# Aspects of wavebreaking in nonlinear plasma oscillations

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8$^{th}$ March 2017

# Introduction

- Strong longitudinal plasma waves can be used to accelerate particles.
- Stronger fields lead to greater acceleration, but there is a limit to how hard we can drive these waves.
- The maximum sustainable electric field depends on a wide variety of physical phenomena.
- Can wavebreaking limits reveal new fundamental physics?

# Electron acceleration and plasma oscillations

An intense laser pulse or particle beam propagating through a plasma creates an 'electrostatic' wake: a modulation in the density of electrons, inducing an electric field

$$E(z,t) = E_0 \cos(\omega_p \zeta), \tag{1}$$

where $\omega_p = \sqrt{nq^2/m}$ is the plasma frequency and $\zeta = t - z/\beta$ with $\beta$ the speed of the driver.

Increasing the strength $E_0$ results in two changes: the oscillations become distorted (non-sinusoidal), and the frequency drops.

These effects lead to a limit on the strength of a propagating wakefield—the 'wavebreaking limit'—and the energy a particle can gain from it.

# Electron response to electrostatic fields

The electron fluid in a cold plasma can be described by its proper density $n$ and 4-velocity $V^\mu$. These evolve according to the continuity equation

$$\partial_\mu (nV^\mu) = 0 \tag{2}$$

and the Lorentz force

$$V^\nu \partial_\nu V^\mu = \frac{q}{m} F^\mu{}_\nu V^\nu. \tag{3}$$

(Ions are assumed immobile on relevant timescales.)

Assuming no transverse motion, the Lorentz force is solved by

$$F_{\mu\nu} = \frac{m}{q} (\partial_\nu V_\mu - \partial_\mu V_\nu), \tag{4}$$

i.e. the velocity can serve as the 4-potential.

$V^\mu$ can now be used to eliminate $F_{\mu\nu}$ from Maxwell's equations.

# Maxwell's equations and the nonlinear oscillator

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Assuming all fields depend only on the wave phase $\zeta$, Maxwell's equations now reduce to an algebraic equation and an ODE.
The algebraic equation yields the density:

$$n = \frac{\beta n_0}{\Gamma(\beta - u)}, \tag{5}$$

where $V^\mu \partial_\mu = \Gamma(\partial_0 + u\partial_1)$. Clearly, the electrons cannot outrun the wave.
The ODE is a nonlinear oscillator equation

$$\frac{d^2\mu}{d\zeta^2} = \omega_p^2 \beta^2 \gamma^2 \left[ \frac{\beta\mu}{\sqrt{\mu^2 - \gamma^{-2}}} - 1 \right], \tag{6}$$

where $\mu = \Gamma(1 - \beta u)$ is the electron energy in the wave frame.

# Cold wavebreaking limit

We can integrate the oscillator equation:

$$\frac{1}{2}\left(\frac{d\mu}{d\zeta}\right)^2 + \omega_p^2\gamma^2\beta^2\left[\mu - \beta\sqrt{\mu^2 - \gamma^{-2}}\right] = C. \tag{7}$$

For maximum amplitude oscillations, $\mu = \gamma^{-1}$ must be a turning point, fixing the constant $C = \omega_p^2\gamma\beta^2$.

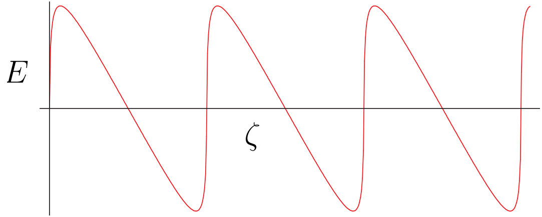
Finally, noting that the electric field is given by $E = -\frac{m}{\beta q}\frac{d\mu}{d\zeta}$ and peaks at $\mu = 1$,

$$E_{\max} = \frac{m\omega_p}{q}\sqrt{2(\gamma - 1)}. \tag{8}$$

Akhiezer and Polovin, JETP **3**, 696 (1956),
Dawson, Phys. Rev. **113**, 383 (1959).

# Plasma pressure

While the mathematics leading to the cold wavebreaking limit is robust, it neglects some important physics. Consider the electric field for a wave close breaking:




From Gauss's law, the rapid jumps in the electric field imply the electrons acquire a very high density. At these densities we can no longer ignore the plasma pressure. But we are far from thermal equilibrium—what equation of state is appropriate?

# Vlasov equation

When momentum spread is non-negligible, we cannot describe the motion of electrons with a single vector field. Instead we use a distribution function $f$, which satisfies the Vlasov equation:

$$\dot{x}^{\mu}\left(\frac{\partial f}{\partial x^{\mu}} + \frac{q}{m}F^{\nu}{}_{\mu}\frac{\partial f}{\partial \dot{x}^{\nu}}\right) = 0. \tag{9}$$

This states that all electrons in the distribution follow orbits of the Lorentz force.

The Vlasov equation is often used to close the hierarchy of equations for moments of the system. This is often simpler than solving the Vlasov equation.

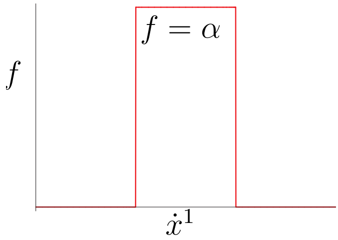
However, the Vlasov equation does admit some very simple solutions.

# 1D Waterbag

Simple solution to the 1D Vlasov equation: $f = \alpha = \text{const}$.
No good: $\varrho = \int f(u) du = \infty$.

However, consider 'waterbag' distribution:

$$f = \alpha \quad \text{for} \quad U_- < \dot{x}^1 < U_+, \text{ f=0 otherwise.}$$




The dynamics of the Vlasov equation is reduced to describing the boundaries $U_\pm$, which both follow the Lorentz force.

# Warm wavebreaking limits in 1D

Since the boundaries of the waterbag follow the Lorentz force, the analysis is very close to the cold plasma.

The nonlinear oscillator equation becomes

$$\frac{d^2\mu}{d\zeta^2} = \omega_p^2\beta^2\gamma^2\left[\frac{\beta}{2a}\left\{\sqrt{(\mu+a)^2-\gamma^{-2}}-\sqrt{(\mu-a)^2-\gamma^{-2}}\right\}-1\right] \tag{10}$$

where $a = \beta n_0/2\alpha \approx \beta\sqrt{3T/m}$ is a measure of the thermal spread.

For $\gamma \gg 1$, this yields a wavebreaking limit

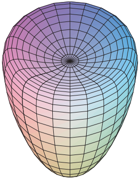
$$E_{\max} = \frac{m\omega_p}{q}\sqrt{\frac{\ln[a\gamma]}{2a}}. \tag{11}$$

Katsouleas and Mori, PRL **61**, 90 (1988),
Trines and Norreys, Phys. Plasmas **13**, 123102 (2006).

# Waterbags in 3D

A waterbag distribution describing thermal spread in 3 velocity dimensions is substantially more complicated.

We must account for the structure of its boundary:



Points on the boundary are no longer required to follow Lorentz orbits (gauge invariance).

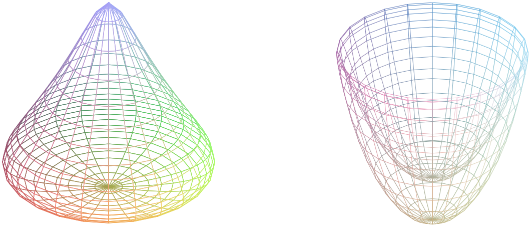
Nevertheless, even simple choices exhibit a rich structure.

Parametrise a class of 'spheroidal' distributions by two constants: $a$ determines longitudinal and $R$ transverse temperatures.

Burton, Noble and Wen, Nuovo Cimento C **32**, 1 (2009).

# Warm wavebreaking limits in 3D

A generalisation of the analysis for the 1D waterbag yields the wavebreaking limits for the spheroidal waterbags. These depend dramatically on the ratio of $a/R$.



As $\beta \to 1$:

- For $a \gg R$, $E_{\max} \approx \frac{m\omega_p}{q}\left(\frac{9m}{20T}\right)^{\frac{1}{4}}$
- For $a \ll R$, $E_{\max} \approx \frac{m\omega_p}{q}\sqrt{2(\gamma-1)}\left(1-\frac{3}{8}\sqrt{5T/m}\right)$

Burton and Noble, AIP Conf. Proc. **1086**, 252 (2009); J. Phys. A **43**, 075502 (2010).

# Nonlinear electrodynamics

Electric fields close to wavebreaking can be very strong. In ultra-strong fields, Maxwell's equations acquire nonlinear corrections:

- Euler-Heisenberg theory—encode $e^+e^-$ loops in $\gamma\gamma$ coupling (well established theoretically, out of reach experimentally?)
- Born-Infeld theory—postulated to avoid self-energy divergences, later rediscovered in string theory (more speculative, involves a free parameter $\kappa$)

Replace inhomogeneous Maxwell equation with

$$\partial_\mu G^{\mu\nu} = j^\nu, \qquad G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial X} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial Y} \widetilde{F}^{\mu\nu}, \tag{12}$$

with $\mathcal{L}$ the Lagrangian, some function of field invariants $X = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ and $Y = -\frac{1}{4} F^{\mu\nu} \widetilde{F}_{\mu\nu}$.

# NLED effects on particle acceleration

Nonlinear effects typically reduce the strength of electromagnetic fields compared to their Maxwellian equivalents.

This reduction in field strength is inherited by the wavebreaking limits. In BI, for example, the cold wavebreaking limit is

$$E_{\mathrm{max}}^{BI} = \frac{1}{\kappa}\sqrt{1 - \left[1 + \frac{\kappa^2}{2} E_{\mathrm{max}}^2\right]^{-2}} \leq E_{\mathrm{max}}. \tag{13}$$

Nevertheless, a compensating increase of the oscillation wavelength ensures the maximum energy a particle can acquire in this field, $\Delta\mathcal{E}_{\mathrm{max}} = m(4\gamma^3 - 3\gamma - 1)$, is the same as found in Maxwell's theory. This result is true for all nonlinear generalisations of Maxwell.

Burton, Trines, Walton and Wen, J. Phys. A **44**, 095501 (2011),
Burton, Flood and Wen, J. Math. Phys. **56**, 042901 (2015).

# Axion electrodynamics

Many extensions of the standard model of particle physics predict the existence of axions, light scalar particles coupling to $\widetilde{F}^{\mu\nu}F_{\mu\nu}$. In the presence of axions, the Maxwell equations become

$$\partial_\mu G^{\mu\nu} = g\partial_\mu \alpha \widetilde{F}^{\mu\nu} + j^\nu, \tag{14}$$

and are supplemented by the equation for the axion

$$(\partial^2 - m_\alpha^2)\alpha = \frac{g}{4}\widetilde{F}^{\mu\nu}F_{\mu\nu}. \tag{15}$$

The electrons follow the Lorentz force as before.

Observations restrict the axion coupling, $g \lesssim 10^{-7}$ GeV$^{-1}$, and mass, $10^{-6}$ eV $\lesssim m_\alpha \lesssim 10^{-2}$ eV.

# Axionic suppression of wakefield acceleration

It is highly unlikely that axions could influence laboratory wakefields. However, it has been suggested that plasma waves in neutron stars may be responsible for ultra-high energy cosmic rays. In such a scenario, axions generated by the plasma wave and the neutron star's magnetic field could play a role.

In the presence of a magnetic field $B$, the energy gain of a charged particle is limited to

$$\frac{\Delta\mathcal{E}_{\text{axion}}}{\Delta\mathcal{E}_{\text{max}}} \simeq 1 - \frac{g^2 B^2}{m_\alpha^2}\left(1 - \frac{\tanh\sigma}{\sigma}\right), \quad \sigma = \sqrt{2\gamma^3}\frac{m_\alpha}{\omega_p}. \tag{16}$$

In a typical neutron star, $B \sim 10^8$ T. Using this and the bounds on the axion mass and coupling, this leads to a 90% reduction in energy gain. However, this depends strongly on the axion mass: increasing this to $10^{-5}$ eV gives only a 3% reduction.

Burton, Noble and Walton, J. Phys. A **49**, 385501 (2016).

# Summary

- Nonlinear plasma waves exhibit wavebreaking phenomena.
- It is relatively straightforward to derive wavebreaking limits; the physical interpretation is sometimes less straightforward.
- Wavebreaking limits in thermal plasmas depend heavily on the details of the electron distribution function.
- 'New' physical phenomena can affect wavebreaking, but how this can be tested remains unclear.

# Acknowledgements

ALPHA-X
http://phys.strath.ac.uk/alpha-x

SUPA

EPSRC
Engineering and Physical Sciences
Research Council

Laserlab
Europe

eli
extreme light infrastructure

## Thank you