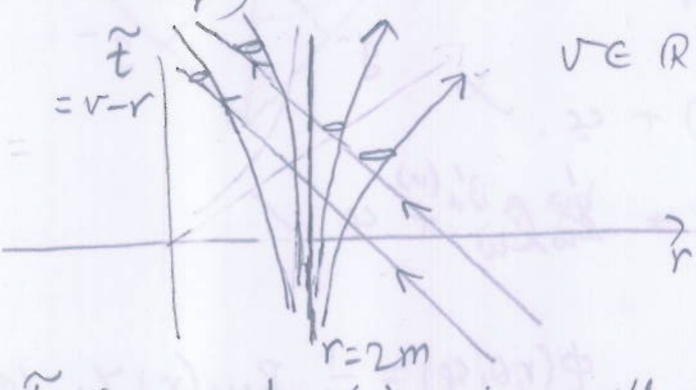
$$= \left(1-\frac{2m}{r}\right)(-du\,dv) + \text{''}$$

$u = t - r_*,\ v = t + r_*$

$$= -\left(1-\frac{2m}{r}\right)dv^2 + 2dv\,dr + \text{''}$$

Eddington-Finkelstein (ingoing)

$v \in \mathbb{R},\ r \in (0, \infty)$



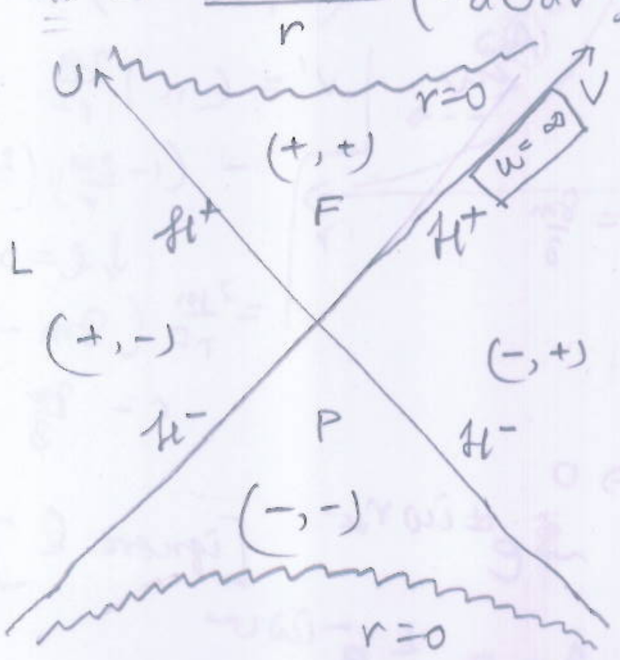
✓

$v \equiv \tilde{t} + r = \text{const}$ (ingoing null lines)

$u = v - 2r_* = \text{Const}$ (outgoing " ")

$\Rightarrow v = \text{const} + 2r_* \rightarrow -\infty$ as $r \to 2m \pm \varepsilon$

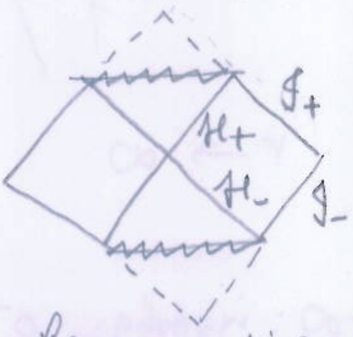
$$ds^2 = \frac{e^{-\frac{r}{2m}}\,16m^2}{r}(-dU\,dV) + r^2 d\Omega^2$$ ✓ Kruskal



$U = \pm\frac{1}{\kappa}e^{-\kappa u}$ ✓

$V = \pm\frac{1}{\kappa}e^{\kappa u}$ $\quad \kappa = \frac{1}{4m}$

cf. Rindler

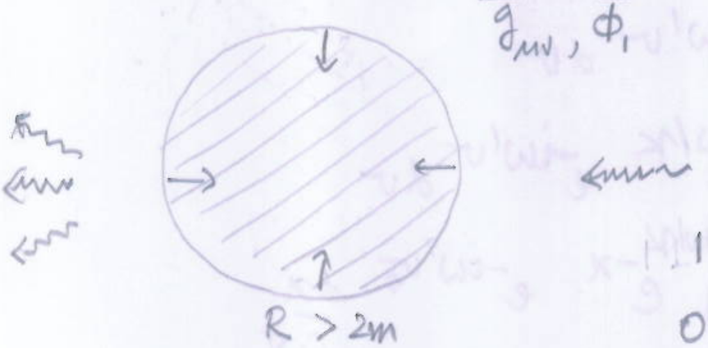


✓

Penrose diagram

Collapse (spherical)

$\bar{g}_{\mu\nu}, \bar{\Phi}_1$ ✓



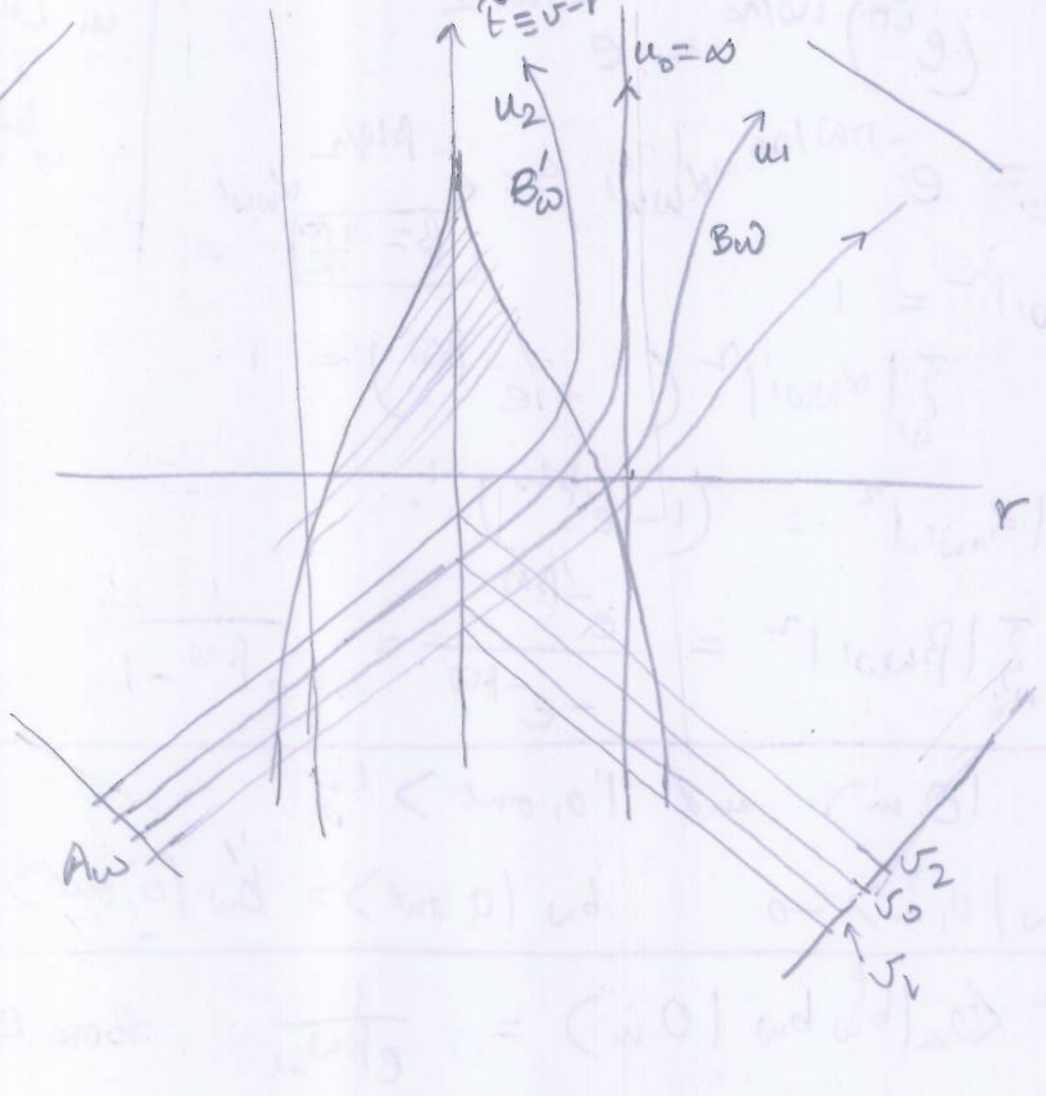
S-matrix

$1 \longrightarrow n$

$0 \longrightarrow n$ cf.

Also $1 \longrightarrow 0$

time-dependent geometry even in the "final stage" (no time like Killing vector inside the horizon)



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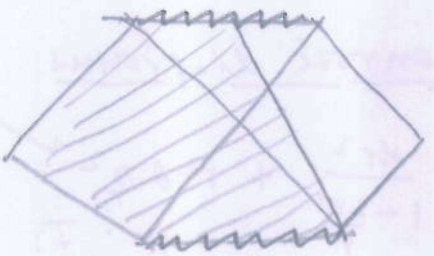
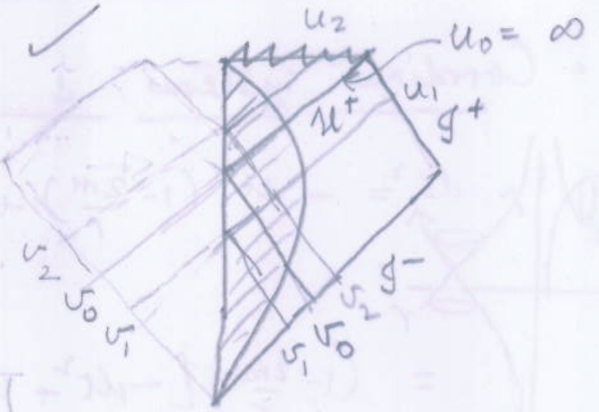
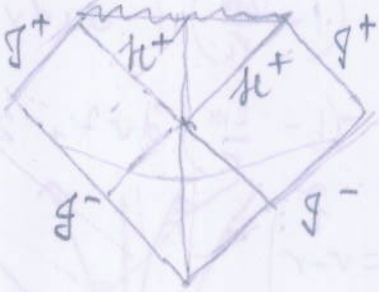
✓  ✓



- Past Cauchy slice $\longrightarrow$ flat space $\mathcal{J}^-$
- Future Cauchy slice $= \mathcal{J}^+ \cup \mathcal{H}^+$



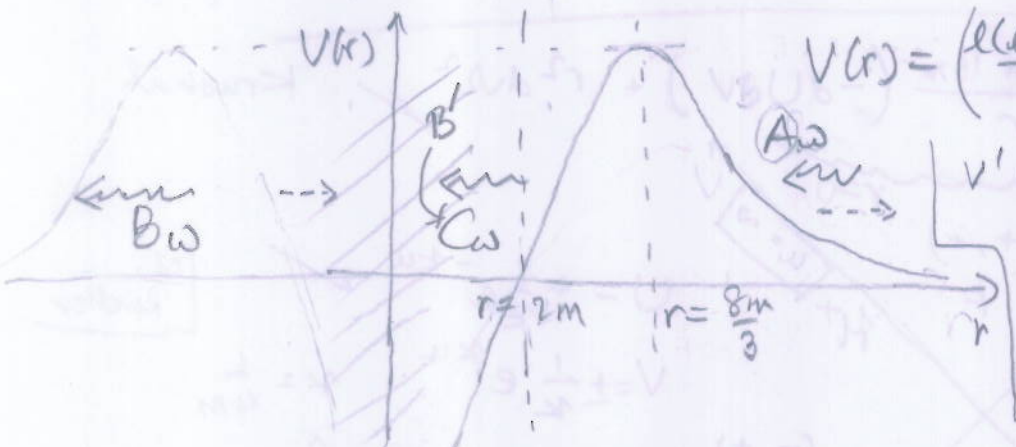
$$\phi = \sum_\omega a_\omega A_\omega(v) + c.c.$$ ✓

$$\phi = \sum_\omega b_\omega B_\omega(u) + \overset{b'_\omega}{c_\omega}\, \overset{B'_\omega(u)}{C_\omega} + c.c.$$ ✓

mode functions

4D $\quad D^\mu \partial_\mu \phi = 0 \qquad \phi(r,\theta,\varphi) = \frac{1}{r} R_{\omega l}(r) Y_{lm}(\theta,\varphi) e^{-i\omega t}$

$$-\frac{d^2 R_{\omega l}(r)}{dr_*^2} + \left(V(r) - \omega^2\right) R_{\omega l}(r) = 0$$



$$V(r) = \left(\frac{l(l+1)}{r^2} + \frac{2m}{r^3}\right)\left(1 - \frac{2m}{r}\right)$$

$$V' = (\;,,\;)\frac{2m}{r^2} - \left(1-\frac{2m}{r}\right)\left(\frac{2l(l+1)}{r^3} + \frac{6m}{r^4}\right)$$

$$\downarrow l = 0$$

$$= \frac{2m}{r^5}(8m - 3r)$$

$$r = \frac{8m}{3}$$

$r \to \infty \qquad V(r) \longrightarrow 0$

$$R_{\omega l}(r_*) \sim e^{\pm i\omega r_*} \quad [\text{ignore } l]$$

$t \to -\infty$: ingoing $e^{-i\omega r_*} \Rightarrow A_\omega = \frac{1}{r} e^{-i\omega v}$

$t \to +\infty$: outgoing $e^{i\omega r_*} \Rightarrow B_\omega = \frac{1}{r} e^{-i\omega u}$, ingoing C_ω

① $\longrightarrow$ $$B_\omega(v) = \sum_{\omega'} \alpha_{\omega\omega'} A_{\omega'}(v) + \beta_{\omega\omega'} \overline{A_{\omega'}(v)}$$ (4a)

$$\alpha_{\omega\omega'} = \int_{-\infty}^{v_0} B_\omega(v)\, e^{i\omega' v}\, dv$$
$$= \int (v-v_0)^{i\omega/\kappa}\, e^{-i\omega' v}\, dv$$
$$= \int_0^\infty \left(\frac{x}{i\omega'}\right)^{i\omega/\kappa} e^{-x}\, e^{-i\omega' v_0}\, \frac{dx}{i\omega'}$$
$$= \left(\frac{1}{i\omega'}\right)^{i\omega/\kappa} \frac{1}{i\omega'}\, \Gamma\!\left(\frac{i\omega}{\kappa}+1\right) e^{-i\omega' v_0}$$

$$\beta_{\omega\omega'} = \alpha_{\omega,-\omega'} \qquad -\omega' = e^{-i\pi}\omega'$$
$$\left(e^{i\pi}\right)^{i\omega/\kappa} = e^{-\pi\omega/\kappa}$$
$$\therefore\ \beta_{\omega\omega'} = e^{-\pi\omega/\kappa}\alpha_{\omega\omega'} =: e^{-\beta\omega/2}\alpha_{\omega\omega'} \qquad \boxed{\beta = \frac{2\pi}{\kappa}}$$

$e^{-i\omega' v}$ analytic in LHP of ω' ∴ large v behaviour is regular

$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1$$
$$\Rightarrow \sum_{\omega'} |\alpha_{\omega\omega'}|^2 \left(1 - e^{-\beta\omega}\right) = 1$$
$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 = \left(1 - e^{-\beta\omega}\right)^{-1}$$
$$\therefore \sum_{\omega'} |\beta_{\omega\omega'}|^2 = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} = \frac{1}{e^{\beta\omega}-1}$$

Define $|0, in\rangle$ and $|0, out\rangle$ by
$$a_\omega |0, in\rangle = 0 \qquad b_\omega |0, out\rangle = b'_\omega |0, out\rangle = 0$$

We get $\langle 0_{in}| b^\dagger_\omega b_\omega |0\ in\rangle = \dfrac{1}{e^{\beta\omega}-1}$ Bose-Einstein

How about entanglement?

2D schwarzschild

e.g. $ds^2 = -dt^2\left(1-ae^{-Q\eta}\right) + \frac{d\eta^2}{1-ae^{-Q\eta}}$, $\Phi = -\frac{Q}{2}\eta$

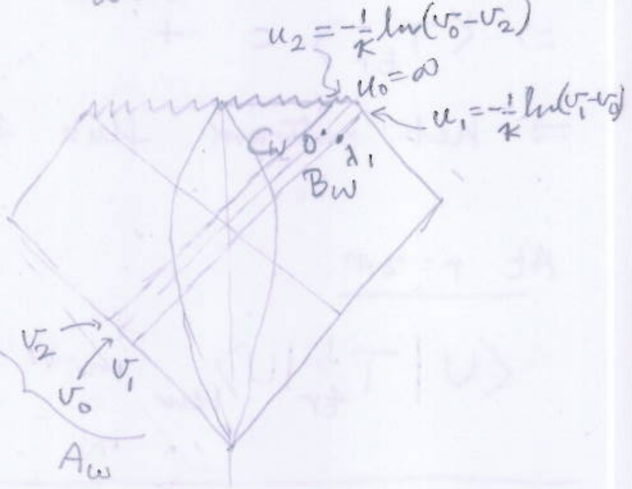
We will take $ds^2 = -dt^2\left(1-\frac{2m}{r}\right) + \frac{dr^2}{1-\frac{2m}{r}}$

→ no back-scattering.

$$\phi = \sum_{\omega}^{b} a_\omega A_\omega(v) + c.c.$$

$$= \sum_{\omega} b_\omega B_\omega(u) + b'_\omega B'_\omega + c.c.$$

$A_\omega(u) = e^{-i\omega v}$



$B_\omega(u) = e^{-i\omega u}$ on $\mathscr{J}^+$

① to pg (4ω) $= e^{i\omega/\kappa \ln(v-v_0)}$ $\mathscr{J}^-$

Pf. in $B_\omega(u)$

Phase change between $(u=u_1)$ and $(u=\infty)$ $= \omega\times\infty = \infty$

Change of affine parameter = finite

$u=u_1 \Rightarrow \lambda_1 = r_1 - 2m \simeq \text{Const}\, e^{-\frac{u_1}{4m}}$ —(i)

$u=u_0=\infty \Rightarrow \lambda_0 = 0$

$\frac{r_*}{2m} \simeq \ln\left(\frac{r}{2m}-1\right)$

$\frac{r-2m}{2m} \simeq e^{\frac{r_*}{2m}}$

$\simeq e^{\frac{v-u}{4m}}$

on a const v line

$r-2m \simeq (\text{Const})e^{-\frac{u}{4m}}$

∴ ∞ change of phase for a finite change of affine parameter

⇒ ∞ blue-shift

⇒ geometrical optics (eikonal approximation)

(Also details of collapsing matter can be ignored)

Ray tracing back to $\mathscr{J}^-$:

Along constant u, $v = \text{const} + 2r_* \simeq \text{const} + 2r$

but $\lambda = \text{Const.} + r$

$\Rightarrow v = \text{Const} + 2\lambda$

$$\Rightarrow \Delta v = v_1 - v_0 = \lambda_1 - \underset{0}{\underset{\shortparallel}{\lambda_0}} = Ce^{-\frac{u_1}{4m}} = Ce^{-\kappa u_1} \quad \text{—(ii)}$$

$\Rightarrow u_1 = -\frac{1}{\kappa}\ln\Delta v$

(5b)

Flux calculation

At ∞

$$\langle U | T_\mu{}^\nu | U \rangle_{ren} \xrightarrow{r\to\infty} \frac{L}{4\pi r^2}\left[\begin{array}{cc|c} -1 & -1 & \\ 1 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right]$$

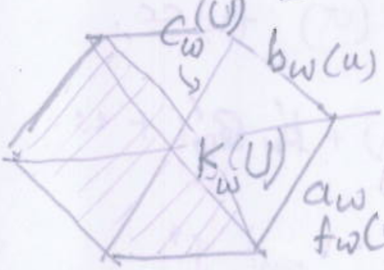
$$L = \frac{1}{2\pi}\int_0^\infty d\omega\, \omega \, \frac{(2l+1)|B_l(\omega)|^2}{e^{\beta\omega}-1}$$

greybody factor

i.e. $\langle T^t{}_r \rangle = -$

$\Rightarrow \langle T_{tr} \rangle = +$

$\Rightarrow$ net outward flux



$$|0_{in}\rangle \longrightarrow |0\rangle_U$$
$$= |0\rangle_{a,\tilde{b}}$$

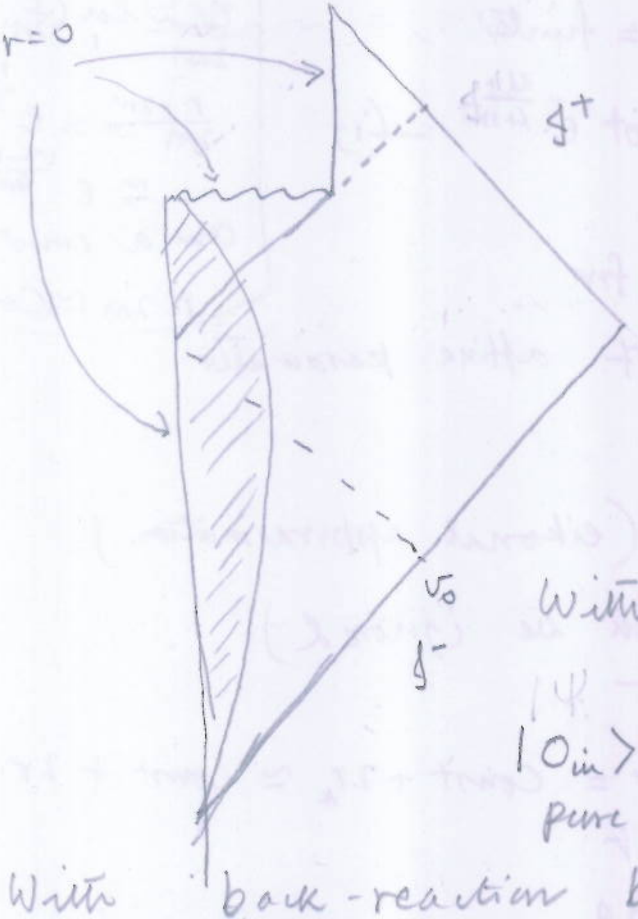
At r = 2M

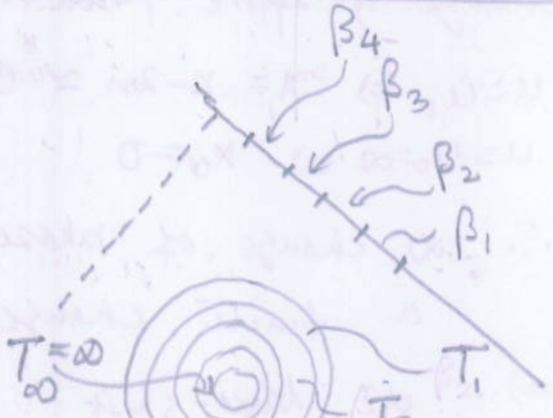
$\langle U | T_{tr} | U \rangle_{ren} \sim L \times$ finite constant (+ve)

1

$$\langle U | b_\omega^\dagger b_\omega | U \rangle = \frac{1}{e^{\beta\omega}-1}$$

Back-reaction





Without back reaction

$$|0_{in}\rangle \longrightarrow e^{-\sum \gamma_\omega \tilde{a}_\omega^\dagger a_\omega'^\dagger} |0\rangle_{sch}$$

pure state → pure state

$$\underline{\underline{\rho_{\tilde{a}_\omega^\dagger}}} = \frac{e^{-\beta H}}{Z}$$

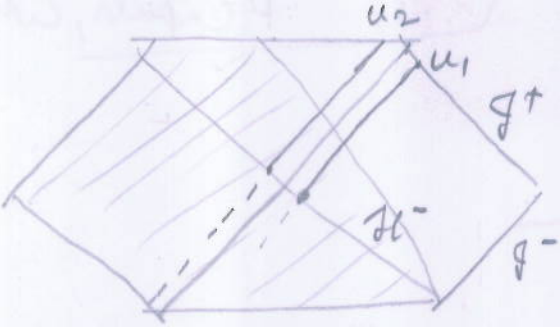
With back-reaction b.h. evaporates:
no $a_\omega'^\dagger$ modes (no inside modes). left
with $\rho_{\tilde{a}_\omega^\dagger} = \frac{e^{-\beta H}}{Z}$

Now observe that near the horizon

$$\text{Eq (ii)} \Rightarrow v_1 - v_0 = C U_1 = C(U_1 - U_0)$$

$$\text{Eq (i)} \Rightarrow \lambda_1 = C' U_1 \qquad \because U_0 = 0 \text{ (value of } U \text{ on the horizon)}$$

$$U = -\frac{1}{\kappa} e^{-\kappa u}$$



$$B_\omega = e^{-i\omega u_1} \quad \text{near } \mathcal{I}^+$$

$$= e^{+i\omega/\kappa \ln(-\kappa U)} \quad \text{near } \mathcal{H}^- \quad \left[U = -\frac{1}{\kappa} e^{-\kappa u}\right]$$

affine parameter

The appropriate 'in'-mode on $\mathcal{H}^-$ is

$$\tilde{K}_\omega = e^{-i\omega U} \qquad (\text{analogous to } e^{-i\omega v} \; \underline{v > v_0})$$

The Unruh vacuum is defined (in terms of the eternal black hole geometry) in terms of the mode expansion

$$\phi = \sum_\omega a_\omega A_\omega(v) + \tilde{k}_\omega \tilde{K}_\omega(U) + c.c.$$

$$A_\omega(v) = e^{-i\omega v}$$

$$\tilde{K}_\omega(U) = e^{-i\omega U}$$

This can be seen to be complete on $\mathcal{H}^- \cup \mathcal{I}^-$

$$|0\rangle_U : \quad a_\omega |0\rangle_U = 0 = \tilde{k}_\omega |0\rangle_U$$

Kruskal modes $\phi = \sum_\omega k_\omega K_\omega(V) + \tilde{k}_\omega \tilde{K}_\omega(U) + c.c.$

$$|0\rangle_K : \quad k_\omega |0\rangle_K = 0 = \tilde{k}_\omega |0\rangle_K$$

$K_\omega = e^{-i\omega V}$

$\tilde{K}_\omega(U) = e^{-i\omega U}$

cf. Minkowski modes

Boulware vacuum: $\phi = \sum a_\omega A_\omega(v) + \tilde{a}_\omega A_\omega + c.c.$

$$|0\rangle_B : \quad a_\omega |0\rangle_B = 0 = \tilde{a}_\omega |0\rangle_B$$

① $|0\rangle_K = \frac{1}{\sqrt{Z}}\sum_i e^{-\beta E_i/2}\,|i, i\rangle$ (5 a)

Israel, ..., Maldacena, Maldacena-Hartman, ..., P Capulā, CM, R. Sinha

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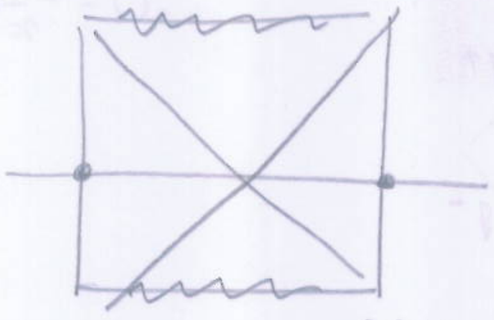
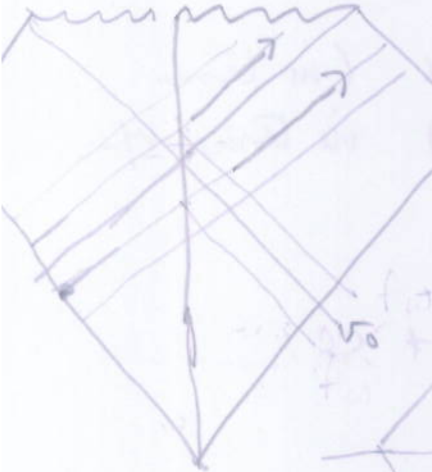
AdS/CFT



$$|\Psi\rangle_{CFT} = \frac{1}{\sqrt{Z_{CFT}}}\sum e^{-\beta E_i/2}\,|i, i\rangle$$

$$|i\rangle = |\{N_k, \tilde{N}_k\}\rangle = |\{N_k\}\rangle \equiv \prod_k \frac{(a_k^{+})^{N_k}}{\sqrt{N_k!}}|0\rangle_S$$

Unruh vacuum



$|0\rangle_U$

$$\boxed{\phi = \sum \tilde{a}_\omega A_\omega(v) + \tilde{k}_\omega \tilde{K}_\omega(U) + \tilde{a}'_\omega A'_\omega(v)}$$

$|0\rangle_{in}$

$$\phi = \sum \underbrace{a_\omega A_\omega(v)}_{v > v_0} + \underbrace{a_\omega A_\omega(v)}_{v < v_0}$$

$|0\rangle_H$, $\phi = \sum_\omega k_\omega K_\omega(V) + \tilde{k}_\omega \tilde{K}_\omega(U)$

$|0\rangle_S \equiv |0\rangle_B$ $\phi = \boxed{a_\omega A_\omega(v) + \tilde{a}_\omega \tilde{A}_\omega(u) + a^{\prime}_\omega A^{\prime}_\omega(v) + \tilde{a}^{\prime}_\omega \tilde{A}^{\prime}_\omega(u)}$

$$|0\rangle_U = e^{-\sum_\omega r_\omega \tilde{a}^{+}_\omega \tilde{a}^{\prime +}_\omega}|0\rangle_B$$

Properties of the various vacua

$$|0\rangle_K = e^{-\sum_\omega e^{-\beta\omega/2}(a_\omega^{R\dagger} a_\omega^{L\dagger} + \tilde{a}_\omega^{R\dagger} \tilde{a}_\omega^{L\dagger})}|0\rangle_S$$

① = entangled state of Boulware modes

← to page (5a)

$\beta = \frac{2\pi}{\kappa}$ $\kappa = \frac{1}{4M}$

Similarly
$$|0\rangle_U = e^{-\sum_\omega e^{-\beta\omega/2} \tilde{a}_\omega^{R\dagger} a_\omega^{L\dagger}}|0\rangle_S$$

no mixing of the v-modes

$|0\rangle_U$

~~Stress tensors:~~

~~Rindler~~

~~$\langle 0_M | T_{UU} | 0_M \rangle = 0$~~

~~$T_{UU} = \frac{1}{U_u^2}\left(T_{uu} + \frac{c}{12}\frac{U_u U_{uuu} - \frac{3}{2}U_{uu}^2}{U_u^2}\right)$~~

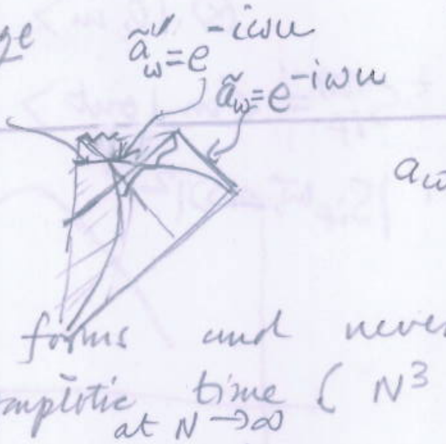
~~$= \frac{1}{U_u^2}\left(T_{uu} - \frac{c}{24}a^2 e^{-2au}\right)$~~

~~$\therefore T_{uu} = \frac{c}{24} 4\pi^2 T^2 \boxed{e^{-2au}}$~~

$U = -\frac{1}{a}e^{-au}$

$T = \frac{a}{2\pi}$

strange



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microscopic model:

- challenges $\tilde{a}'^{\dagger}_\omega$ modes ?
- time black hole never forms and never evaporates in asymptotic time (N^3 time) at $N \to \infty$
- D1-D5 $|\text{pure}\rangle \longrightarrow |\text{pure}\rangle \xrightarrow[\text{(by hand)}]{\text{averaging}} \rho = \frac{e^{-\beta H}}{Z}$

Cosmology

$$(a_\omega - \gamma_\omega a_\omega^\dagger)|0_{in}\rangle = 0 \quad \forall \omega$$

$$|0_{in}\rangle = \underbrace{e^{\sum_\omega \gamma_\omega a_\omega^\dagger a_\omega} |0_{out}\rangle}_{\text{pure state}}$$

| pure state⟩

Rindler

$$\overset{|\text{pure}\rangle}{|0_{Min}\rangle} = e^{\sum_\omega \gamma_\omega (a_\omega^\dagger a_\omega'^\dagger + \tilde{a}_\omega^\dagger \tilde{a}_\omega'^\dagger)} \overset{|\text{pure}\rangle}{|0_{Rin}\rangle}$$

$$= \sum_{i \in \mathcal{H}} e^{-\beta E_i/2} |i\rangle_L |i\rangle_R$$

$\uparrow a_\omega'$ $\uparrow a_\omega$

$$\downarrow$$

$$\rho_R = \frac{1}{Z_R} e^{-\beta H_R}$$ reduced density matrix is thermal

$|0_{Min}\rangle$ provides a purification to ρ_R

a_ω' (system) $\times$ system a_ω

Black hole

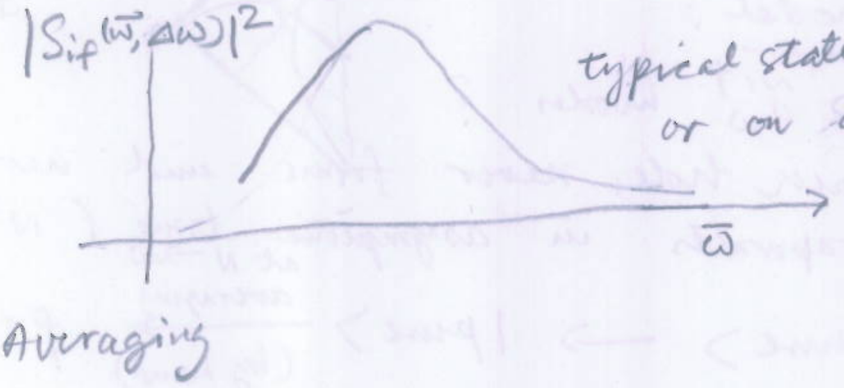
$$\overset{|0_{unruh}\rangle}{|0_{in}\rangle} \Longrightarrow e^{\sum_\omega \gamma_\omega \tilde{a}_\omega^\dagger \tilde{a}_\omega'^\dagger} |0_{sch}\rangle \xrightarrow[\mathrm{Tr}_{\tilde{a}_\omega'^\dagger}]{} \rho_a = \frac{1}{Z} e^{-\beta H_a}$$

|pure⟩ |pure⟩

D1-D5

$$|in\rangle = |\overbrace{\{N_\omega^L, \tilde{N}_\omega\}}^{E}\rangle \otimes |0, in\rangle \xrightarrow{U_t} |out\rangle = \overbrace{|\{N_\omega', \tilde{N}_\omega'\}\rangle}^{E-\omega} \otimes |1_{\bar{\omega}}\rangle$$

$$S_{if}^{(\bar{\omega})} = |\langle in | out \rangle|^2 \propto \sqrt{N_{\bar{\omega}/2}} \sqrt{\tilde{N}_{\bar{\omega}/2}}$$ S-matrix



typical state or on averaging

Averaging