Plasma Based particle Acceleration

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Need for Plasma based accelerators

Limitation of conventional linear accelerators:

- Material breakdown limit 100MV/m.

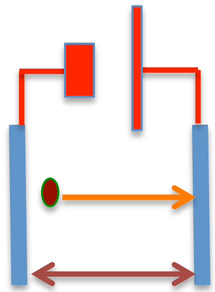
  SLAC length = 3.2 Km produces 50 GeV electrons!

Plasma based accelerators:

- No breakdown limitation.
- Short length high energy accelerators.

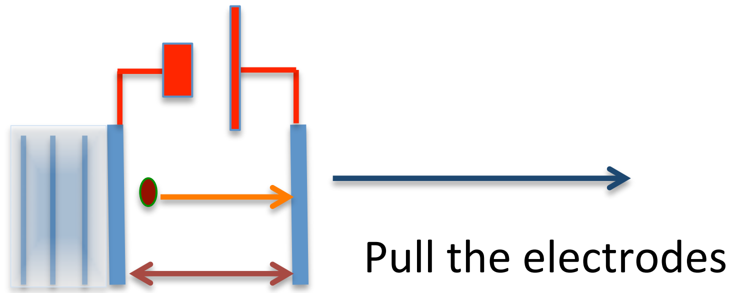
  30 Billion Volts/m (e.g. 1000 times superior)

Principle of particle Accelerator



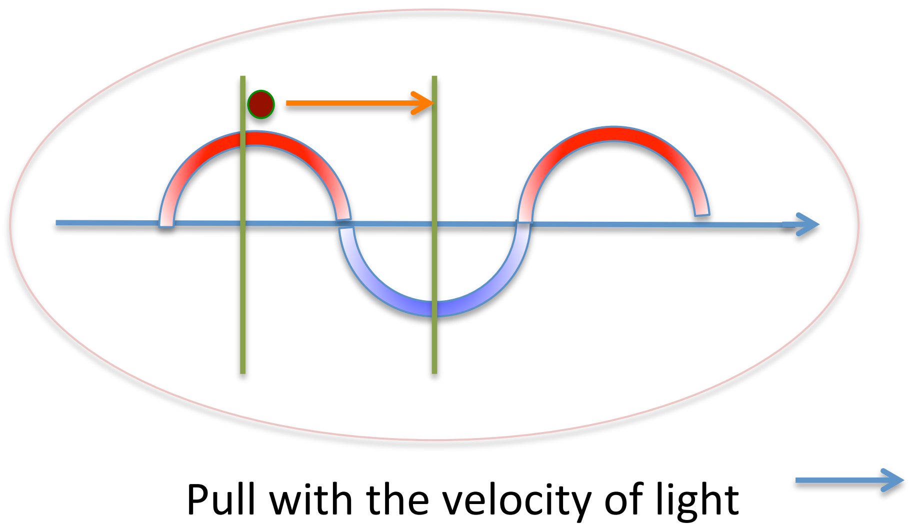
(A) Static accelerator

Acceleration length: Distance between the electrodes



(B) Electrodes are pulled at the particle velocity.

Acceleration length can be quite large!!



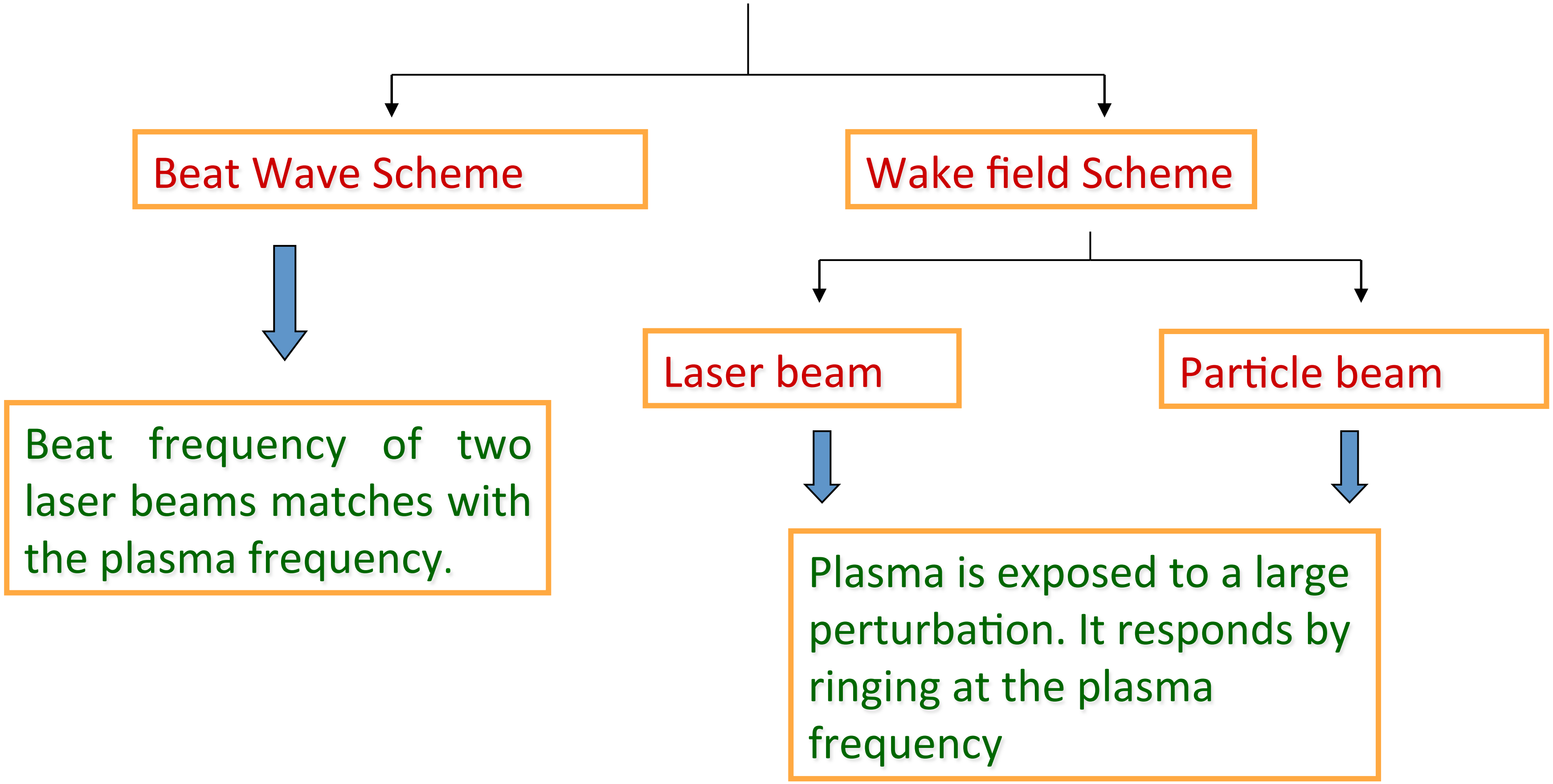
(C) Plasma based accelerators.

The density hump and density crest of the plasma wave acts as the two electrodes. For laser driven plasma wave the phase velocity is equal to the speed of light - the acceleration length to be quite large.

One requires …

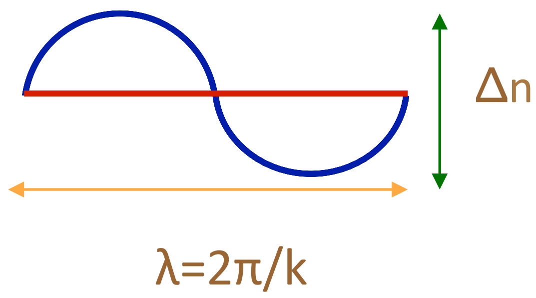
- Excitation of plasma wave moving with a phase speed close to ‘c’.
- Higher amplitude wave – for high potential difference.
- Particle injection at appropriate phase …

Plasma wave driving



Creation of a large space charge field

How large an electric field can be generated in plasma ?



Using Gauss's law

$$\frac{E}{\lambda} = 4\pi en\left(\frac{\delta n}{n}\right)$$

$$V_{ph} = \frac{\omega_p}{k} \approx c$$

Maximum field when all particles are expelled i.e when **Δn/n ~ 1** is

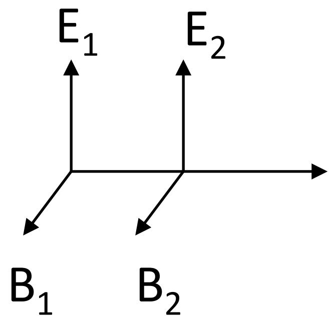
$$E \approx 8\pi^2 en\left(\frac{c}{\omega_p}\frac{\delta n}{n}\right)$$

$$E(Volts/cm) \approx \sqrt{n(cm^{-3})}$$

Thus for n =10^{16} cm^{-3}; E is 10GV/m

Plasma Beat Wave Acceleration

(PBWA: Physical Mechanism)



Laser frequencies ω_1 and ω_2 such that

$$\omega_1 - \omega_2 = \omega_p; \qquad k_1 - k_2 = \Delta k$$

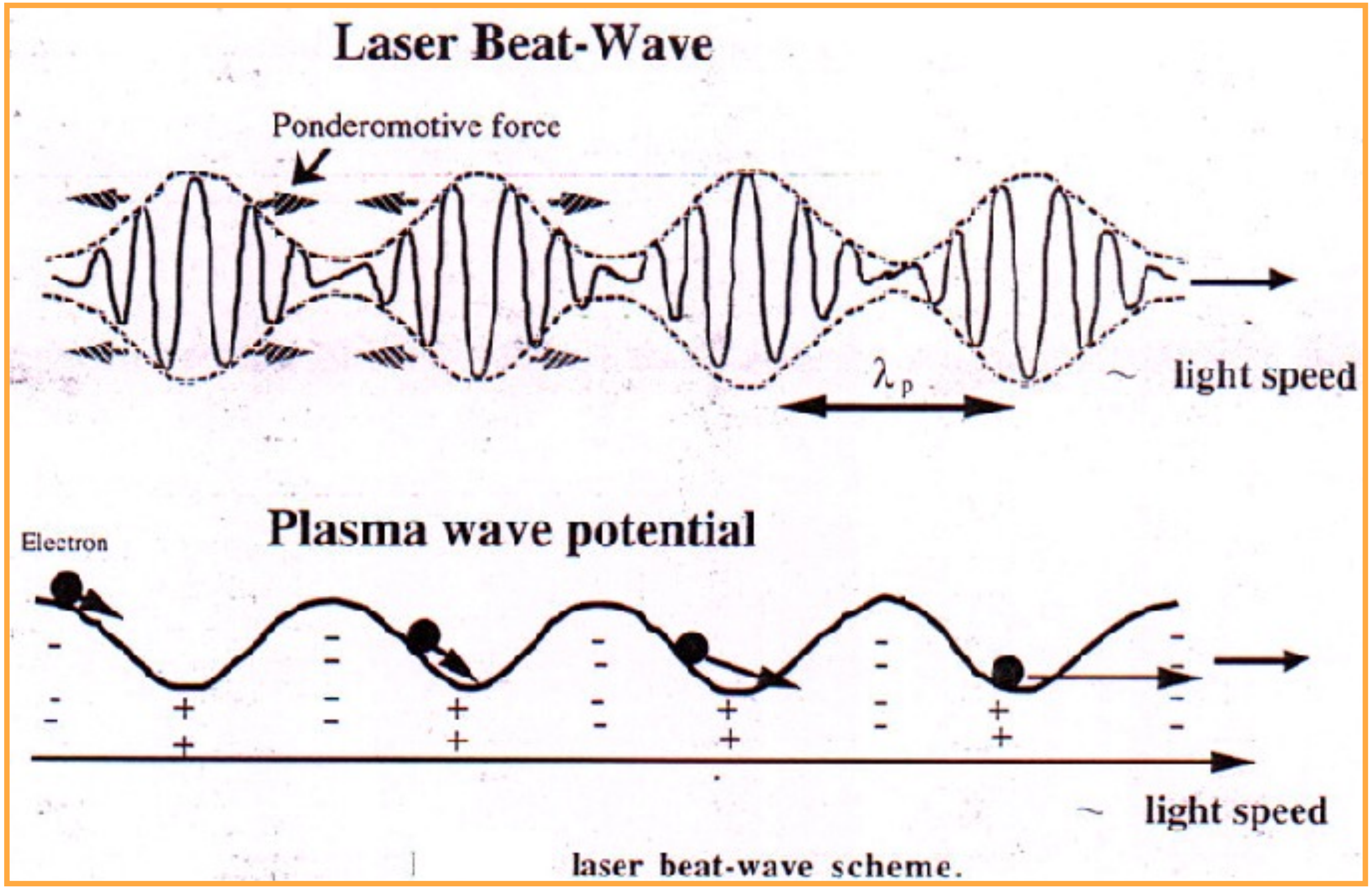
Longitudinal force $\quad j_1 \times B_2 + j_2 \times B_1$

At $\omega_1 - \omega_2 = \omega_p$; can pump plasma waves !

$$V_{ph}^{pl\ wave} \approx \frac{\omega_p}{\Delta k} \approx \frac{\Delta\omega}{\Delta k} \approx V_{gr}^{EM\ wave}$$

→ Phase coherence between plasma wave and EM beats. Leads to efficient generation

Schematic of Beat Wave acceleration



Plasma Beat Wave Acceleration

(Analytical derivation of resonance)

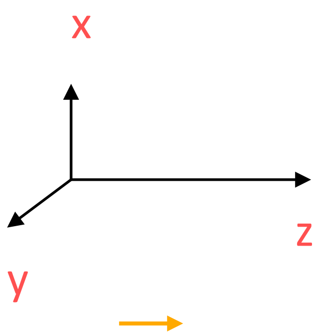
Assumptions:

- Non relativistic Laser intensity.
- Homogeneous plasma
- One dimensional variations along the propagation direction.

Choose: Slab geometry, propagation along Z direction.

Let X and Y directions be that of electric and magnetic field of the light wave.

PBWA (Contd.)



Propagation direction

$$\partial/\partial y = \partial/\partial x = 0$$

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

Electrostatic field due to plasma wave

Electromagnetic field of the light wave

Relevant Equations:

- Continuity Equation
- Equation of motion for electrons
- Maxwell's Equation for the field

Ions do not respond at fast times scales!

PBWA (Contd.)

Choose

$$\vec{A} = \alpha_1 Cos(k_1 z - \omega_1 t)\hat{x} + \alpha_2 Cos(k_2 z - \omega_2 t)\hat{x}$$

→ Electric field along 'x' and magnetic field along 'y'

Seek solution for small v/c and weak plasma response !

Electron momentum equation

$$m\frac{d\vec{v}}{dt} = -e\left(E + \frac{\vec{v} \times \vec{B}}{c}\right)$$

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots$$

Dominant response

$$m\frac{\partial \vec{v}_1}{\partial t} = -e\vec{E}_{EM} = \frac{e}{c}\frac{\partial \vec{A}}{\partial t} \quad \rightarrow \quad \vec{v}_1 = \frac{e\vec{A}}{mc}$$

PBWA (Contd.)

In second order

$$m\frac{\partial \vec{v}_2}{\partial t} = -m\vec{v}_1 \bullet \nabla \vec{v}_1 - e v_1 \times \left(\nabla \times \vec{A}\right) + e\nabla\phi$$

$$m\frac{\partial \vec{v}_2}{\partial t} = -\frac{e^2}{2mc^2}\nabla A^2 + e\nabla\phi \qquad (1)$$

Ponderomotive force

Space charge field

Scalar potential determined by Poisson's equation

$$\nabla^2\phi = 4\pi e\tilde{n} \qquad (2)$$

Density perturbation by continuity equation

$$\frac{\partial \tilde{n}}{\partial t} + n_0 \nabla \bullet v = 0 \qquad (3)$$

PBWA (Contd.)

For one dimensional variations along z Eqs.(1), (2) and (3) can be written as :

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + \omega_p^{\,2} \tilde{n} = \frac{e^2 n_0}{2m^2 c^2} \frac{\partial^2}{\partial z^2} A^2 \qquad (4)$$

Equation of Harmonic oscillator with a driver

$\omega_p^{\,2}$ the natural frequency of the oscillator.

Right hand side is the driving force.

Recall the resonance condition:

The natural oscillator frequency should match the frequency of the driving force !

PBWA (Contd.)

Under what condition is resonance possible?

$$A^2 = \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2\right) + \frac{1}{2}\left[\alpha_1^2 Cos\,2\left(k_1 z - \omega_1 t\right) + \alpha_2^2 Cos\,2\left(k_2 z - \omega_2 t\right)\right]$$

$$+\alpha_1\alpha_2\left[Cos\left\{\left(k_1 + k_2\right)z - \left(\omega_1 + \omega_2\right)t\right\} + Cos\left\{\left(k_1 - k_2\right)z - \left(\omega_1 - \omega_2\right)t\right\}\right]$$

This is the only part which can satisfy the resonance condition

Underdense plasma

Hence both $(\omega,k)_1$ & $(\omega,k)_2 > (\omega,k)_p$

⇒ For resonance we must have
$(\omega,k)_1 - (\omega,k)_2 = (\omega,k)_p$

PBWA (Contd.)

Thus Eq.(4) can be written as

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + \omega_p{}^2 \tilde{n} = \frac{e^2 n_0}{2m^2 c^2} \alpha_1 \alpha_2 (\Delta k)^2 Cos\left\{\Delta k z + \omega_p t\right\} + \ldots \quad (5)$$

$\omega_1 - \omega_2 = \omega_p$;

$k_1 - k_2 = \Delta k$

Resonant forcing

non resonant terms

Solution of (5)

$$\tilde{n} = C_1 Cos\left(\Delta k\ z + \omega_p t\right) + \left(\frac{e^2 n_0 (\Delta k)^2}{2m^2 c^2}\right) \frac{\alpha_1 \alpha_2 t}{2\omega_p} Sin\left(\Delta k\ z + \omega_p t\right)$$

Plasma waves gets resonantly driven

PBWA (Contd.)

Phase velocity of the space charge wave

$$V_{ph} = \frac{\omega_p}{\Delta k} = \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} = V_{gr\ EM}$$

= Group velocity of EM wave

EM wave dispersion relation

$$\omega^2 = \omega_p^2 + k^2 c^2$$

$$\frac{\partial\omega}{\partial k} = \frac{c^2 k}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right)$$

Expansion valid for small ω_p^2/ω^2

Thus smaller ω_p^2/ω^2 implies that the phase velocity of space charge wave is close to 'c' the speed of light

PBWA: Maximum possible energy gain ?

Dephasing

- Particle velocity typically very close to 'c'

  e.g. A 3Mev electron has $\gamma \approx 7$, $\beta = v/c \approx 0.99$
- The phase velocity of space charge wave differs from 'c'

$$\frac{\Delta V}{c} = \frac{c - V_{ph}}{c} \approx \frac{c - c\left(1 - \frac{\omega_p^2}{2\omega_{1,2}^2}\right)}{c} = \frac{\omega_p^2}{2\omega_{1,2}^2}$$

ΔV causes a phase slip between the particle and the space charge wave, causing dephasing of the particle from the accelerating field!!

PBWA (Energy limitation)

Dephasing time τ_D

Time taken by ΔV to propagate a distance of the order of half plasma wavelength.

$$\tau_D = \frac{\pi}{\Delta k}\frac{1}{\Delta V} = \frac{2\pi}{\Delta k}\frac{\omega_{1,2}^2}{c\,\omega_p^2}$$

Dephasing length L_D

$$L_D = c\tau_D = \frac{\pi}{\Delta k}\frac{1}{\Delta V} = \frac{2\pi}{\Delta k}\frac{\omega_{1,2}^2}{\omega_p^2}$$

$$\Delta k = \frac{\omega_p}{V_{ph}} \approx \frac{\omega_p}{c}$$

$$L_D = 2\pi\frac{\omega_{1,2}^2}{\omega_p^2}\frac{c}{\omega_p}$$

Beyond this length the particle will not gain any energy

PBWA (Energy limitation)

Maximum energy gain

$$W_{max} = eEL_D \quad \text{- - - - (6)}$$

Typical space charge field amplitude

E_{max} can be estimated from Gauss' s law

$$|E_{max}| = \frac{4\pi e n_0}{\Delta k} = \frac{4\pi e n_0 c}{\omega_p}$$

Substituting for E_{max} and L_D in (6)

$$W_{max} \le 2\pi (mc^2) \frac{\omega_{1,2}^2}{\omega_p^2}$$

½ MeV

Thus if $\omega_1^2 / \omega_p^2 \approx 1000$

Gives $W_{max} \approx$ GeV

Thus Underdense plasma helps

Surfratron Mechanism of overcoming energy limitation

Add a transverse magnetic field

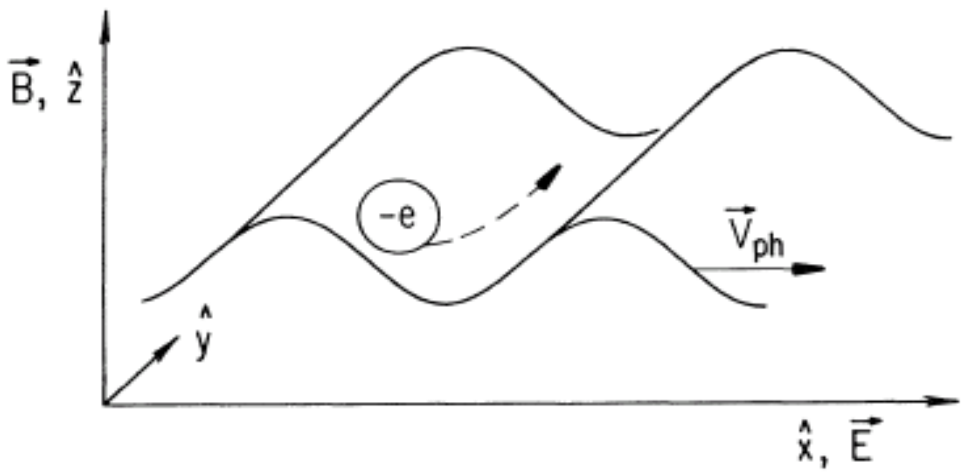


FIG. 1. An electron trapped by a potential trough moving at $\vec{V}_{ph}$ sees an electric field from the Lorentz transformation $\gamma_{ph}\vec{V}_{ph} \times \vec{B}/c$ which accelerates it across the wave front.

Here x has been chosen as the propagation direction of the wave.

Surfratron Mechanism of overcoming energy limitation

Katsouleas and Dawson (1983) PRL 392.

Consider a longitudinal plane wave and a transverse magnetic field

$$\vec{E} = E_0 \sin(kx - \omega t)\hat{x}, \quad \vec{B} = B\hat{z}.$$

Represents the longitudinal accelerating electric field.

Equation of motion for the charged particle

$$d(\gamma V_x)/dt = (qE_0/m)\sin(kx - \omega t) + \omega_c V_y,$$
$$d(\gamma V_y)/dt = -\omega_c V_x,$$
$$\gamma = (1 - V_x^2/c^2 - V_y^2/c^2)^{-1/2},$$

Charged particles can be accelerated so long as $V_x \leq \omega/k$

Surfratron Mechanism (Contd.)

Assume $V_x = V_{ph}$ then integrating the y component and substituting for γ

$$V_y = \frac{-\omega_c V_{ph} t}{\gamma_{ph}(1+\omega_c^2 t^2 V_{ph}^2/c^2)^{1/2}}$$

→ Acceleration across the front

Exact numerical integration of the coupled set also shows that V_y keeps increasing upto c.

Unlimited acceleration possible

Surfratron Mechanism (Contd.)

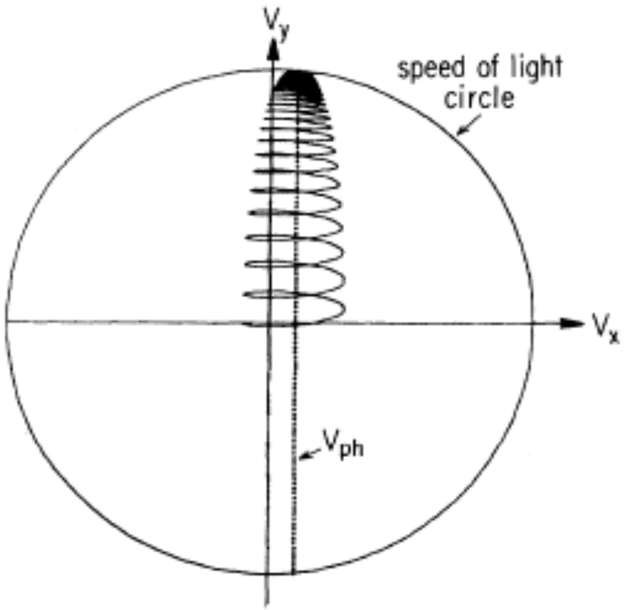


FIG. 2. Velocity-space trajectory of a particle in a low-phase-velocity wave ($V_{ph} = 0.1\,c$, $E_0/B = 1.5$, $\omega/\omega_c = 2$).

However, larger cross section of radiation required. …

PBWA (Other limitations)

- Detuning due to

  (a) Inhomogeneous plasma (most severe constraint for PBWA)

  (b) Plasma nonlinearity.
- Instabilities.
- Beam Diffraction.
- Pump depletion

Wake field accelerators

Uses a fast varying ($t \sim \omega_p^{-1}$) laser pulse (LWFA) or beam of charged particles (PWFA) to excite plasma waves. After the laser/particle pulse leaves, the plasma is left with residual oscillations.

Unlike PBWA Resonant matching is not necessary!!

High intensity laser and/or particle beams required!

Wake field accelerators

- Except for resonant matching condition, other limitations same as in beat wave accelerators. (viz. energy limitation, instabilities etc.)
- Require high intensity laser beam, hence diffraction and pump erosion are important issues of concern.

Laser Wake field Acceleration (LWFA)

Relevant Equations:

Continuity Equation

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\vec{v}) = 0 \quad \text{- - - - - (1)}$$

Relativistic Electron Momentum Equation

$$\frac{\partial}{\partial t}(\gamma m\vec{v}) + \vec{v} \bullet \nabla(\gamma m\vec{v}) = e\nabla\phi + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} - e\frac{\vec{v} \times \nabla \times \vec{A}}{c} \quad \text{- - - (2)}$$

Poisson's Equation

$$\nabla^2\phi = 4\pi e(n_0 - n) \quad \text{- - - (3)}$$

Wave Equation

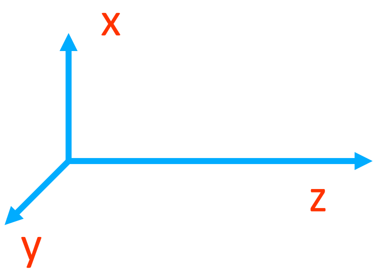
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}ne\vec{v} \quad \text{- - - - (4)}$$

LWFA (Basic Equations)

Equation (2) can also be written as

$$\frac{\partial}{\partial t}\left(\gamma m\vec{v} - e\vec{A}/c\right) = \nabla e\phi - \nabla m\gamma\vec{v}\bullet\vec{v} + \vec{v}\times\nabla\times\left(\gamma m\vec{v} - e\vec{A}/c\right) \quad \text{(2a)}$$

Considering only longitudinal 1 dim variations along the propagation direction 'z'



For light wave

$$\vec{A} = A_\perp e_\perp$$

$e_\perp$ is in the x- y plane

LWFA (Basic Equations)

Perpendicular component of Eq.(2a) is

$$\frac{d}{dt}\left(\gamma m v_{\perp} - eA_{\perp}/c\right) = \left(\frac{\partial}{\partial} + \vec{v}\bullet\nabla\right)\left(\gamma m v_{\perp} - eA_{\perp}/c\right) = 0$$

Upon integration and using the boundary condition

$$\Rightarrow \quad v_{\perp} = \frac{eA_{\perp}}{m\gamma c} \quad \text{- - - - - (a)}$$

Wave Eq.(4) has only perpendicular component, substituting v from (a) gives

$$\frac{\partial^2}{\partial z^2}A_{\perp} - \frac{1}{c^2}\frac{\partial^2 A_{\perp}}{\partial t^2} = \frac{4\pi n e^2 m}{c^2}\frac{A_{\perp}}{\gamma} \quad \text{- - - - - (b)}$$

LWFA (Basic Equations)

The relativistic factor ϒ can be written as

$$\gamma^2 = \frac{1}{1-\left(v_z^2+v_\perp^2\right)/c^2} = \frac{1}{1-v_z^2/c^2-\left(eA_\perp/m\gamma c^2\right)^2}$$

$$\Rightarrow \quad \gamma^2 = \frac{1+\left(eA_\perp/mc^2\right)}{\left(1-v_z^2/c^2\right)}$$

Using the above expression for ϒ, the Z component of Eq.(2a) becomes

$$\frac{\partial}{\partial t}\left(m\gamma v_z\right) = e\frac{\partial\phi}{\partial z} - v_z\frac{\partial}{\partial z}\left(m\gamma v_z\right) - \frac{e^2}{2mc^2\gamma}\frac{\partial}{\partial z}A_\perp^2$$

LWFA (Basic Equations)

We choose the following normalizations

- Velocity by c
- Time by ω_p^{-1}
- Length by c/ω_p
- Density by background plasma density n_0
- Scalar φ and vector potential A by mc^2/e

LWFA (Basic Equations)

Normalized final set of equations

$$\frac{\partial}{\partial t}(\gamma u) = e\frac{\partial \phi}{\partial z} - u\frac{\partial}{\partial z}(\gamma u) - \frac{e^2}{2\gamma}\frac{\partial}{\partial z}a^2$$

$$\frac{\partial^2}{\partial z^2}\vec{a} - \frac{\partial^2 \vec{a}}{\partial t^2} = \frac{n\vec{a}}{\gamma}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0$$

$$\frac{\partial^2 \phi}{\partial z^2} = n - 1$$

$$\gamma^2 = \frac{1 + a^2}{(1 - u^2)}$$

LWFA (Contd.)

Transforming coordinates as
(i.e. in the phase velocity frame of the light wave)

$$\xi = z - t; \tau = t$$

For simplicity consider circularly polarized light of the following form

$$\vec{a} = \left\{ \hat{x}\tilde{a}(\xi) + i\hat{y}\tilde{a}(\xi) \right\} exp(-i\omega\tau) + c.c$$

where $a^2 = 4|\tilde{a}|^2$ ← Note it is independent of τ

⟹ ϒ is independent of τ

LWFA (Contd.)

Assuming quasi-static plasma response i.e. density n and u to be independent of τ we obtain upon integration

$$n = 1/(1-u)$$

$$\gamma(1-u) = 1+\phi$$

$$\left(2\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \tau}\right)\frac{\partial \vec{a}}{\partial \tau} = \frac{n\vec{a}}{\gamma}$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = n - 1$$

$$\gamma^2 = \frac{1+a^2}{(1-u^2)}$$

Density can be eliminated from the Poisson's equation

$$\frac{\partial^2 \phi}{\partial \xi^2} = -\frac{1}{2}\left[1 - \frac{1+a^2}{(1+\phi)^2}\right]$$

Thus φ the scalar potential (for particle acceleration) can be determined from a given profile of light wave intensity.

LWFA (Numerical results.)

Sprangle et al. PRL 64 2011(1990)

Scalar potential profile for a given '*a*' profile.

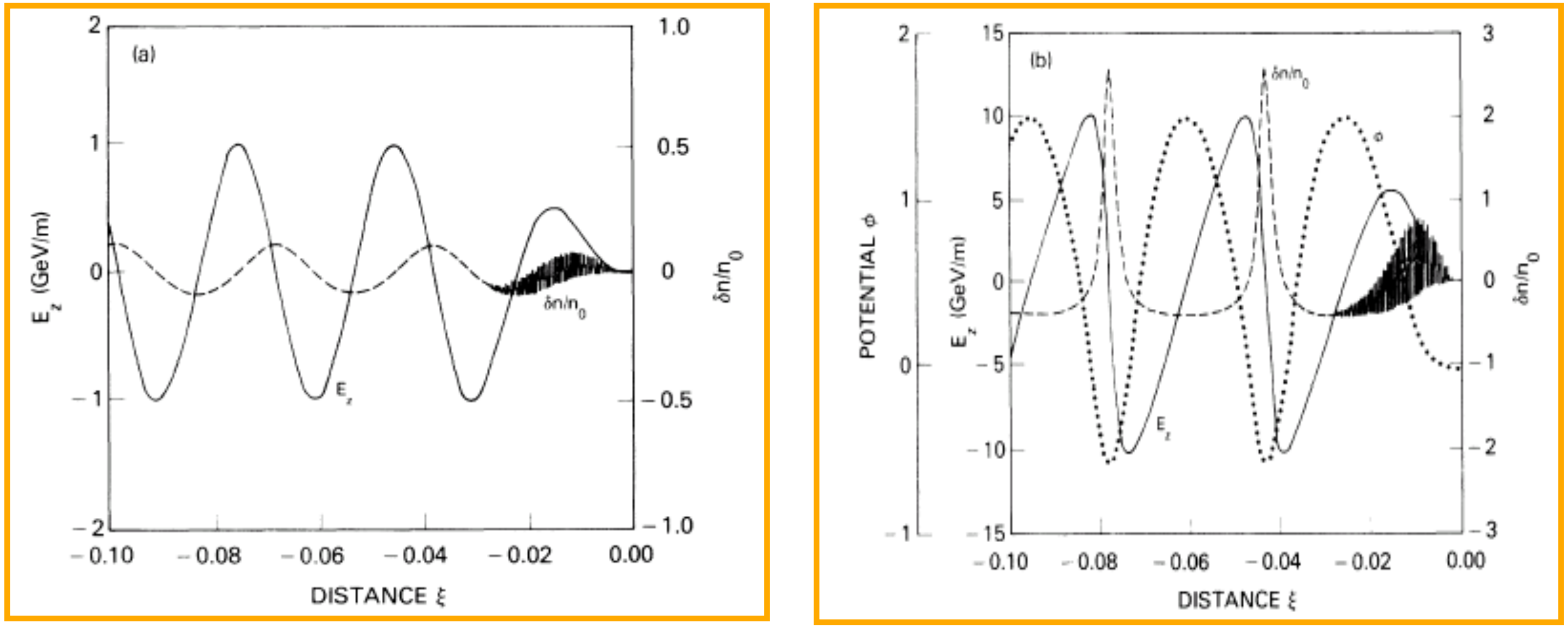


FIG. 1. Density variation $\delta n/n_0 = n/n_0 - 1$ (dashed line), axial electric field E_z in GeV/m (solid line), and electrostatic potential ϕ (dotted line) for a laser pulse located within the region $-L \leq \xi \leq 0$, where $L = \lambda_p = 0.03$ cm and (a) $a_{L0} = 0.5$ and (b) $a_{L0} = 2$.

LWFA (Contd.)

- For producing large amplitude wake fields high intensity of laser light is suitable.
- Severe limitation on the laser intensity due to diffractive spreading. Vacuum diffraction length is given by the Rayleigh length $Z_R = \pi r_0^2/\lambda_0$. (where r_0 is the laser spot size and λ_0. is the vacuum wavelength).
- Relativistic and optical guiding help in reducing this spread.

LWFA (Contd.)

Relativistic and optical guiding:

Laser beam intensity falls in the transverse direction producing.

- Hollow electron density profile radially
- The relativistic factor decreases with radius.

$$\Rightarrow \quad \frac{\partial \varepsilon}{\partial r} < 0 \qquad \text{where} \qquad \varepsilon = \frac{ck}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

Plasma acts like a convex focusing lense!

LWFA (Contd.)

- Competition between plasma focusing action and diffraction of the beam!!
- Critical power P_c necessary for focusing to balance diffraction.
- The front of the pulse for which $P < P_c$ will continuously erode.

Tajima and Dawson 43 (1979) 267

The mean oscillatory energy gained by the electrons in the transverse electric field of the EM wave:

$$\langle \Delta W_T \rangle \cong m\langle v_y^2 \rangle / 2 = e^2 \langle E_y^2 \rangle / 2m\omega^2$$ $\equiv$ the energy lost by the EM wave

$\rightarrow$

The loss of light wave momenta

$$\langle \Delta p_x \rangle = \langle \Delta W_T \rangle / c$$

$\equiv$ The gain of electron momenta

Here x is the propagation direction!!

Derivation: Tajima and Dawson ..

The x displacement of the electron

$$\Delta x = \langle \Delta p_x \tau / m \rangle$$

τ is the light pulse duration

Once the light pulse has passed the space charge due to this displacement pulls the electron back and a plasma oscillation is set up!!

Electron energy gain in the generated plasma wave?

Transform to the rest frame of light wave induced plasmon.

$\beta = V_{ph\ plasmon}/c = V_{gr\ light}/c$, hence $\gamma = \omega/\omega_p$.

Derivation: Tajima and Dawson ..

Transformed wave vector and energy for the light and plasma wave.
This frame is also the rest frame of photons in the plasma

$$\begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} k_x \\ i\,\omega/c \end{pmatrix} = \begin{pmatrix} 0 \\ i\,\omega_p/c \end{pmatrix}$$

Light wave

$$\begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} k_p \\ i\,\omega_p/c \end{pmatrix} = \begin{pmatrix} k_p/\gamma \\ 0 \end{pmatrix}$$

Plasma wave

Lab frame

Wave frame

$$\omega = (\omega_p{}^2 + k_x{}^2 c^2)^{1/2}$$

Here ω_p is like the rest mass of photon in plasma.

Derivation: Tajima and Dawson ..

- Longitudinal electric field associated with plasmon is invariant. $(E_L^{\text{wave}} = E_L)$
- Critical amplitude is determined by the wave breaking limit. The oscillation length by the plasmon in one plasma period should not exceed the wavelength. $k_p x_L \simeq 1$, where $x_L \simeq eE_L / m\omega_p^2$.
- From Gauss' s law $eE_L^{\text{cr}} \cong mc\omega_p$

Since $1/k_{pL} = c/\omega_p$
$e\varphi_L = mc^2$

- Invariance of E_L and transformation of wave vector gives

$$e\varphi^{\text{wave}} = \gamma e\varphi \approx \gamma mc^2$$

← Energy gain in wave frame

Tajima and Dawson ..

Transforming energy and velocity back in the lab frame give the maximum energy gain. W^{max} in the lab frame

$$\begin{pmatrix} \gamma & -i\beta\gamma \\ i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \gamma\beta mc \\ i\gamma mc \end{pmatrix} = \begin{pmatrix} 2\gamma^2\beta mc \\ i\,mc\gamma^2(1+\beta^2) \end{pmatrix}$$

Hence

$$W^{max} \equiv \gamma^{max} mc^2 \simeq 2\gamma^2 mc^2$$

$$\gamma^{max} = 2\omega^2/\omega_p^2$$

Energy limitation

Force *distance = energy; $l_a = W^{max}/eE_L^{cr}$

The time t_a and length l_a to reach this energy is given by

$$l_a \cong 2\omega^2 c/\omega_p^3$$

$$t_a \cong W^{max}/ceE_L^{cr}$$

So far …

Simplistic order of magnitude estimates in 1-D.

- Plasma waves moving with phase speed can be created.
- Amplitude dependence, maximum energy gain, …. Estimated
- Particle injection – external. For high power lasers plasma itself solves the problem – wave breaking and self injection.

Other features

- 2D effects
- Plasma nonlinearity
- Wave particle interactions.
- Optimized configurations
- Possible scalings
- Ion acceleration through overdense medium

Such details can be studied through simulations and/or experiments – and would be presented in the school by other authors.

Thank You