An Invitation to Open Quantum Systems: A Density Matrix Approach

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Overview of Talk

- Here we will attempt to provide an introduction to Open Quantum Systems.

- After motivating the need for a study of open quantum systems, laying down the basic tools required along with a brief description of density matrices, the master equation commonly known as the Lindblad equation will be developed and the physical assumptions underlying its derivation discussed. This equation has wide applicability, specially in studies in quantum optics and quantum information.

- The theory developed will be applied to a few examples relevant to Quantum Information.

- We then discuss some aspects of non-Markovian evolution. Again our discussions will evolve around a physical model.

Open Quantum Systems: A Brief Preview and Motivation

- The theory of open quantum systems addresses the problems of damping and dephasing in quantum systems by the assertion that all real systems of interest are 'open' systems, surrounded by their environments (U. Weiss: (1999); H. -P. Breuer and F. Petruccione: (2002)).

- Quantum optics provided one of the first testing grounds for the application of the formalism of open quantum systems (W. H. Louisell: (1973)). Application to other areas was intensified by the works of (Caldeira and Leggett: (1983)) and (Zurek: (1993)), among others.

- The recent upsurge of interest in the problem of open quantum systems is because of the spectacular progress in manipulation of quantum states of matter, encoding, transmission and processing of quantum information, for all of which understanding and control of the environmental impact are essential (Turchette *et al.*: (2000); Myatt *et al.*: (2000); Haroche *et al.* (1996)). This increases the relevance of open system ideas to quantum computation and quantum information.

- Hamiltonian of the total (closed system):

$$H = H_S + H_R + H_{SR}.$$

- S- system, R- reservoir (bath), $S - R$-interaction between them.

- System-reservoir complex evolves unitarily by:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}.$$

- We are interested in the reduced dynamics of the system S, taking into account the influence of its environment. This is done by taking a trace over the reservoir degrees of freedom, making the reduced dynamics *non-unitary*:

$$\rho^s(t) = \mathrm{Tr_R}(\rho(t)) = \mathrm{Tr_R}\left[e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}\right].$$

Open Quantum Systems: *continued...*

- Open quantum systems can be broadly classified into two categories:
  (A). Quantum non-demolition (QND), where $[H_S, H_{SR}] = 0$ resulting in decoherence without any dissipation (Braginsky *et al.*: (1975), (1980); Caves *et al.*: (1980); G. Gangopadhyay, S. M. Kumar and S. Duttagupta: (2001); SB and R. Ghosh: (2007)) and
  (B). Quantum dissipative systems, where $[H_S, H_{SR}] \neq 0$ resulting in decoherence with dissipation (Caldeira and Leggett: (1983); H. Grabert, P. Schramm and G-L. Ingold: (1988); SB and R. Ghosh: (2003), (2007)).

- In the parlance of quantum information theory, the noise generated by a QND open system would be a "phase damping channel", while that generated by a dissipative (Lindblad) evolution would be a "(generalized) amplitude damping channel".

Density Matrix: A Brief Interlude

- Language of density operator is often used in studies related to Open Quantum Systems.

- Average of an operator M in Schrödinger picture (SP) (where the state vector evolves with time but not the operator) , in the state $|\psi_S(t)\rangle$, is

$$
\begin{aligned}
\langle M \rangle &= \langle \psi_S(t)|M_S|\psi_S(t)\rangle \\
&= \mathrm{Tr} M_S |\psi_S(t)\rangle\langle\psi_S(t)|.
\end{aligned}
$$

- In many cases, it is not possible to determine exactly the state $|\psi_S(t)\rangle$ to which the system belongs. The best one can have is the probability p_ψ of the system being in the state $|\psi_S(t)\rangle$. Then the above expression for the operator average becomes modified to

$$
\begin{aligned}
\langle\langle M \rangle\rangle &= \Sigma_\psi p_\psi \mathrm{Tr} M_S |\psi_S(t)\rangle\langle\psi_S(t)| \\
&= \mathrm{Tr} M_S \rho_S(t),
\end{aligned}
$$

where the density matrix $\rho_S(t)$ is

$$
\rho_S(t) = \Sigma_\psi p_\psi |\psi_S(t)\rangle\langle\psi_S(t)|,
$$

and $\Sigma_\psi p_\psi = 1$.

Density Matrix: *continued...*

- Density matrix satisfies two properties:

  (a).

  $$\mathrm{Tr}\rho = 1,$$

  (b).

  $$\mathrm{Tr}\rho^2 \leq 1,$$

  with equality for pure and inequality for mixed states.

Density Matrix: Some Interesting examples

- Density matrix relects lack of complete knowledge of the system. This has been exploited by Jaynes (E. T. Jaynes: (1957)) to get the density by maximizing the entropy: $S = -k \mathrm{Tr}\rho \ln(\rho)$ subjected to some constraints.

- Examples:
  (A). Maximising S subject to the constraints $\mathrm{Tr}(\rho) = 1$ and that the average energy of the system is known, i.e., $\langle E \rangle = \mathrm{Tr}(\rho \mathrm{H})$, we get the density of a harmonic oscillator of frequency ω and temperature T (this system is equivalent to a mode of a cavity filled with electromagnetic radiation in thermal equilibrium with the walls at temperature T) as:

$$\rho = \frac{\exp(-\frac{H}{kT})}{\mathrm{Tr}\exp(-\frac{\mathrm{H}}{\mathrm{kT}})}.$$

  (B). Maximising S subject to the constraints $\mathrm{Tr}(\rho) = 1$ and the spin polarization $s = \mathrm{Tr}(\rho\sigma)$ (where σ stands for the three Pauli spin matrices), yields the density matrix of an ensemble of spin-$\frac{1}{2}$ particles as:

$$\rho = \frac{1}{2}(I + s.\sigma).$$

Examples of Open Quantum Systems

- Quantum Brownian motion (QBM), wherein the quantum mechanical system is taken as a harmonic oscillator coupled linearly via its displacement x to a fluctuating environment, serves as a paradigm of quantum open systems in that it provides a model wherein the concepts of the system plus reservoir are elucidated. QBM being a generalization of classical Brownian motion into the quantum regime, gives us a physical realization of dissipation reconciled with quantization. Interest in this has been motivated by observation of macroscopic effects in quantum systems such as dissipation in tunneling and problems of quantum measurement theory (for example, the loss of quantum coherence due to a system's interaction with its environment). This has also been used to gain useful insight into problems which are not exactly solvable (Caldeira and Leggett: (1983); H. Grabert, P. Schramm and G-L. Ingold: (1988)).

- The diversity of QBM can be gauged from the fact that it has been used to address issues in quantum gravity (B. L. Hu, J. P. Paz and Y. Zhang: (1992, 1993, 1994)) such as the interconnection of some basic quantum statistical processes like decoherence, dissipation, particle creation, noise and fluctuation. The understanding of many quantum statistical processes in the early universe and black holes (B. L. Hu: (1992)) requires an extension of the existing framework of quantum field theory in the setup of quantum open systems represented by the QBM.

Examples of Open Quantum Systems: continued...

- These ideas have been applied to the analysis of some basic issues in quantum cosmology (J. P. Paz and S. Sinha: (1991, 1992)), and the foundation of quantum mechanics, such as the uncertainty principle (A. Anderson and J. J. Halliwell: (1993)) and decoherence (W. H. Zurek: (1991)) in the quantum to classical transition problem.

- Open system ideas have been used by (R. K. Gupta *et al.*: (1984); A. Isar *et al.*: (1994)) to study the charge equilibration process in deep inelastic collisions and the damping of the proton and neutron asymmetry degrees of freedom have been treated within the framework of Lindbladian master equations.

- Open system ideas have been applied extensively in quantum optics (W. H. Louisell: (1973); F. Haake: (1973); G. S. Agarwal: (1974)).

- These ideas have been used in quantum information theoretic processes (SB and R. Srikanth: (2007)).

- Ideas developed by R. Landauer: (1961) and C. H. Bennett: (1988), established a deep connection between information and thermodynamics.

Examples of Open Quantum Systems: continued...

- Open system ideas applied to transport processes in the context of non-equilibrium statistical mechanics (Butticker: (1992); Y. Meir and N. S. Wingreen; (1992); S. Datta: (1996); A. Dhar and D. Sen: (2006)). They have been used in the study of mesoscopic systems (Y. Imry: (1997)). Entanglement between charge qubits induced by a common dissipative environment was recently analyzed using concurrence as the measure (L. D. Contreras-Pulido and R. Aguado: (2008)) in the context of quantum information processing using solid state nanostructures. Entanglement between two qubits mediated by the interaction with the reservoir for both purely dephasing as well as dissipative interactions has been studied (SB, V. Ravishankar and R. Srikanth: (2009)). A review of coherent and collective quantum optical effects in mesoscopic systems has been presented in (T. Brandes: (2005)).

- These ideas have been applied to the problem of escape from a metastable state, separated from a continuum or quasi-continuum of states by a free energy barrier, which play a central role in low-temperature physics, nuclear physics, chemical kinetics and transport in biomolecules (H. A. Kramers: (1940); A. J. Leggett: (1980); H. Grabert, P. Olschowski and U. Weiss: (1987)).

Examples of Open Quantum Systems: continued...

- Another important model serving as a paradigm for describing a number of phenomena is the dissipative two-state system (A. J. Leggett *et al*). An important example of a dissipative two-state system is an rf SQUID (Superconducting Quantum Interference Device) ring threaded by an external flux near half-a-flux quantum. Such a superconducting device might be appropriate for the observation of macroscopic quantum coherence effects (A. J. Leggett: (1986)). Quantum effects in the current-voltage characteristics of a small Josephson junction are described by this model (V. Ambegaokar *et al*: (1982)). In such systems the most important issue is the computation of transport properties (M. Sassetti *et al.*: (1992); M. P. A. Fisher and W. Zwerger: (1985); U. Eckern and F. Pelzer: (1987)).

System-Reservoir Initial Conditions

- The open system evolution starts from the system and the reservoir being initially separated or correlated.

  (A). <u>Separable Initial Condition</u>: it is assumed that the system and the environment (reservoir) are initially uncorrelated (Feynman and Vernon: (1963); Caldeira and Leggett: (1983)). In such a situation the initial density matrix factorizes so that

  $$\rho(0) = \rho_S(0).\rho_R(0),$$

  where $\rho_S(0)$ stands for the initial system density matrix and $\rho_R(0)$ stands for the initial reservoir density matrix.

(B). <u>Non-Separable Initial Condition</u>: in many applications, the system and the reservoir are integral parts of the same system and their interaction is not at our disposal. These considerations lead to the introduction of a class of initial conditions, the `generalized initial conditions' (Hakim and Ambegaokar: (1985); H. Grabert, P. Schramm and G-L. Ingold: (1988); SB and R. Ghosh: (2003)). A very general class of initial conditions are of the form

$$\rho_0 = \sum_j O_j \rho_\beta O_j',$$

where

$$\rho_\beta = Z_\beta^{-1} \exp(-\beta H)$$

is the canonical density matrix describing the equilibrium of the interacting system in the presence of a time-independent potential V and Z_β^{-1} is the partition function. Here $\beta = (k_B T)^{-1}$, with T being the equilibrium temperature of the interacting system. The operators O_j, O_j' act upon the system coordinate only and leave the environment (reservoir) coordinates unchanged but can be chosen arbitrarily otherwise.

Time scales associated with the Open System Evolution

- The open system evolution is characterized by a number of time-scales, the salient ones being:

- Scale associated with the natural frequency of the system.

- Relaxation time scale determined by the S-R coupling strength.

- Reservoir correlation time (memory time) associated with the high-frequency cutoff in the reservoir spectral density and the time scale associated with the reservoir temperature, which measures the relative importance of quantum to thermal effects.

Quantum Non-Demolition (QND)

- Generic Hamiltonian:

$$
\begin{aligned}
H \;=\;& H_S + H_R + H_{SR} = H_S \\
& + \sum_k \hbar\omega_k b_k^\dagger b_k + H_S \sum_k g_k(b_k + b_k^\dagger) + H_S^2 \sum_k \frac{g_k^2}{\hbar\omega_k} \;.
\end{aligned}
$$

- S- system, R- reservoir (bath), $S - R$-interaction between them.

$$
[H_S,\; H_{SR}] = 0 \Rightarrow \; QND,
$$

- Dephasing without dissipation...

- Use made of the above Hamiltonian in the context of the influence of dephasing in quantum computation—(Unruh: (1995)), (Palma et al.: (1996)), (DiVincenzo: (1995)).

- Also used by (Turchette et al.: (2000)) in context of engineered reservoir.

Quantum Non-Demolition (QND) Measurements

- QND measurement of observable $\hat{A}$ would be a sequence of precise measurements of $\hat{A}$ such that each measurement is completely predictable from the result of the first measurement, i.e., the system to be measured is independent of the backaction of the measuring apparatus. This implies $[\hat{A}, \hat{H}_{int}] = 0$, where $\hat{H}_{int}$ is the interaction term between the observable and the measuring apparatus. Historically introduced to design gravitational-wave antennas (Braginsky *et al.* : (1975); (1980)).

- Further, $[\hat{A}(t_i), \hat{A}(t_j)] = 0$ for all times t_i, t_j. This would protect $\hat{A}$ from contamination by noncommuting (with $\hat{A}$) observables. This is guaranteed if $\hat{A}$ is a constant of the free evolution, i.e., $[\hat{A}, \hat{H}_S] = 0$, where $\hat{H}_S$ is the Hamiltonian responsible for the free evolution of $\hat{A}$ (Caves *et al.* : (1980)).

- If $\hat{H}_{int} = \kappa \hat{A} \hat{P}_R$, where κ is a constant and $\hat{P}_R \in \mathcal{H}_R$, $\mathcal{H}_R$ being the Hilbert space of the apparatus or probe, then the evolution of $\hat{A}$ with coupling turned on is identical to its free evolution and it is free from contamination. Then $\hat{A}$ is the *pointer observable* and the interaction $\hat{H}_{int}$ corresponds to a measurement of $\hat{A}$. For $\hat{A} = \hat{H}_S$, this would correspond to the measurement of energy (Unruh: (1978)).

Dynamics of the Reduced Density Matrix for QND systems

(SB and R. Ghosh: (2007))

- System-reservoir complex evolves unitarily by:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht},$$

  where $\rho(0) = \rho^s(0)\rho_R(0)$.

- We are interested in the reduced dynamics of the system S, taking into account the influence of its environment. Reduced density matrix in system eigenbasis is

$$\rho^s_{nm}(t) = e^{-\frac{i}{\hbar}(E_n - E_m)t}e^{i(E_n^2 - E_m^2)\eta(t)}e^{-(E_n - E_m)^2\gamma(t)}\rho^s_{nm}(0).$$

  Here the system eigenbasis, formed from the Wigner-Dicke states for the qubit system and the number states for the oscillator system, would be

$$H_S|n\rangle = E_n|n\rangle.$$

A Derivation of the reduced density matrix

- We consider the Hamiltonian:

$$
\begin{aligned}
H &= H_S + H_R + H_{SR} \\
&= H_S + \sum_k \hbar\omega_k b_k^\dagger b_k + H_S \sum_k g_k (b_k + b_k^\dagger) + H_S^2 \sum_k \frac{g_k^2}{\hbar\omega_k}.
\end{aligned}
$$

- Here H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system-reservoir interaction, respectively. H_S is a generic system Hamiltonian which can be specified depending on the physical situation. $b_k^\dagger$, b_k denote the creation and annihilation operators for the reservoir oscillator of frequency ω_k, g_k stands for the coupling constant (assumed real) for the interaction of the oscillator field with the system. The last term on the right-hand side of Eq. (1) is a renormalization inducing `counter term'. Since $[H_S, H_{SR}] = 0$, the Hamiltonian (1) is of QND type.

A Derivation of the reduced density matrix: continued...

- The system-plus-reservoir composite is closed and hence obeys a unitary evolution given by

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar},$$

where

$$\rho(0) = \rho^s(0)\rho_R(0),$$

i.e., we assume separable initial conditions.

- In order to obtain the reduced dynamics of the system alone, we trace over the reservoir variables. The matrix elements of the reduced density matrix in the system eigenbasis are

$$\rho^s_{nm}(t) \quad = \quad e^{-i(E_n-E_m)t/\hbar} \; e^{-i(E_n^2-E_m^2)/\hbar \sum_k (g_k^2 t/\hbar\omega_k)}$$
$$\times \mathrm{Tr_R}\left[e^{-iH_n t/\hbar}\rho_R(0)e^{iH_m t/\hbar}\right]\rho^s_{nm}(0),$$

where E_n's are the eigenvalues of the system Hamiltonian.

A Derivation of the reduced density matrix: continued...

- Here $\rho_R(0)$ is the initial density matrix of the reservoir which we take to be a thermal bath given by:

$$\rho_R(0) = \rho_{th} = \prod_k \left[1 - e^{-\beta\hbar\omega_k}\right] e^{-\beta\hbar\omega_k b_k^\dagger b_k}$$

  This is the density matrix of a thermal bath at temperature T, with $\beta \equiv 1/(k_B T)$, k_B being the Boltzmann constant.

- In the reduced density matrix

$$H_n = \sum_k \left[\hbar\omega_k b_k^\dagger b_k + E_n g_k (b_k + b_k^\dagger)\right].$$

- Using the identity

$$\mathrm{Tr_R} \left[\rho_{\mathrm{th}} e^{tb^\dagger + ub} \right] = \exp \left[\frac{tu}{2} \coth \left(\frac{\beta \hbar \omega_k}{2} \right) \right],$$

the reduced density matrix is obtained. There the terms

$$\eta(t) = -\frac{\gamma_0}{\pi} \tan^{-1}(\omega_c t),$$

and

$$\gamma(t) = 2 \sum_k \frac{g_k^2}{\hbar^2 \omega_k^2} \coth \left(\frac{\beta \hbar \omega_k}{2} \right) \sin^2(\frac{\omega_k t}{2}),$$

are due to the effect of the bath on the system.

Details pertaining to the reservoir

Explicit forms of $\eta(t)$, $\gamma(t)$:

- assuming the bath to have large number of degrees of freedom such that information going out of the system of interest does not return to it, we take a `quasi-continuous' bath spectrum with spectral density $I(\omega)$ such that

$$\sum_k \frac{g_k^2}{\hbar^2} f(\omega_k) \longrightarrow \int_0^\infty d\omega I(\omega) f(\omega),$$

- in case of Ohmic bath with spectral density:

$$I(\omega) = \frac{\gamma_0}{\pi} \omega e^{-\omega/\omega_c},$$

where γ_0 and ω_c two bath parameters, $\eta(t)$ and $\gamma(t)$ can be seen to be:

$$\eta(t) = -\frac{\gamma_0}{\pi} \tan^{-1}(\omega_c t),$$

Details pertaining to the reservoir   continued ...

and $\gamma(t)$ at $T = 0$

$$\gamma(t) = \frac{\gamma_0}{2\pi} \ln(1 + \omega_c^2 t^2),$$

where $t > 2a$, and for high T

$$\gamma(t) = \frac{\gamma_0 k_B T}{\pi \hbar \omega_c} \left[2\omega_c t \tan^{-1}(\omega_c t) + \ln\left(\frac{1}{1 + \omega_c^2 t^2} \right) \right],$$

where, again, $t > 2a$.

Two-Level System

Hamiltonian

$$H_S = \frac{\hbar\omega}{2}\sigma_z,$$

σ_z being the usual Pauli matrix.

- System eigenbasis: $|j, m\rangle$— Wigner-Dicke States

$$
\begin{aligned}
H_S|j, m\rangle &= \hbar\omega m|j, m\rangle \\
&= E_{j,m}|j, m\rangle,
\end{aligned}
$$

where $-j \le m \le j$.

- Initial system state:

$$|\psi(0)\rangle = \cos\left(\frac{\theta_0}{2}\right)|1\rangle + e^{i\phi_0}\sin\left(\frac{\theta_0}{2}\right)|0\rangle.$$

- Reduced Density Matrix:

$$\rho_{m,n}^s(t) = \begin{pmatrix} \cos^2(\frac{\theta_0}{2}) & \frac{1}{2}\sin(\theta_0)e^{-i(\omega t+\phi_0)}e^{-(\hbar\omega)^2\gamma(t)} \\ \frac{1}{2}\sin(\theta_0)e^{i(\omega t+\phi_0)}e^{-(\hbar\omega)^2\gamma(t)} & \sin^2(\frac{\theta_0}{2}) \end{pmatrix}.$$

- Bloch vectors:

$$
\begin{aligned}
\langle \sigma_x(t) \rangle &= \sin(\theta_0) \cos(\omega t + \phi_0) e^{-(\hbar\omega)^2 \gamma(t)}, \\
\langle \sigma_y(t) \rangle &= \sin(\theta_0) \sin(\omega t + \phi_0) e^{-(\hbar\omega)^2 \gamma(t)}, \\
\langle \sigma_z(t) \rangle &= \cos(\theta_0).
\end{aligned}
$$

- QND Evolution—Coplanar, fixed by the polar angle θ_0, in-spiral towards the z-axis of the Bloch sphere. This is the characteristic of a **phase-damping channel** (M. Nielsen and I. Chuang: (2000)).

Dissipative form of evolution equation: Lindblad form

- A very useful form of the evolution equation of the reduced density matrix, of the system of interest, for the case where the system and interaction Hamiltonians do not commute is called the Lindblad master equation.

- The physical assumptions underlying the Lindblad form of the master equation are the *Born* (weak coupling), *Markov* (memoryless) and *Rotating Wave Approximation* (fast system dynamics compared to the ralaxation time).

Dynamics of the Reduced Density Matrix for Dissipative systems: An Illustration

- Here we illustrate the Lindblad evolution by means of a practical example, i.e., decay of a two-level system interacting with a radiation field (bath) in the weak Born-Markov, rotating wave approximation.

- $H_S = \frac{1}{2}\hbar\omega_0\sigma_z$, ω_0 is the transition frequency. System interacts with bath of harmonic oscillators via the atomic dipole operator (in the interaction picture)

$$\vec{D}(t) = \vec{d}\sigma_- e^{-i\omega t} + \vec{d}^*\sigma_+ e^{i\omega t},$$

where $\vec{d} = \langle g|\vec{D}|e\rangle$: transition matrix elements of dipole operator and the $S - R$ coupling term is:

$$H_{SR} = -\vec{D}.\vec{E}.$$

Here $\vec{E}$ is the electric field operator, which in the Schrodinger picture is:

$$\vec{E} = i\sum_{\vec{k}}\sum_{\lambda=1,2}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\vec{e}_\lambda(\vec{k})\left(b_\lambda(\vec{k}) - b_\lambda^\dagger(\vec{k})\right).$$

Here the field modes are represented by $\vec{k}$ and two corresponding, transverse unit polarization vectors $\vec{e}_\lambda(\vec{k})$.

- The Pauli operators σ_+, σ_- satisfy: $[H_S, \sigma_-] = -\omega_0 \sigma_-$, $[H_S, \sigma_+] = \omega_0 \sigma_+$, i.e., they lower/raise the atomic energy by $\mp \omega_0$.

- This process thus has two Lindblad operators: $\vec{A}(\omega_0) \equiv \vec{A} = \vec{d} \sigma_-$, $\vec{A}(-\omega_0) \equiv \vec{A^\dagger} = \vec{d}^* \sigma_+$.

- The Lindblad master equation for the reduced density matrix operator in the interaction picture (neglecting the Lamb shift terms) becomes:

$$
\frac{d}{dt}\rho^S(t) \;=\; \gamma_0(N_{th}+1)\left(\sigma_-\rho^S(t)\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho^S(t) - \frac{1}{2}\rho^S(t)\sigma_+\sigma_-\right)
$$
$$
+\;\gamma_0 N_{th}\left(\sigma_+\rho^S(t)\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho^S(t) - \frac{1}{2}\rho^S(t)\sigma_-\sigma_+\right).
$$

- Here γ_0 is spontaneous emission rate

$$
\gamma_0 = \frac{4\omega^3|\vec{d}|^2}{3\hbar c^3},
$$

and σ_+, σ_-: standard raising and lowering operators, respectively given by

$$\sigma_+ = |1\rangle\langle 0| = \frac{1}{2}\left(\sigma_x + i\sigma_y\right); \quad \sigma_- = |0\rangle\langle 1| = \frac{1}{2}\left(\sigma_x - i\sigma_y\right).$$

- The first term on the RHS of the Lindblad equation containing $\gamma_0(N_{th}+1)$ is responsible for spontaneous (γ_0) plus thermal ($\gamma_0 N_{th}$) emission while the second term containing $\gamma_0 N_{th}$ is responsible for thermal absorption.

- The master equation may be expressed in manifestly Lindblad form

$$\frac{d}{dt}\rho^s(t) = \sum_{j=1}^{2}\left(2R_j\rho^s R_j^\dagger - R_j^\dagger R_j\rho^s - \rho^s R_j^\dagger R_j\right),$$

where $R_1 = (\gamma_0(N_{\mathrm{th}}+1)/2)^{1/2}\sigma_-$, $R_2 = (\gamma_0 N_{\mathrm{th}}/2)^{1/2}\sigma_+$. (If $T = 0$, a single Lindblad operator suffices)

$$N_{th} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}.$$

Here N_{th}: Planck distribution giving the number of thermal photons at the frequency ω.

- The Lindblad equation obtained can be easily solved by invoking the representation of the two-level density matrix in terms of Pauli operators:

$$\rho^S(t) = \frac{1}{2}\left(I + \langle \vec{\sigma}(t)\rangle . \vec{\sigma}\right)$$

$$= \begin{pmatrix} \left(\frac{1}{2}\right)(1 + \langle \sigma_z(t)\rangle) & \langle \sigma_-(t)\rangle \\ \\ \langle \sigma_+(t)\rangle & \left(\frac{1}{2}\right)(1 - \langle \sigma_z(t)\rangle) \end{pmatrix}.$$

- Using, for e.g.:

$$\frac{d}{dt}\langle \sigma_z(t) = \mathrm{Tr}\left(\sigma_z \frac{\mathrm{d}}{\mathrm{dt}}\rho^S(t)\right),$$

and likewise for the other two Pauli operators, we get three linear differential equations which can be easily solved to yield the Bloch vectors

$$\langle \sigma_x(t)\rangle = e^{-\frac{\gamma_0}{2}(2N_{th}+1)t}\langle \sigma_x(0)\rangle,$$

$$\langle \sigma_y(t)\rangle = e^{-\frac{\gamma_0}{2}(2N_{th}+1)t}\langle \sigma_y(0)\rangle,$$

$$\langle \sigma_z(t)\rangle =$$

$$e^{-\gamma_0(2N_{th}+1)t}\langle \sigma_z(0)\rangle - \frac{1}{(2N_{th}+1)}\left(1 - e^{-\gamma_0(2N_{th}+1)t}\right).$$

Quantum Operations

- Any evolution consistent with the general rules of quantum mechanics can be described by a linear, completely positive map, called quantum operation ($\mathcal{E}$). (M. A. Nielsen and I. L. Chuang: (2000))

- Complete positivity: Consider any positive map $\mathcal{E}$ on the system Q_1: if an extra system R of arbitrary dimensionality is introduced, and $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive on any positive operator A on the combined system RQ_1, where $\mathcal{I}$ denotes the identity map on system R: then $\mathcal{E}$ is completely positive.

- A unitary evolution is a special case of a quantum operation: general quantum operations can describe non-unitary evolutions, due to coupling with environment.

Quantum Operations   *continued...*

- Any such quantum operation can be composed from elementary operations:
  - unitary transformations: $\mathcal{E}_1(\rho) = U\rho U^\dagger$
  - addition of an auxiliary system: $\mathcal{E}_2(\rho) = \rho \otimes \sigma$: here ρ is the original system and σ is the auxiliary one
  - partial traces: $\mathcal{E}_3(\rho) = \mathrm{Tr}_\mathrm{B}(\rho)$
  - projective measurements: $\mathcal{E}_4(\rho) = P_k \rho P_k / \mathrm{Tr}(\mathrm{P_k}\rho)$, with $P_k^2 = P_k$.

Connection to quantum noise processes

- Interpret our results in terms of familiar noisy channels. How these environmental effects can affect quantum computing.
  In operator-sum representation, action of superoperator $\mathcal{E}$ due to environmental interaction

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_k \langle e_k | U(\rho \otimes |f_0\rangle\langle f_0|) U^\dagger | e_k \rangle = \sum_j E_j \rho E_j^\dagger,$$

  unitary U acts jointly on system-environment $|f_0\rangle$: environment's initial state; $\{|e_k\rangle\}$ a basis for the environment.

- environment-system assumed to start in a separable state.

- $E_j \equiv \langle e_k | U | f_0 \rangle$ are the Kraus operators; partition of unity: $\sum_j E_j^\dagger E_j = \mathcal{I}$. Any transformation representatable as operator-sum is a completely positive (CP) map.

Connection to quantum noise processes: QND interactions

- Here we give some illustrations of single qubit quantum noisy channels.

- QND interactions yields quantum phase damping channel: uniquely non-classical quantum mechanical noise process, describing the loss of quantum information without the loss of energy.

- Kraus operator elements (SB and R. Ghosh: (2007))

$$
E_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\beta(t)}\sqrt{1-\lambda} \end{bmatrix} \quad ; \qquad E_1 \equiv \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix},
$$

  where $\beta(t)$ encodes the free evolution of the system and λ the effect of the environment.

- Applying this to initial state yields

$$
\rho^s(t) = \mathcal{E}(\rho^s(0)) \quad =
$$

$$
\begin{pmatrix} \cos^2\left(\frac{\theta_0}{2}\right) & \left(\frac{1}{2}\right) e^{-i(\beta(t)+\phi_0)}\sin(\theta_0)\sqrt{1-\lambda} \\[2ex] \left(\frac{1}{2}\right) e^{i(\beta(t)+\phi_0)}\sin(\theta_0)\sqrt{1-\lambda} & \sin^2\left(\frac{\theta_0}{2}\right) \end{pmatrix}.
$$

Connection to quantum noise processes: QND interactions

- Comparing with QND interaction with bath of harmonic oscillators

$$\lambda(t) = 1 - \exp\left[-2(\hbar\omega)^2\gamma(t)\right]; \qquad \beta(t) = \omega t.$$

- QND interaction with a bath of two level systems

$$\lambda(t) = 1 - (1 + 4\omega_c^2 t^2)^{(-\gamma_0/2\pi)(\hbar\omega)^2}; \qquad \beta(t) = \omega t.$$

- $\lambda(t) \longrightarrow 1$ as $t \longrightarrow \infty$ (exponentially for high T and as power law for $T = 0$)

Connection to quantum noise processes: Dissipative interactions

- Squeezed generalized amplitude damping channel: extends the concept of generalized amplitude damping channel by allowing for finite bath squeezing along with dissipation (R. Srikanth and SB: (2007))

- It is characterized by the Kraus operators

$$
E_0 \equiv \sqrt{p_1} \begin{bmatrix} \sqrt{1-\alpha(t)} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1 \equiv \sqrt{p_1} \begin{bmatrix} 0 & 0 \\ \sqrt{\alpha(t)} & 0 \end{bmatrix},
$$

$$
E_2 \equiv \sqrt{p_2} \begin{bmatrix} \sqrt{1-\mu(t)} & 0 \\ 0 & \sqrt{1-\nu(t)} \end{bmatrix},
$$

$$
E_3 \equiv \sqrt{p_2} \begin{bmatrix} 0 & \sqrt{\nu(t)} \\ \sqrt{\mu(t)}e^{-i\Phi} & 0 \end{bmatrix}.
$$

Connection to quantum noise processes: Dissipative interactions

- Here

$$
\nu(t) \;=\; \frac{N}{p_2(2N+1)}(1 - e^{-\gamma_0(2N+1)t}),
$$

$$
\mu(t) \;=\; \frac{2N+1}{2p_2 N}\,\frac{\sinh^2(\gamma_0 a t/2)}{\sinh(\gamma_0(2N+1)t/2)}\,\exp\left(-\frac{\gamma_0}{2}(2N+1)t\right),
$$

$$
\alpha(t) \;=\; \frac{1}{p_1}\left(1 - p_2[\mu(t) + \nu(t)] - e^{-\gamma_0(2N+1)t}\right),
$$

where $p_2 = 1 - p_1$, and

$$
\begin{aligned}
p_2 \;=\;& \frac{1}{(A + B - C - 1)^2 - 4D} \\
\times\;& \left[A^2 B + C^2 + A(B^2 - C - B(1 + C) - D) \right. \\
-\;& (1 + B)D - C(B + D - 1) \\
\pm\;& 2\,(D(B - AB + (A - 1)C + D) \\
\times\;& \left. (A - AB + (B - 1)C + D))^{1/2} \right],
\end{aligned}
$$

Connection to quantum noise processes: Dissipative interactions

- with

$$
\begin{aligned}
A &= \frac{2N+1}{2N}\frac{\sinh^2(\gamma_0 a t/2)}{\sinh(\gamma_0(2N+1)t/2)}\exp\left(-\gamma_0(2N+1)t/2\right), \\
B &= \frac{N}{2N+1}(1-\exp(-\gamma_0(2N+1)t)), \\
C &= A+B+\exp(-\gamma_0(2N+1)t), \\
D &= \cosh^2(\gamma_0 a t/2)\exp(-\gamma_0(2N+1)t).
\end{aligned}
$$

- If squeezing parameter r is set to zero, the Kraus operators reduce to that of a generalized amplitude damping channel, with $\nu(t) = \alpha(t)$, $\mu(t) = 0$ and p_1 and p_2 becoming time-independent. If further $T = 0$, then $p_2 = 0$, resulting in two Kraus operators, corresponding to an amplitude damping channel.
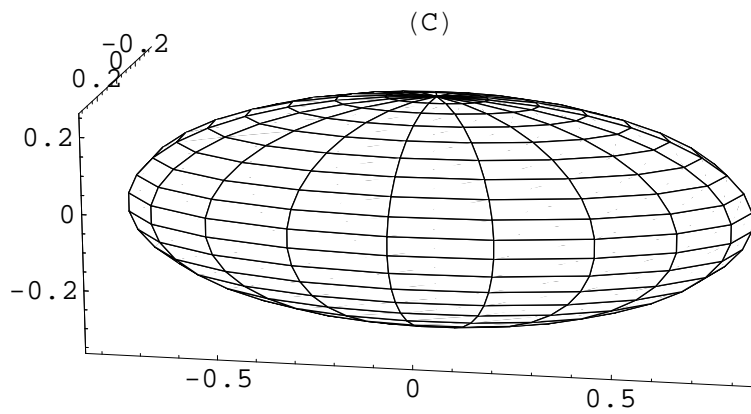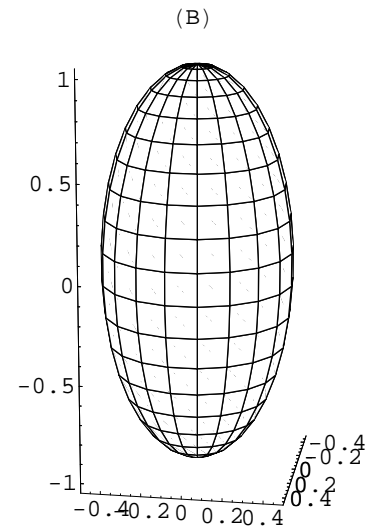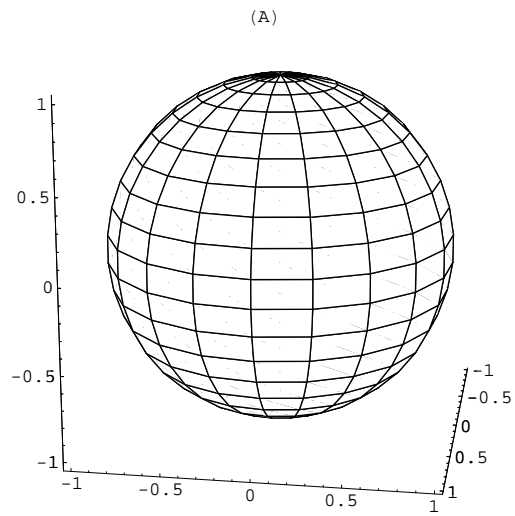
(A)

(B)

(C)

(D)
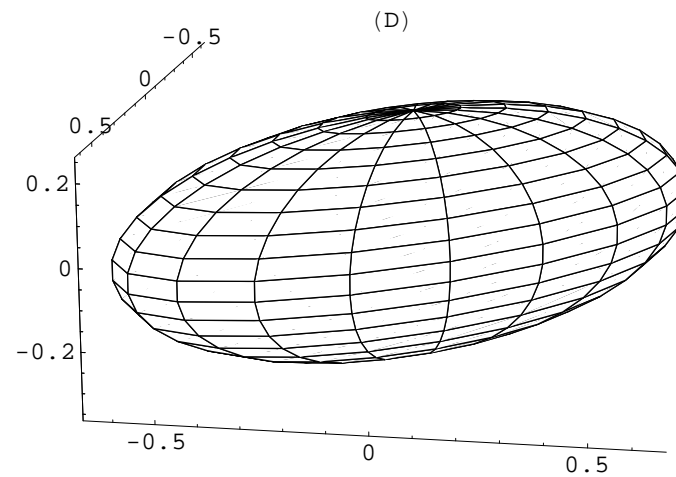
Fig. : Effect of QND and dissipative interactions on the Bloch sphere: (A) the full Bloch sphere; (B) the Bloch sphere after time $t = 20$, with $\gamma_0 = 0.2$, $T = 0$, $\omega = 1$, $\omega_c = 40\omega$ and the environmental squeezing parameter $r = a = 0.5$, evolved under a QND interaction ; (C) and (D) the effect of the Born-Markov type of dissipative interaction with $\gamma_0 = 0.6$ and temperature $T = 5$, on the Bloch sphere – the x and y axes are interchanged to present the effect of squeezing more clearly. (C) corresponds to $r = 0.4$, $\Phi = 0$ and $t = 0.15$ while (D) corresponds to $r = 0.4$, $\Phi = 1.5$ and $t = 0.15$.
(SB and R. Ghosh: (2007))

Derivation of Lindbladians

A Simple Intuitive Proof:

- Let us look at the dynamics of the system on a timescale δt. It should satisfy two conditions:

  (A). $\delta t \ll \tau_S$: the timescale is small compared to the characteristic timescale of the system τ_S; the system density matrix evolves only a little bit in this time interval;

  (B). $\delta t \gg \tau_B$: At the same time δt is long compared to the time over which the environment/bath forgets its information about the system τ_B.

- Since we look for dynamics beyond time τ_B, the evolution through time δt should be described by a quantum operation on the current system density matrix. Thus:

$$\rho_S(\delta t) = \mathcal{E}(\rho_S(0)) = \sum_k E_k \rho_S(0) E_k^\dagger = \rho_S(0) + O(\delta t).$$

- Thus it follows that the Kraus operators should be of the form

$$E_0 = \mathcal{I}_S + (K - \frac{i}{\hbar}H)\delta t,$$

$$E_k = \sqrt{\delta t}L_k, \ \ k \geq 1.$$

K, H are arbitrary Hermitian operators, L_k are also arbitrary and are called the Lindblad operators.

- The normalization of Kraus operators gives

$$\mathcal{I}_S = \mathcal{I}_S + (2K + \sum_k L_k^\dagger L_k)\delta t + O((\delta t)^2),$$

implying that $K = -\frac{1}{2}\sum_k L_k^\dagger L_k$.

- Therefore

$$\rho_S(\delta t) = \rho_S(0) - \left\{ \frac{i}{\hbar}[H, \rho_S] - \sum_k \left[L_k \rho_S(0) L_k^\dagger - \frac{1}{2}\{\rho_S(0), L_k^\dagger L_k\} \right] \right\} \delta t + O((\delta t)^2).$$

- $\{A, B\} = AB + BA$. Taking the limit $\delta t \longrightarrow 0$, the Lindblad master equation is obtained as

$$\frac{d\rho_S}{dt} = \frac{1}{i\hbar}[H, \rho_S] + \sum_k \left[L_k \rho_S(0) L_k^\dagger - \frac{1}{2}\{\rho_S(0), L_k^\dagger L_k\} \right].$$

- If the evolution were unitary, there are no Lindblad operators, then the above master equation reduces to $\frac{d\rho_S}{dt} = \frac{1}{i\hbar}[H, \rho_S]$, the usual Schr*ö*dinger-vonNeumann equation.

- This derivation gives no clue to the microscopic origins of the Lindbladians. That would require a more detailed derivation.

Derivation of Lindbladians   *continued...*

A Detailed Derivation:

- Starting with the interaction picture von Neumann equation
  $\frac{d}{dt}\rho(t) = -i[H_{SR}(t), \rho(t)]$, for the total density matrix $\rho(t)$. This gives:

$$\rho(t) = \rho(0) - i \int_0^t ds[H_{SR}(s), \rho(s)].$$

- Inserting the integral into the von Neumann equation and tracing over the bath:

$$\frac{d}{dt}\rho^S(t) = - \int_0^t ds \mathrm{Tr}_{\mathrm{R}}[\mathrm{H}_{\mathrm{SR}}(\mathrm{t}), [\mathrm{H}_{\mathrm{SR}}(\mathrm{s}), \rho(\mathrm{s})]],$$

  where $\mathrm{Tr}_R[H_{SR}(s), \rho(0)] = 0$ is assumed. The RHS of the equation depends on the full density matrix $\rho(s)$. This is where the first approximation is made:

- *Born approximation*: assumes that the coupling between S and R is weak, ρ_R is negligibly affected by the interaction and the total system after time t is:
  $\rho(t) \equiv \rho^S(t) \otimes \rho_R$. This gives:

$$\frac{d}{dt}\rho^S(t) = - \int_0^t ds \mathrm{Tr}_R[H_{SR}(t), [H_{SR}(s), \rho^S(s) \otimes \rho_R]].$$

Derivation of the Lindblad form   *contin-ued ...*

- A further simplification: $\rho^S(s) \longrightarrow \rho^S(t)$. Thus the evolution equation of the system at t depends only on the present state. This is *Redfield* equation (A. G. Redfield: (1957)).

- Redfield equation is local in time, but depends on the choice of the initial preparation at $t = 0$, hence is not *Markovian*. To make it Markovian replace s by $t - s$ in the integrand and let the upper limit go to infinity. This gives:

$$\frac{d}{dt}\rho^S(t) = -\int_0^\infty ds \, \mathrm{Tr}_\mathrm{R}[\mathrm{H}_\mathrm{SR}(t), [\mathrm{H}_\mathrm{SR}(t-s), \rho^\mathrm{S}(t) \otimes \rho_\mathrm{R}]].$$

This is a Markovian equation and the approximation is called the Markovian approximation. It is justified when the time scale associated with the reservoir correlations τ_R is much smaller than the time scale τ_{rel} over which the state varies appreciably. Thus the Markovian evolution is defined on a coarse-grained time scale, where the dynamical behaviour over times of the order of τ_R are not resolved. Since τ_R depends on the reservoir temperature and τ_{rel} on the $S - R$ coupling strength, the Markovian approximation is easily justified for weak $S - R$ coupling and high T.

- The approximations made till now would be collectively called the *Born-Markov* approximation. However, they do not guarantee a quantum dynamical semigroup evolution (E. B. Davies: (1974); R. Dumcke and H. Spohn: (1979)). A further approximation involving the averaging over the rapidly oscillating terms in the master equation is performed, the *Rotating Wave Approximation*.

- The interaction Hamiltonian H_{SR} is decomposed into eigenoperators of the system Hamiltonian H_S. A generic interaction Hamiltonian in the interaction picture can be written as:

$$H_{SR}(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t),$$

where A, B denote operators belonging to the system and reservoir, respectively. Also:

$$A_\alpha(\omega) = \sum_{\epsilon'-\epsilon=\omega} \Pi(\epsilon) A_\alpha \Pi(\epsilon'),$$

where $\Pi(\epsilon)$ projects the operator onto the eigenspace of H_S belonging to the eigenvalue ϵ. Thus $[H_S, A_\alpha(\omega)] = -\omega A_\alpha(\omega)$, i.e., $A_\alpha(\omega)$ lowers the energy of H_S by ω while $A_\omega^\dagger$ raises it by ω.

Derivation of the Lindblad form   *contin-ued ...*

- From this:

$$e^{iH_S t} A_\alpha(\omega) e^{-iH_S t} = e^{-i\omega t} A_\alpha(\omega),$$

and

$$e^{iH_R t} B_\alpha e^{-iH_R t} = B_\alpha(t).$$

- The earlier condition $\mathrm{Tr}_R [H_{SR}(s), \rho(0)] = 0$ now implies $\langle B_\alpha(t) \rangle = \mathrm{Tr}_R(B_\alpha(t) \rho_R) = 0$.

- This leads to the following form of the Born-Markov equation obtained earlier:

$$\frac{d}{dt} \rho^S(t) = \sum_{\omega, \omega'} \sum_{\alpha, \beta} e^{i(\omega'-\omega)t} \Gamma_{\alpha,\beta}(\omega) [A_\beta(\omega) \rho^S(t) A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega') A_\beta(\omega) \rho^S(t)] + h.c.$$

- Here

$$\Gamma_{\alpha,\beta}(\omega) = \int_0^\infty ds\, e^{i\omega s} \langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle,$$

is the one-sided Fourier transform of reservoir correlation functions

$$\langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle = \mathrm{Tr}_R(B_\alpha^\dagger(t) B_\beta(t-s) \rho_R).$$

- If ρ_R is a stationary state of the reservoir, $[H_R, \rho_R] = 0$, the reservoir correlation functions are homogeneous in time

$$\langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle = \langle B_\alpha^\dagger(s) B_\beta(0) \rangle,$$

thus they do not depend on time.

- In the above evolution equation, $|\omega - \omega'|^{-1}$ defines the typical time-scale associated with the intrinsic evolution of the system. If the systematic evolution of the system is very quick, then it goes through many cycles during the relaxation time. Thus the non-secular terms, i.e., those for which $\omega' \neq \omega$, may be neglected. This is the rotating wave approximation. With this, the evolution equation becomes:

$$\frac{d}{dt}\rho^S(t) = \sum_\omega \sum_{\alpha,\beta} \Gamma_{\alpha,\beta}(\omega)[A_\beta(\omega)\rho^S(t)A_\alpha^\dagger(\omega) - A_\alpha^\dagger(\omega)A_\beta(\omega)\rho^S(t)] + h.c.$$

Derivation of the Lindblad form   *contin-ued ...*

- In the above equation, the term $\Gamma_{\alpha,\beta}$ can be rearranged as:

$$\Gamma_{\alpha,\beta}(\omega) = \frac{1}{2}\gamma_{\alpha,\beta}(\omega) + iS_{\alpha,\beta}(\omega),$$

where

$$\gamma_{\alpha,\beta}(\omega) = \Gamma_{\alpha,\beta}(\omega) + \Gamma_{\beta,\alpha}^{*}(\omega) = \int_{-\infty}^{\infty} ds\, e^{i\omega s}\langle B_{\alpha}^{\dagger}(s)B_{\beta}(0)\rangle,$$

and

$$S_{\alpha,\beta}(\omega) = \frac{1}{2i}(\Gamma_{\alpha,\beta}(\omega) - \Gamma_{\beta,\alpha}^{*}(\omega)).$$

- With these, the evolution equation can be written as:

$$\frac{d}{dt}\rho^S(t) = -i[H_{LS}, \rho_S(t)] + \mathcal{D}(\rho^S(t)),$$

where

$$H_{LS} = \sum_\omega \sum_{\alpha,\beta} S_{\alpha,\beta} A_\alpha^\dagger(\omega) A_\beta(\omega),$$

is called the Lamb shift as it leads to a Lamb-type renormalization of the unperturbed energy levels due to the $S - R$ coupling and provides a Hamiltonian contribution to the dynamics. The term $\mathcal{D}(\rho^S(t))$ is called the dissipator and takes the form

$$\mathcal{D}(\rho^S(t)) = \sum_\omega \sum_{\alpha,\beta} \gamma_{\alpha,\beta} \left(A_\beta(\omega)\rho^S A_\alpha^\dagger(\omega) - \frac{1}{2}\{A_\alpha^\dagger(\omega) A_\beta(\omega), \rho^S\} \right).$$

Here $\{A, B\} = AB + BA$.

- The term $\gamma_{\alpha,\beta}$, in the dissipator is the Fourier transform of the homogeneous reservoir correlation functions, is positive by Bochner's theorem and hence can be diagonalized. With that the evolution equation takes the form of the standard Lindblad equation.

Derivation of the Lindblad form   *contin-ued ...*

- We thus stress that the physical assumptions underlying the Lindblad form of the master equation are the *Born* (weak coupling), *Markov* (memoryless) and *Rotating Wave Approximation* (fast system dynamics compared to the ralaxation time).

Simple Applications of Developed Formalism

A. Classical Capacity of Squeezed Generalized Amplitude Damping Channel
(SB and R. Srikanth: (2008))

- A quantum communication channel can be used to perform a number of tasks: transmitting classical or quantum information.

- How information communicated over squeezed generalized amplitude damping channel is degraded...

- Consider the following situation: there is a sender A and receiver B; A has a classical information source producing symbols $X = 0, \cdots, n$ with probabilities $p_0, \cdots, p_n$ which are encoded as quantum states ρ_j $(0 \leq j \leq n)$ and communicated to B, whose optimal measurement strategy maximizes the accessible information, which is bounded above by the Holevo quantity

$$\chi = S(\rho) - \sum_j p_j S(\rho_j),$$

where $\rho = \sum_j p_j \rho_j$, and ρ_j are various initial states and S(ρ) is the von Neumann entropy.

Classical Capacity of Squeezed Generalized continued ...

- Here assume A encodes binary symbols of 0 and 1 in terms of pure, orthogonal states of the form $|\psi(0)\rangle = \cos(\frac{\theta_0}{2})|1\rangle + e^{i\phi_0}\sin(\frac{\theta_0}{2})|0\rangle$, and transmits them over the squeezed generalized amplitude damping channel $(\mathcal{E})$.

- Further assume that A transmits messages as product states, i.e., without entangling them across multiple channel use. Then, the (product state) classical capacity C of the quantum channel is defined as the maximum of $\chi(\mathcal{E})$ over all ensembles $\{p_j, \rho_j\}$ of possible input states ρ_j .
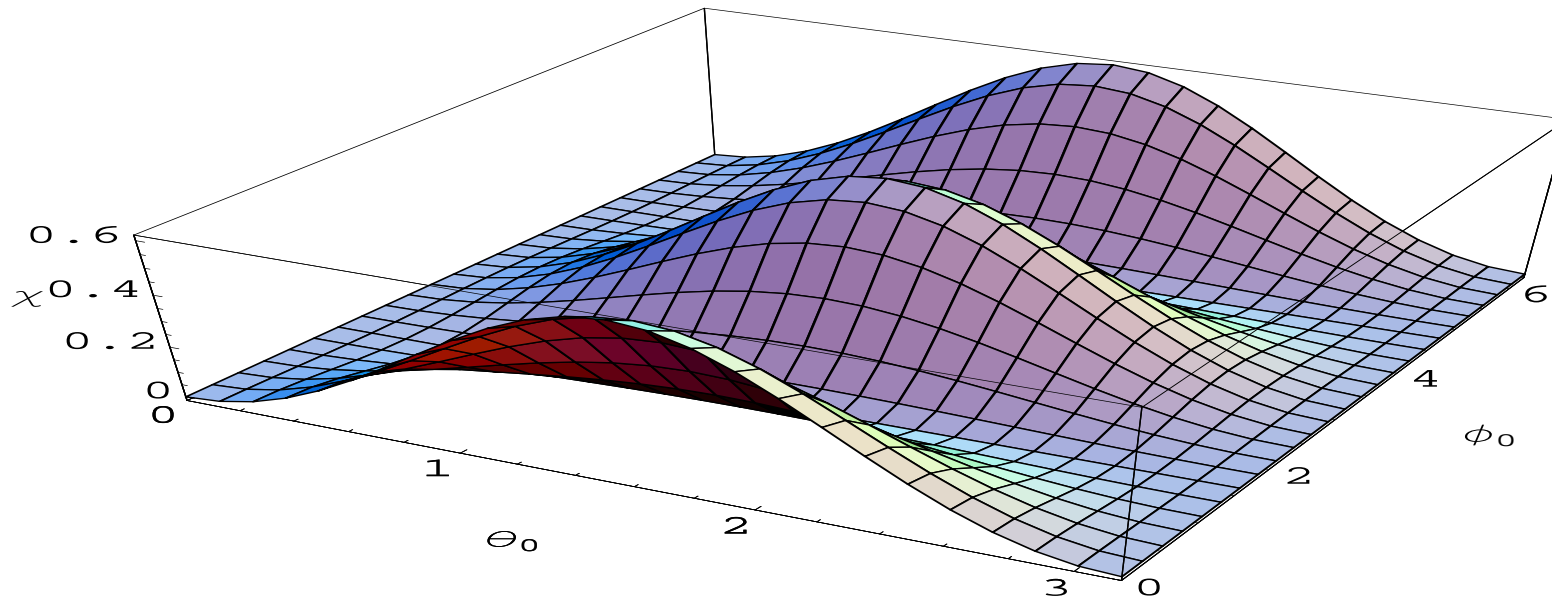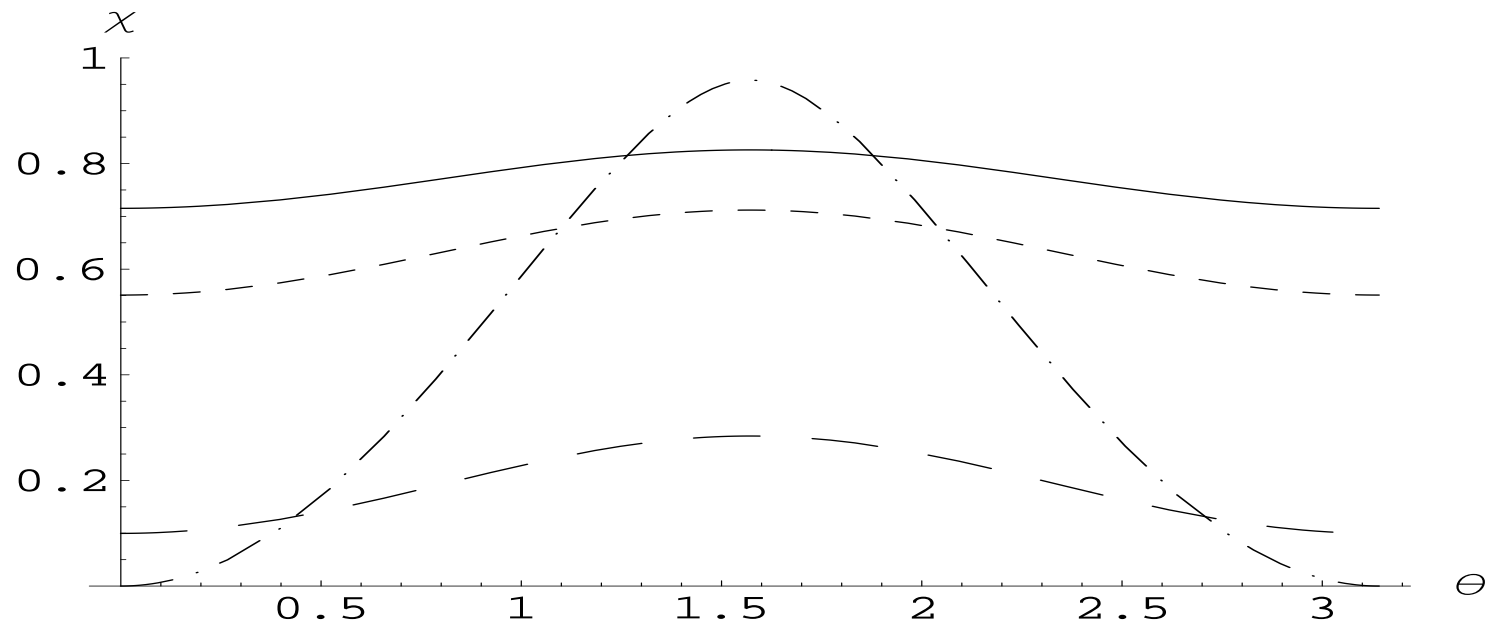
Fig. 2 : Holevo bound χ for a squeezed generalized amplitude damping channel with $\Phi = 0$, over the set $\{\theta_0, \phi_0\}$, which parametrizes the ensemble of input states $\{(\theta_0, \phi_0),\ (\theta_0 + \pi, \phi_0)\}$, corresponding to the symbols 0 and 1, respectively, with probability of the input symbol 0 being $f = 0.5$. Here temperature $T = 5, \gamma_0 = 0.05$, time $t = 5.0$ and bath squeezing parameter $r = 1$. The channel capacity C is seen to correspond to the optimal value of $\theta_0 = \pi/2$ (i.e., the input states $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ for $\phi_0 = 0$).

- Fig. 3 illustrates optimal source coding for the squeezed amplitude damping channel, with χ plotted against θ_0 corresponding to the "0" symbol. Here $\Phi = 0$, $\gamma_0 = 0.05$ and $f = 0.5$. It is seen that χ is maximized for states of the form when the pair of input states are given by $(\theta_0 = \frac{\pi}{2}, \phi_0 = 0)$ and $(\theta_0 = \frac{\pi}{2} + \pi, \phi_0 = 0)$ (i.e., states $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$). The solid and small-dashed curves represent temperature $T = 0$ and bath squeezing parameter $r = 0$, but $t = 1$ and 2, respectively. The large-dashed and dot-dashed curves represent $T = 5$ and $t = 2$, but with $r = 0$ and 2, respectively.

- A comparison of the solid and small-dashed (small-dashed and large-dashed) curves demonstrates the expected degrading effect on the accessible information, of increasing the bath exposure time t (increasing T).

- A comparison of the large-dashed and dot-dashed curves demonstrates the dramatic effect of including squeezing. In particular, whereas squeezing improves the accessible information for the pair of input states $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, it is detrimental for input states (θ_0, ϕ_0) given by $(0,0)$ (i.e., $|1\rangle$) and $(\pi, 0)$ (i.e., $|0\rangle$).
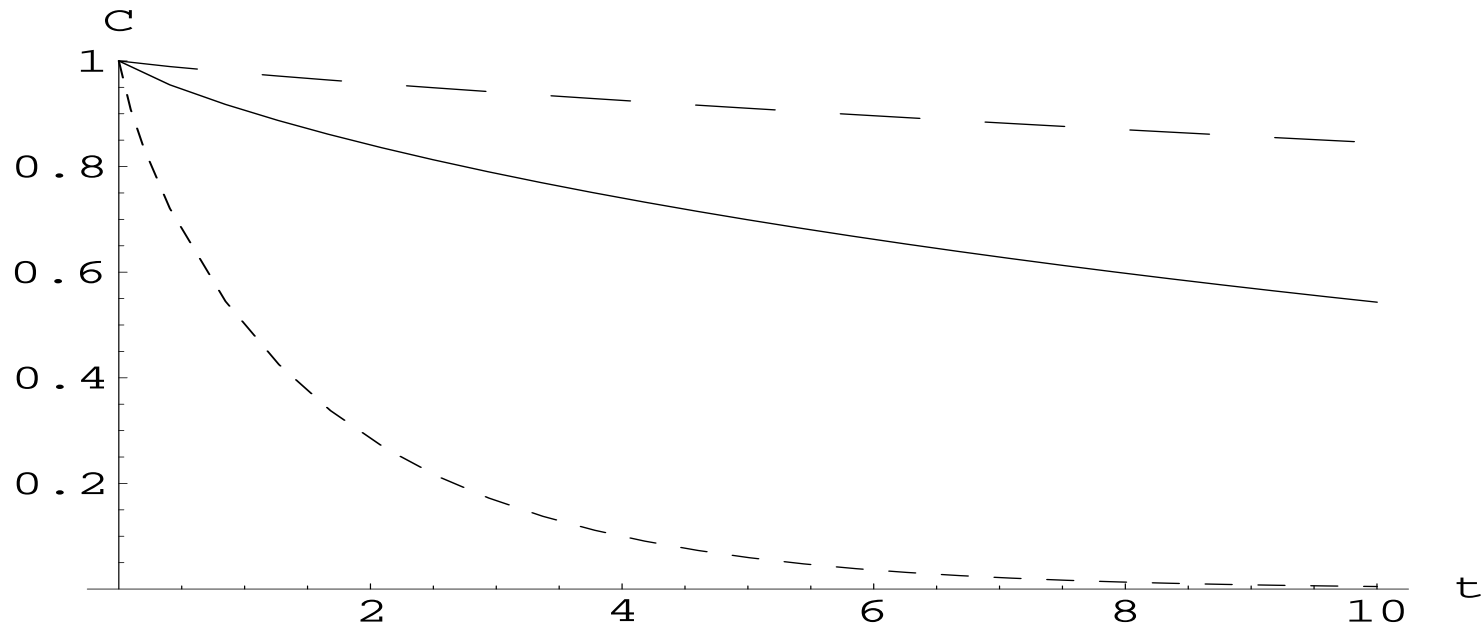
Fig. 4 : Interplay of squeezing and temperature on the classical capacity C of the squeezed amplitude damping channel (with input states $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $f = 1/2$, corresponding to the optimal coding). Here $\Phi = 0$ and $\gamma_0 = 0.05$. The solid and small-dashed curves correspond to zero squeezing r, and temperature $T = 0$ and 5, respectively. The large-dashed curve corresponds to $T = 5$ and $r = 2$. A comparison between the solid and large-dashed curves shows that squeezing can improve C. This highlights the possible usefulness of squeezing to noisy quantum communication.

B. Environment-Mediated Quantum Deleter

(R. Srikanth and SB: (2007))

- Quantum computation is well known to solve certain types of problems more efficiently than classical computation (M. A. Nielsen and I. L. Chuang: (2000)).

- Although quantum mechanical linearity endows a quantum computer with greater-than-classical power (P. Shor: (1995); L. K. Grover: (1997)), it also imposes certain restrictions, such as the prohibition on cloning (W. K. Wooters and W. H. Zurek: (1982)) and on deleting (A. K. Pati and S. Braunstein: (2000)). The latter result means that quantum mechanics does not allow us to delete a copy of an arbitrary quantum state perfectly.

Requirement of Open Quantum System

- A quantum computational task can be broadly divided into three stages:
  - (A). Initializing the quantum computer, by putting all qubits into a standard 'blank state';
  - (B). Executing the unitary operation that performs the actual computation. This is the area where "decoherence" is an obstacle. A variety of techniques, including quantum error correction (S. Calderbank and P. Shor: (1996); A. Steane: (1996)), dynamic decoupling (L. Viola and S. Lloyd: (1998); D. Vitali and P. Tombesi: (2001)) , fault tolerant quantum computation (P. Shor: (1996)), decoherence-free subspaces (D. A. Lidar, I. L. Chuang and K. B. Whaley: (1998)), among others exist to combat decoherence;
  - (C). Performing measurements to read off results.

- In step (A), we must be able to erase quantum memory at the end of a computational task, in order to prepare the state of a quantum computer for a subsequent task. What is required is a quantum mechanism that with high probability allows us to prepare standard 'blank states'. It is clear that no unitary process can achieve this, since true deletion would be irreversible, and hence non-unitary. Further, the no-deleting theorem implies that no qubit state can be erased against a copy (A. K. Pati and S. Braunstein: (2000)).

- A direct method for initializing the quantum computer would be to measure all qubits in the computational basis. This results in a statistical mixture of $|0\rangle$'s and $|1\rangle$'s, and there is no unitary way in a closed system to convert the $|1\rangle$'s while retaining the $|0\rangle$'s. However, open quantum systems, in particular a decohering environment, can effect non-unitary evolution on a sub-system of interest. We are thus led to conclude that decoherence is in fact *necessary* for step (A), since there would be no other way to delete quantum information.

- Here this insight is used to argue that decoherence can be useful to quantum computation. In particular, it is shown that a dissipative environment, the *amplitude-damping channel* in the parlance of quantum information theory, can serve as an effective deleter of quantum information.

Fidelity as a function of Temperature

- Fidelity is defined as

$$
\begin{aligned}
f(t) &= \sqrt{\langle 0|\rho^s(t)|0\rangle} = \sqrt{\frac{1 - \langle \sigma_3(t)\rangle}{2}} \\[2mm]
&= \frac{1}{\sqrt{2}}\left[\left(1 - e^{-\Gamma t}\langle \sigma_3(0)\rangle\right) + \frac{\left(1 - e^{-\Gamma t}\right)}{2N + 1}\right]^{1/2},
\end{aligned}
$$

where $\Gamma \equiv \gamma_0(2N + 1)$ and $\langle \sigma_3(0)\rangle$ is the expectation value of σ_3 at time $t = 0$.
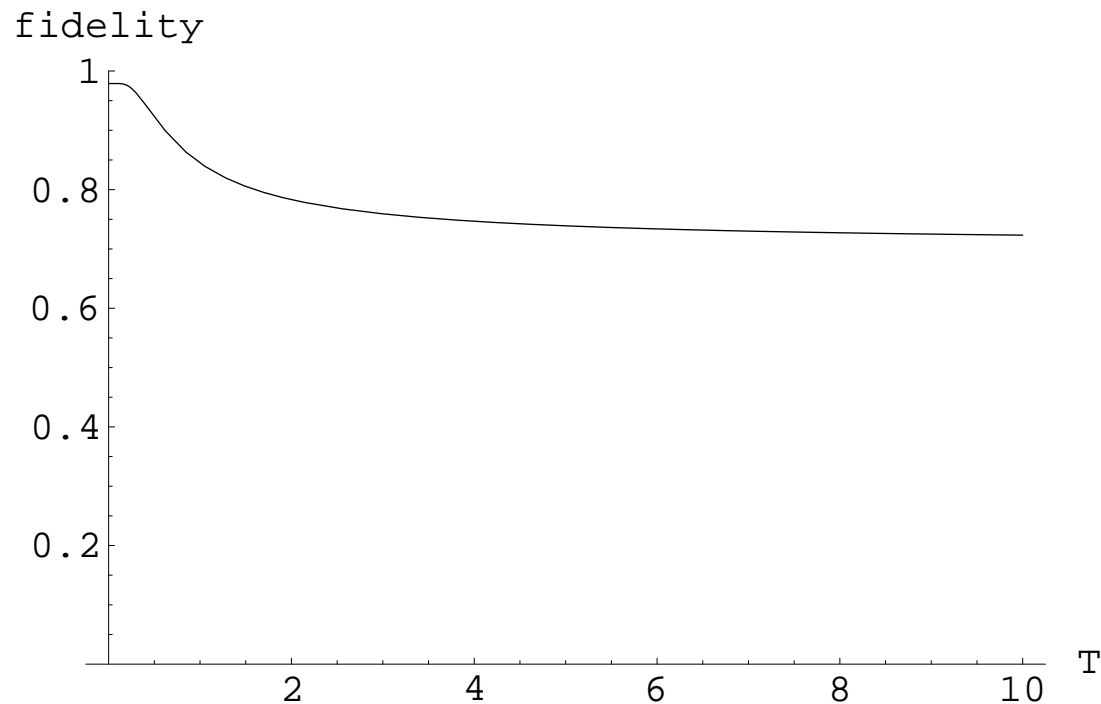
Fig. 5 : Fidelity ($f(t)$) falls as a function of temperature (T, in units where $\hbar \equiv k_B \equiv 1$) until it reaches the value $1/\sqrt{2}$ corresponding to a maximally mixed state. The case shown here corresponds to $\theta_0 = 0$, $\gamma_0 = 0.5$, $\omega = 1.0$ and time $t = 10$. Here we set the squeezing parameters r and Φ to zero.

Non-Markovian Aspects of Open Quantum Systems:Some Features

- We now make a brief excursion into non-Markovian Open Quantum Systems (H-P Breuer (2012)).

- This is a bigger class than the ones discussed till now in these lectures.

- We will illustrate our discussions using a model, introduced by Garraway (PRA 1997), of a two-level system system decaying spontaneously into a vacuum bath.

- Only the RWA approximation is employed, the model is solvable for all system-bath couplings.

Non-Markovian Aspects:Model

- The system Hamiltonian is

$$H_S = \omega_0 \sigma_+ \sigma_-,$$

  describing a two-state system (qubit) with ground state $|0\rangle$, excited state $|1\rangle$ and transition frequency ω_0, where $\sigma_+ = |1\rangle\langle0|$ and $\sigma_- = |0\rangle\langle1|$ are the raising and lowering operators of the qubit.

- The Hamiltonian of the environment is

$$H_R = \sum_k \omega_k b_k^\dagger b_k,$$

  represents a reservoir of harmonic oscillators with creation and annihilation operators $b_k^\dagger$ and b_k. The interaction Hamiltonian takes the form

$$H_{SR} = \sum_k \left(g_k \sigma_+ \otimes b_k + g_k^* \sigma_- \otimes b_k^\dagger \right).$$

- Due to the RWA, the total number of excitations in the system,

$$N = \sigma_+ \sigma_- + \sum_k b_k^\dagger b_k,$$

is a conserved quantity.

Non-Markovian Aspects:Model

- Assuming the environment to be in the vacuum state $|0\rangle$ initially one finds:

$$\begin{aligned}
\rho_{11}(t) &= |c(t)|^2 \rho_{11}(0), \\
\rho_{00}(t) &= \rho_{00}(0) + (1 - |c(t)|^2)\rho_{11}(0), \\
\rho_{10}(t) &= c(t)\rho_{10}(0), \\
\rho_{01}(t) &= c^*(t)\rho_{01}(0),
\end{aligned}$$

where the $\rho_{ij}(t) = \langle i|\rho_S(t)|j\rangle$ denote the matrix elements of $\rho_S(t)$.

- The function $c(t)$ is the solution of the integro-differential equation

$$\frac{d}{dt}c(t) = -\int_0^t dt_1\, f(t - t_1)c(t_1),$$

corresponding to the initial condition $c(0) = 1$,

Non-Markovian Aspects:Model

- where the kernel $f(t - t_1)$ represents a reservoir two-point correlation function,

$$
\begin{aligned}
f(t - t_1) &= \langle 0|R(t)R^\dagger(t_1)|0\rangle e^{i\omega_0(t-t_1)} \\
&= \sum_k |g_k|^2 e^{i(\omega_0 - \omega_k)(t - t_1)},
\end{aligned}
$$

of the environmental/reservoir operators

$$
R(t) = \sum_k g_k b_k e^{-i\omega_k t}.
$$

Non-Markovian Aspects:Model

- These results hold for a generic environmental spectral density and the corresponding two-point correlation function. Taking, for example, a Lorentzian spectral density in resonance with the transition frequency of the qubit we find an exponential two-point correlation function

$$f(\tau) = \frac{1}{2}\gamma_0 \lambda e^{-\lambda|\tau|},$$

  where γ_0 describes the strength of the system-environment coupling and λ the spectral width which is related to the environmental correlation time by $\tau_R = \lambda^{-1}$.

- Using this we find

$$c(t) = e^{-\lambda t/2}\left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right],$$

  where $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$.

Non-Markovian Aspects:Time-Local Master Equations

- A time-local master equation, providing a generalization of the usual Lindbladian type of equation would be of the form

$$\frac{d}{dt}\rho_S(t) = \mathcal{K}(t)\rho_S(t).$$

- The generator $\mathcal{K}(t)$ of the time-local master equation must of course preserve the Hermiticity and the trace.

- From these requirements it follows that the generator must be of the following general form

$$\begin{aligned}
\mathcal{K}(t)\rho_S \quad = \quad & -i\left[H_S(t), \rho_S\right] \\
& + \sum_i \gamma_i(t)\left[A_i(t)\rho_S A_i^\dagger(t) - \frac{1}{2}\left\{A_i^\dagger(t)A_i(t), \rho_S\right\}\right].
\end{aligned}$$

  The structure of the generator provides a natural generalization of the Lindblad structure, in which the Hamiltonian $H_S(t)$, the Lindblad operators $A_i(t)$ as well as the various decay rates $\gamma_i(t)$ may dependent on time.

- When $\gamma_i(t) \geq 0$, the resulting dynamics is completely positive, since the generator is then in Lindblad form for each fixed $t \geq 0$.

Non-Markovian Aspects

- For the Garraway model, the time-local generator takes the form

$$
\mathcal{K}(t)\rho_S \;=\; -\frac{i}{2} S(t)[\sigma_+\sigma_-, \rho_S]
$$

$$
+\gamma(t)\left[\sigma_-\rho_S\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S\}\right],
$$

where $\gamma(t) = -2\Re\left(\frac{\dot{c}(t)}{c(t)}\right)$, $S(t) = -2\Im\left(\frac{\dot{c}(t)}{c(t)}\right)$.

- The quantity $S(t)$ plays the role of a time-dependent frequency shift, and $\gamma(t)$ can be interpreted as a time-dependent decay rate. Due to the time dependence of these quantities the process does not generally represent a dynamical semigroup.

Non-Markovian Aspects: Approach To Markovian Behaviour

- In the limit of small $\alpha = \gamma_0/\lambda$ we may approximate $c(t) \approx e^{-\gamma_0 t/2}$.

- $S(t) = 0$ and $\gamma(t) = \gamma_0$, i.e., the generator $\mathcal{K}(t)$ assumes the form of a Lindblad generator of a quantum dynamical semigroup.

- α can also be written as the ratio of the environmental correlations time $\tau_R = \lambda^{-1}$ and the relaxation time $\tau_{rel} = \gamma_0^{-1}$ of the system $\alpha = \frac{\tau_R}{\tau_{rel}}$.

- Thus we see that the standard Markov condition $\gamma_0 \ll \lambda$ indeed leads to a Markovian semigroup here.

Non-Markovian Aspects: Divisibility of dynamical Maps

- A family of dynamical maps $\Phi(t, 0)$ is defined to be divisible if for all $t_2 \geq t_1 \geq 0$ there exists a CPT map $\Phi(t_2, t_1)$ such that the relation $\Phi(t_2, 0) = \Phi(t_2, t_1)\Phi(t_1, 0)$ holds.

- The simplest example of a divisible quantum process is given by a dynamical semigroup. For a semigroup $\Phi(t, 0) = \exp[\mathcal{L}t]$ and divisibility is satisfied with the CPT map $\Phi(t_2, t_1) = \exp[\mathcal{L}(t_2 - t_1)]$.

- Consider now a quantum process given by the time-local master equation with a time dependent generator. The dynamical maps can then be represented in terms of a time-ordered exponential,

$$\Phi(t, 0) = \mathrm{T} \exp\left[\int_0^t dt' \mathcal{K}(t')\right], \quad t \geq 0,$$

where T denotes the chronological time-ordering operator.

Non-Markovian Aspects: Divisibility of dynamical Maps

- We can also define the maps

$$\Phi(t_2, t_1) = \mathrm{T} \exp \left[\int_{t_1}^{t_2} dt' \mathcal{K}(t') \right], \quad t_2 \geq t_1 \geq 0,$$

such that the composition law $\Phi(t_2, 0) = \Phi(t_2, t_1)\Phi(t_1, 0)$ holds by construction. The maps $\Phi(t_2, t_1)$ are completely positive, as is required by the divisibility condition, if and only if the rates $\gamma_i(t)$ of the generator are positive functions. Thus divisibility is equivalent to positive rates in the time-local master equation (Laine et al. (2010)).

Non-Markovian Aspects: Divisibility of dynamical Maps

- For the Garraway model, the necessary and sufficient condition for the complete positivity of $\Phi(t_2, t_1)$ is given by $|c(t_2)| \leq |c(t_1)|$.

- Thus the dynamical map of the model is divisible if and only if $|c(t)|$ is a monotonically decreasing function of time.

- The rate $\gamma(t)$ can be written as

$$\gamma(t) = -\frac{2}{|c(t)|} \frac{d}{dt} |c(t)|.$$

This shows that any increase of $|c(t)|$ leads to a negative decay rate in the corresponding generator, and illustrates the equivalence of the non-divisibility of the dynamical map and the occurrence of a temporarily negative rate in the time-local master equation.

Non-Markovian Aspects: Information Flow

- Consider two parties, Alice and Bob. Alice prepares a quantum system in one of two states ρ^1 or ρ^2 with probability $\frac{1}{2}$ each, and then sends the system to Bob. It is Bob's task to find out by a single measurement on the system whether the system state was ρ^1 or ρ^2. It turns out that Bob cannot always distinguish the states with certainty, but there is an optimal strategy which allows him to achieve the maximal possible success probability given by

$$
P_{\mathrm{max}} = \frac{1}{2} \left[1 + D(\rho^1, \rho^2) \right].
$$

- The trace distance $D(\rho^1, \rho^2) = \frac{1}{2}||\rho^1 - \rho^2|| = \frac{1}{2}\mathrm{tr}|\rho^1 - \rho^2|$ can therefore be interpreted as a measure for the distinguishability of the quantum states ρ^1 and ρ^2. Here $\mathrm{tr}|A| = \mathrm{tr}\sqrt{A^\dagger A}$.

Non-Markovian Aspects: Properties of Trace Distance

- The trace distance between any pair of states satisfies $0 \leq D(\rho^1, \rho^2) \leq 1$.

- The trace distance is sub-additive with respect to tensor products of states

$$D(\rho^1 \otimes \sigma^1, \rho^2 \otimes \sigma^2) \leq D(\rho^1, \rho^2) + D(\sigma^1, \sigma^2).$$

- The trace distance is invariant under unitary transformations U,

$$D(U\rho^1 U^\dagger, U\rho^2 U^\dagger) = D(\rho^1, \rho^2).$$

More generally, all trace preserving and completely positive maps, i.e., all trace preserving quantum operations Λ are contractions of the trace distance,

$$D(\Lambda\rho^1, \Lambda\rho^2) \leq D(\rho^1, \rho^2).$$

Non-Markovian Aspects

- No quantum process describable by a family of CPT dynamical maps can ever increase the distinguishability of a pair of states over its initial value.

- When a quantum process reduces the distinguishability of states, information is flowing from the system to the environment. Correspondingly, an increase of the distinguishability signifies that information flows from the environment back to the system.

- The definition for quantum non-Markovianity, discussed here, is based on the idea that for Markovian processes any two quantum states become less and less distinguishable under the dynamics, leading to a perpetual loss of information into the environment.

- Quantum memory effect thus arise if there is a temporal flow of information from the environment to the system. The information flowing back from the environment allows the earlier open system states to have an effect on the later dynamics of the system, which implies the emergence of memory effects (Breuer et al. (2009)).

Non-Markovian Aspects: Measure

- A quantum process described in terms of a family of quantum dynamical maps $\Phi(t,0)$ is non-Markovian if there is a pair of initial states $\rho_S^{1,2}(0)$ such that the trace distance between the corresponding states $\rho_S^{1,2}(t)$ increases at a certain time $t > 0$:

$$\sigma(t, \rho_S^{1,2}(0)) \equiv \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) > 0,$$

where $\sigma(t, \rho_S^{1,2}(0))$ denotes the rate of change of the trace distance at time t corresponding to the initial pair of states.

- This implies that all divisible families of dynamical maps are Markovian, including the class of quantum dynamical semigroups.

Non-Markovian Aspects: Measure

- To prove this statement suppose that $\Phi(t,0)$ is divisible. For any pair of initial states $\rho_S^{1,2}(0)$ we then have

$$\rho_S^{1,2}(t+\tau) = \Phi(t+\tau,t)\rho_S^{1,2}(t), \quad t,\tau \geq 0.$$

Since $\Phi(t+\tau,t)$ is a CPT map we can apply the contraction property to obtain:

$$D(\rho_S^1(t+\tau), \rho_S^2(t+\tau)) \leq D(\rho_S^1(t), \rho_S^2(t)).$$

This shows that for all divisible dynamical maps the trace distance decreases monotonically and that, therefore, these processes are Markovian.

- Thus non-Markovian quantum processes must necessarily be described by non-divisible dynamical maps and by time-local master equations whose generator involves at least one temporarily negative rate $\gamma_i(t)$.

Non-Markovian Aspects: Measure

- This suggests defining a measure $\mathcal{N}(\Phi)$ for the non-Markovianity of a quantum process through (Breuer et al. (2009))

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma > 0} dt\, \sigma(t, \rho_S^{1,2}(0)).$$

- The time integration is extended over all time intervals (a_i, b_i) in which σ is positive and the maximum is taken over all pairs of initial states. The measure can be written as

$$\mathcal{N}(\Phi) = \max_{\rho^{1,2}(0)} \sum_i \left[D(\rho_S^1(b_i), \rho_S^2(b_i)) - D(\rho_S^1(a_i), \rho_S^2(a_i)) \right].$$

To calculate this quantity one first determines for any pair of initial states the total growth of the trace distance over each time interval (a_i, b_i) and sums up the contribution of all intervals. $\mathcal{N}(\Phi)$ is then obtained by determining the maximum over all pairs of initial states.

Non-Markovian Aspects: Measure: An Illustration

- Considering the Garraway model, the time evolution of the trace distance corresponding to any pair of initial states $\rho_S^1(0)$ and $\rho_S^2(0)$ is given by

$$D(\rho_S^1(t), \rho_S^2(t)) = |c(t)|\sqrt{|c(t)|^2 a^2 + |b|^2},$$

where $a = \rho_{11}^1(0) - \rho_{11}^2(0)$ and $b = \rho_{10}^1(0) - \rho_{10}^2(0)$.

- The time derivative of this expression yields

$$\sigma(t, \rho_S^{1,2}(0)) = \frac{2|c(t)|^2 a^2 + |b|^2}{\sqrt{|c(t)|^2 a^2 + |b|^2}} \frac{d}{dt}|c(t)|.$$

- From this we conclude that the trace distance increases at time t if and only if the function $|c(t)|$ increases at this point of time. It follows that the process is non-Markovian, $\mathcal{N}(\Phi) > 0$, if and only if the dynamical map is non-divisible, which in turn is equivalent to a temporarily negative rate $\gamma(t)$.

Conclusion

- We have discussed the basic ideas of Open Quantum Systems from a broad perspective, both Markovian as well as non-Markovian evolutions.

- We have motivated the need for a study of Open Quantum Systems in Quantum Optics and Information, as well as introduced some useful techniques in this context.

- The semigroup Lindblad evolution was discussed in some detail.

- The physical assumptions underlying the methods were discussed and elucidated by applying each of them to an open system model.

- Some physical examples, based on the Lindbladian evolution, were also elaborated.

- We then talked about some aspects of non-Markovian behaviour in Open Quantum Systems.