A METHOD TO TEST THE BINARY BLACK HOLE NATURE OF COMPACT BINARY COALESCENCE

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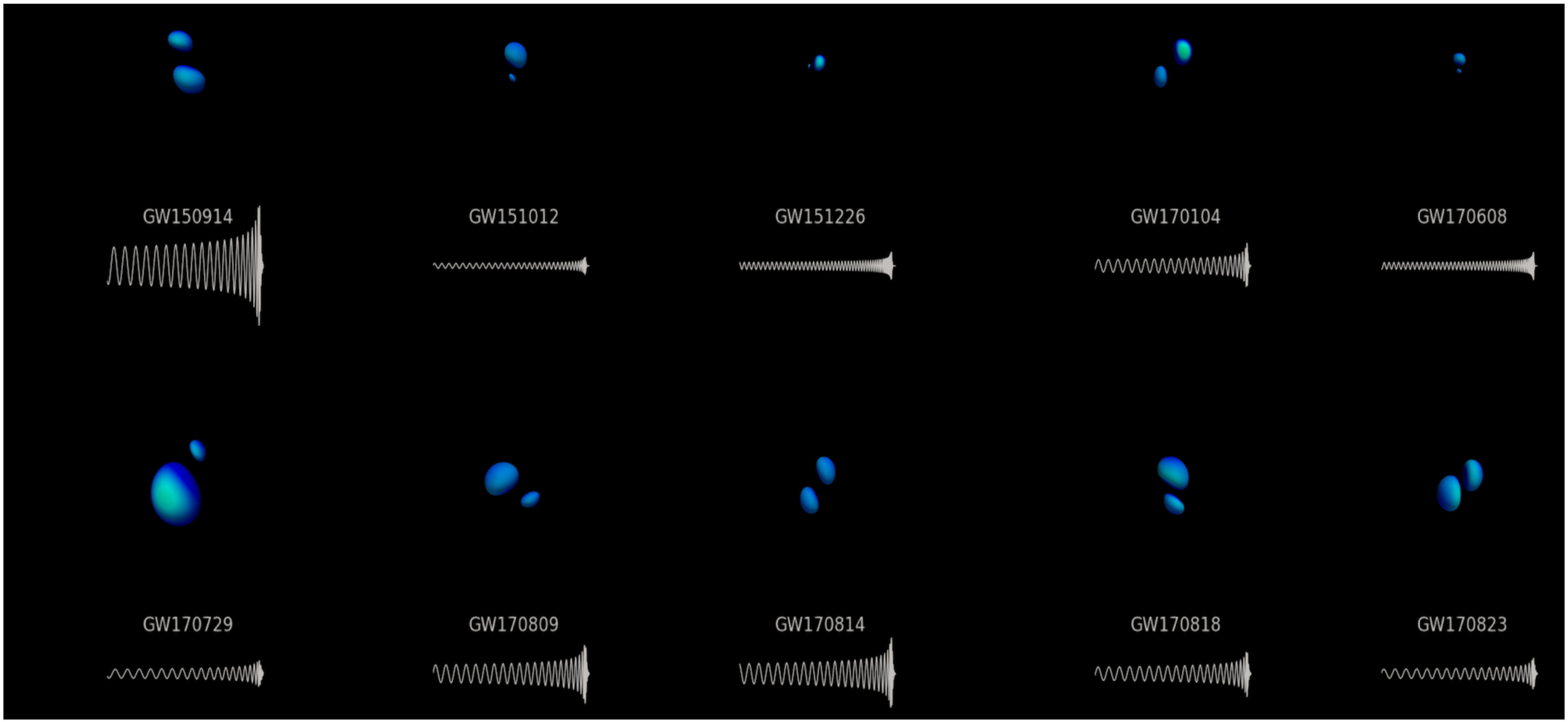
ERA OF GRAVITATIONAL WAVE ASTRONOMY!



https://www.ligo.caltech.edu/

Firm detections of ten binary black holes and a binary neutron star merger

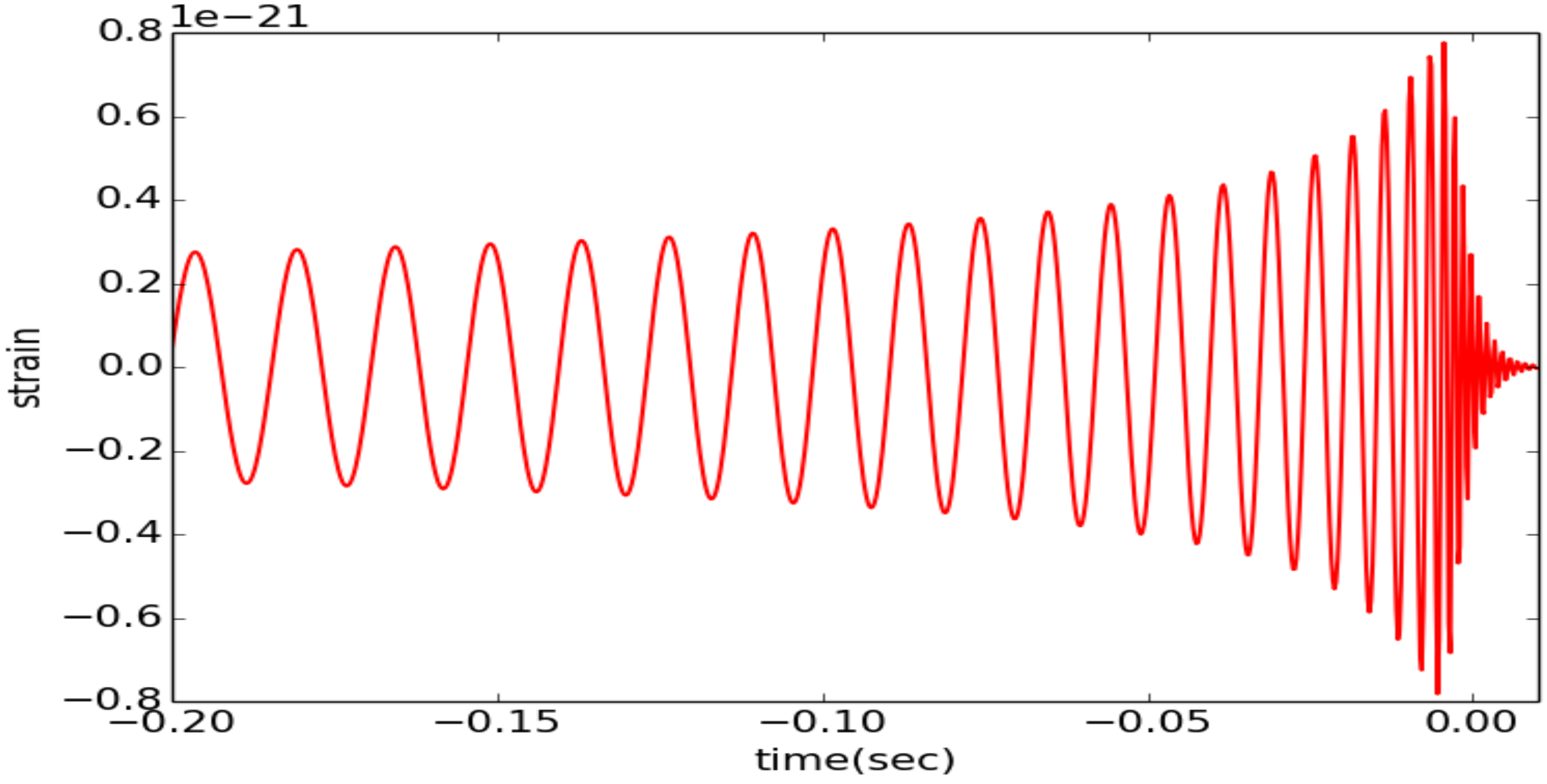
BINARY BLACKHOLE DYNAMICS

Inspiral

(Post-Newtonian theory)

Merger

(Numerical relativity)



Ringdown

(Black hole perturbation theory)

MULTIPOLE MOMENTS OF A KERR BLACKHOLE

- The multipole moments of a Kerr blackhole are completely described by its mass and spin: [Hansen 1974; Carter 1971]

Current type moments

$$Q_l + i\, S_l = M^{l+1}\, (i\, \chi)^l$$

Mass type moments

where $|\vec{\chi}| = \frac{|\vec{S}|}{M^2}$,

M is the total mass

$\vec{\chi}$ is the spin parameter

$\vec{S}$ is the spin angular momentum vector

- Deformations due to the spin of a compact object lead to ***spin-induced multipole moments*** which are unique for a Kerrs blackhole given mass and spin.

Spin-induced multipole moments

- Leading order effect is the spin-induced quadrupole moment,

$$Q_2 = -\kappa\,\chi^2\,M^3,\ \kappa_{\rm BH} = 1 \qquad |\vec{\chi}| = \frac{|\vec{S}|}{M^2}$$

- Next higher order effect is the spin-induced octupole moment,

$$S_3 = -\lambda\,\chi^3\,M^4,\ \lambda_{\rm BH} = 1$$

Neutronstars	$\kappa \sim 2 - 14$ $\lambda \sim 4 - 30$ [Laarakkers 1997; Pappas 2012]
Bosonstars	$\kappa \sim 10 - 150$ $\lambda \sim 10 - 200$ [Ryan; 1997]

For a non BH compact object the value depends on the Equation of State.

We can measure the **spin-induced multipole moment parameters** to test the black hole nature of the compact object

Alternative: Measure the tidal deformability parameter to distinguish black holes from black hole mimickers

ICTS Group

SPIN-INDUCED MULTIPOLE MOMENTS IN POST NEWTONIAN WAVEFORMS

- Schematic representation of frequency domain gravitational waveform,

$$\tilde{h}(\theta^j,\ f) = \mathcal{A}(\theta^j,\ f)\,\mathrm{Exp}[i\,\psi(\theta^j,\ f)]$$

$$\mathcal{A}(\theta^j,\ f) \propto \frac{\mathcal{M}_c^{5/6}}{D_L}$$

$$\psi(\theta^j,\ f) = \frac{3}{128\,\eta\,v^5}\sum_{\alpha=0}^{N}\Psi_\alpha\,v^\alpha$$

$$v = (\pi\,M\,f)^{\frac{1}{3}}$$

v is the PN expansion parameter, $v^2 \sim 1\,\mathrm{PN}$

θ^j includes masses, spins, source location and orientation

SPIN-INDUCED MULTIPOLE MOMENTS IN POST-NEWTONIAN WAVEFORMS

- 2 PN phasing coefficient with spin-induced effects, $\Psi_4^{\rm spin} = \Psi_4^{\rm SS} + \Psi_4^{\rm qm}$, where,

[Poisson; 1998]

$$\Psi_4^{\rm qm} = -\frac{5}{2} \sum_{A=1,2} \frac{Q_A}{m_A M^2} \left(3\left(\hat{L}.s_A\right)^2 - 1\right),$$ where $s_A = \chi_A \, m_A^2$

- Quadrupole moment scalar, $Q_A = -\kappa_A \, \chi_A^2 \, M_A^3$

For Kerr black holes $\kappa_A = 1$

TESTING THE BINARY BLACKHOLE NATURE OF A COMPACT BINARY SYSTEM

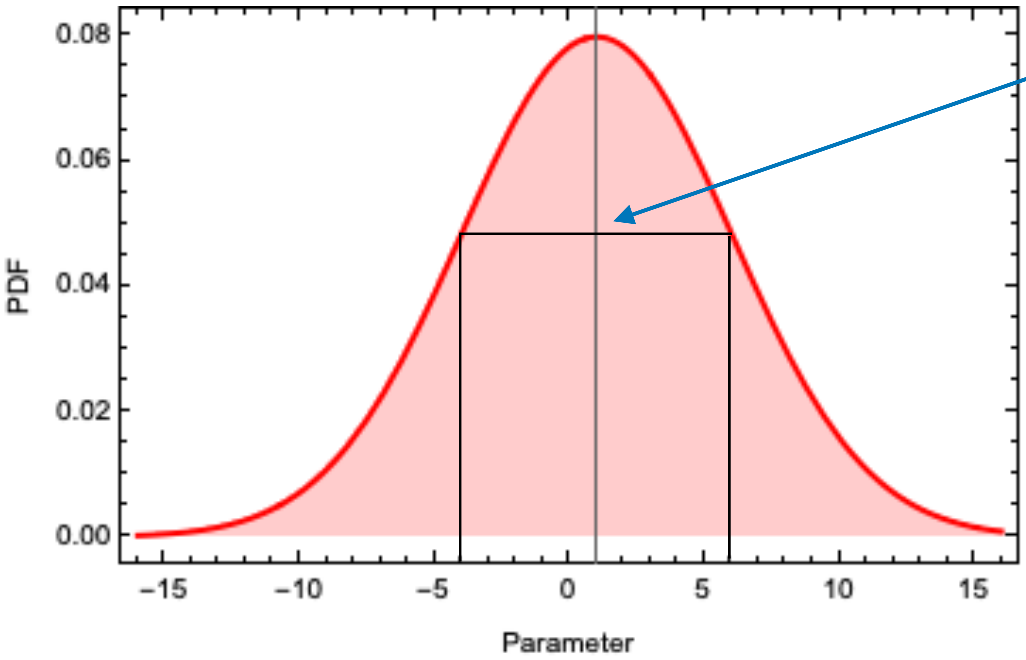
Formulating the parameter estimation problem

Define,

$$\kappa_s = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$\kappa_a = \frac{1}{2}(\kappa_1 - \kappa_2)$$

→ For binary black holes $\kappa_s = 1,\ \kappa_a = 0$ → Measure κ_s keeping $\kappa_a = 0$



Error bar at a particular confidence interval

Error bar at a fixed confidence interval can be interpreted as possible upper bound on the parameter value

How accurately can we measure κ_s from the observed gravitational wave signal and thereby constrain the black hole mimicker parameter space?

Formulating the parameter estimation problem

We measure the spin-induced quadrupole moment parameter using,

1. **Fisher information matrix analysis**:
    - This semi-analytical parameter estimation technique provides the 1-sigma error bars on the spin-induced multipole moment parameter for a given waveform family and detector sensitivity.
2. **Bayesian inference**:
    - We obtain the posterior distributions on the spin-induced multipole moment parameters using Bayesian inference

TESTING THE BINARY BLACKHOLE NATURE OF A COMPACT BINARY SYSTEM

Parameter estimation using Fisher information matrix

- We describe a binary black hole system using 9 parameters

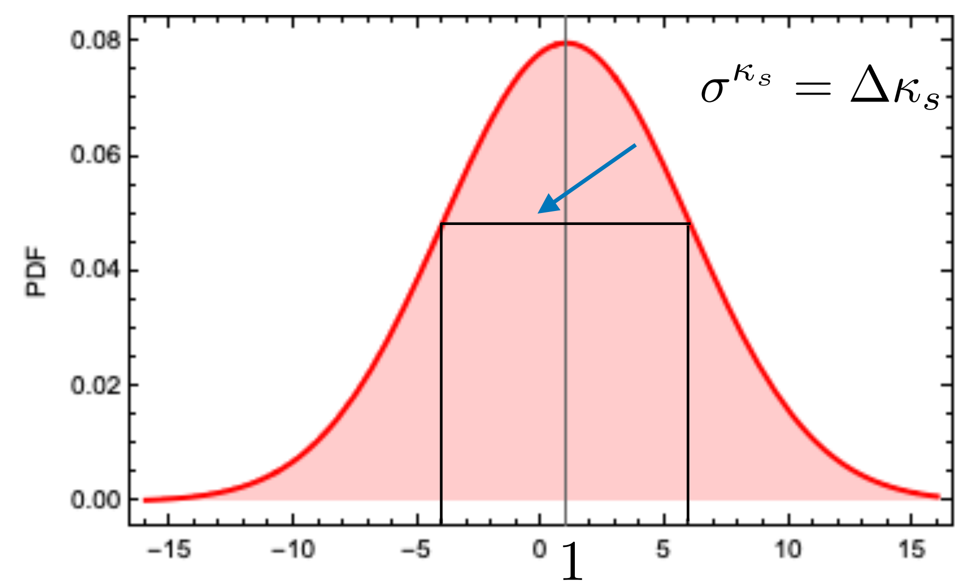
$$\theta^j = \{\kappa_s,\, \mathcal{M}_c,\, \delta,\, \chi_1,\, \chi_2,\, t_c,\, \phi_c,\, \cos\iota,\, \mathcal{D}_\mathrm{L}\}$$

- If we assume high signal to noise ratio and the noise in the detector is Gaussian-stationary, then the errors in estimation of each parameter follow a Gaussian distribution,

$$P(\Delta\theta^j) \sim e^{-\frac{1}{2}\Gamma_{kl}\theta^k\theta^l}$$ where,

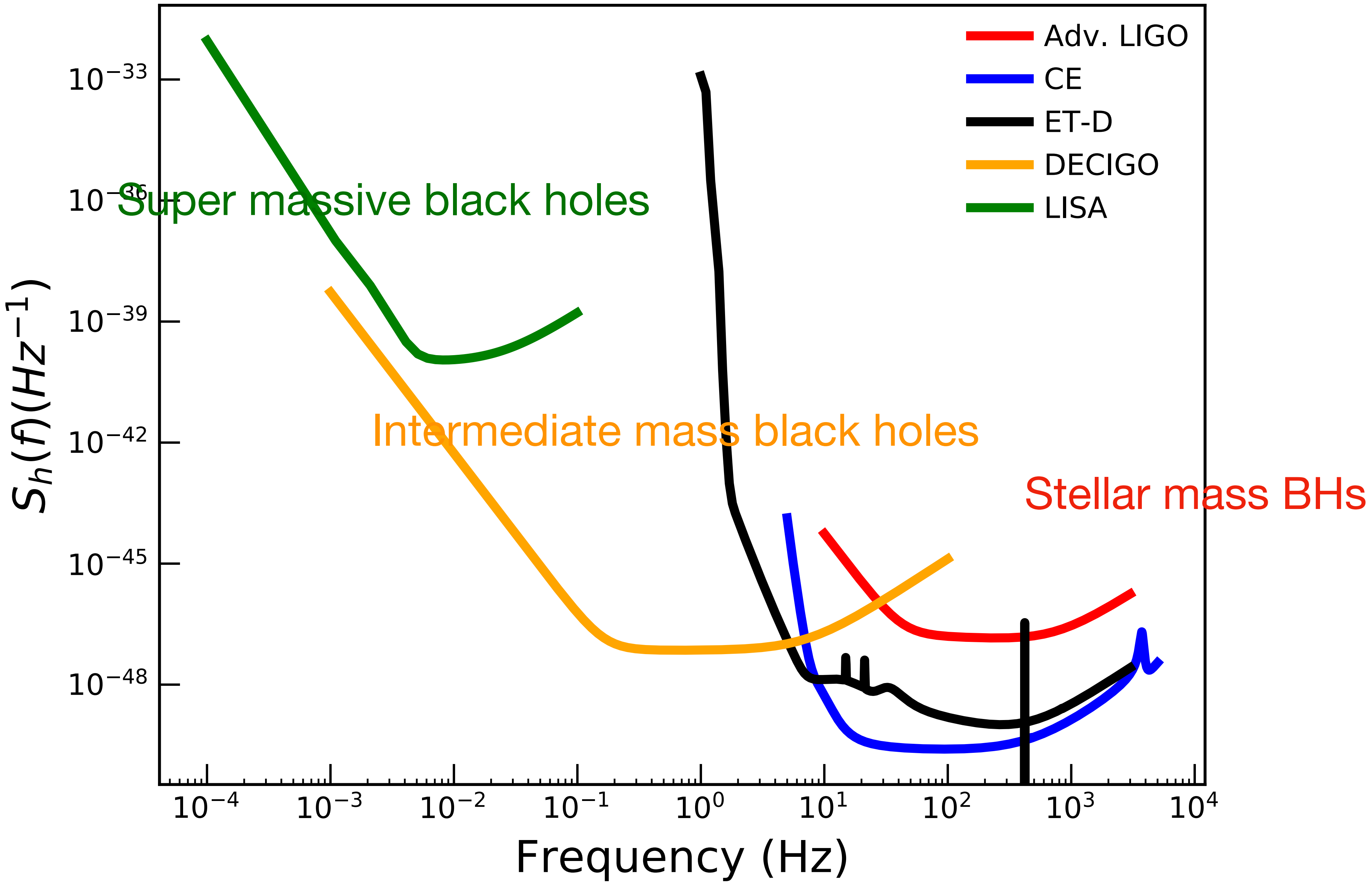
$$\mathbf{\Gamma}_{ij} = 2\int_{f_{low}}^{f_{up}} df\, \frac{\tilde{h}_i \tilde{h}_j^* + \tilde{h}_j \tilde{h}_i^*}{S_n(f)},\; h^i = \frac{\partial}{\partial\theta^i}\tilde{h}(f,\,\theta^i)$$

is the Fisher information matrix



- $1-\sigma$ **error bars on each parameter** is given by, $\sigma^j = \sqrt{(\Gamma^{-1})^{jj}}$

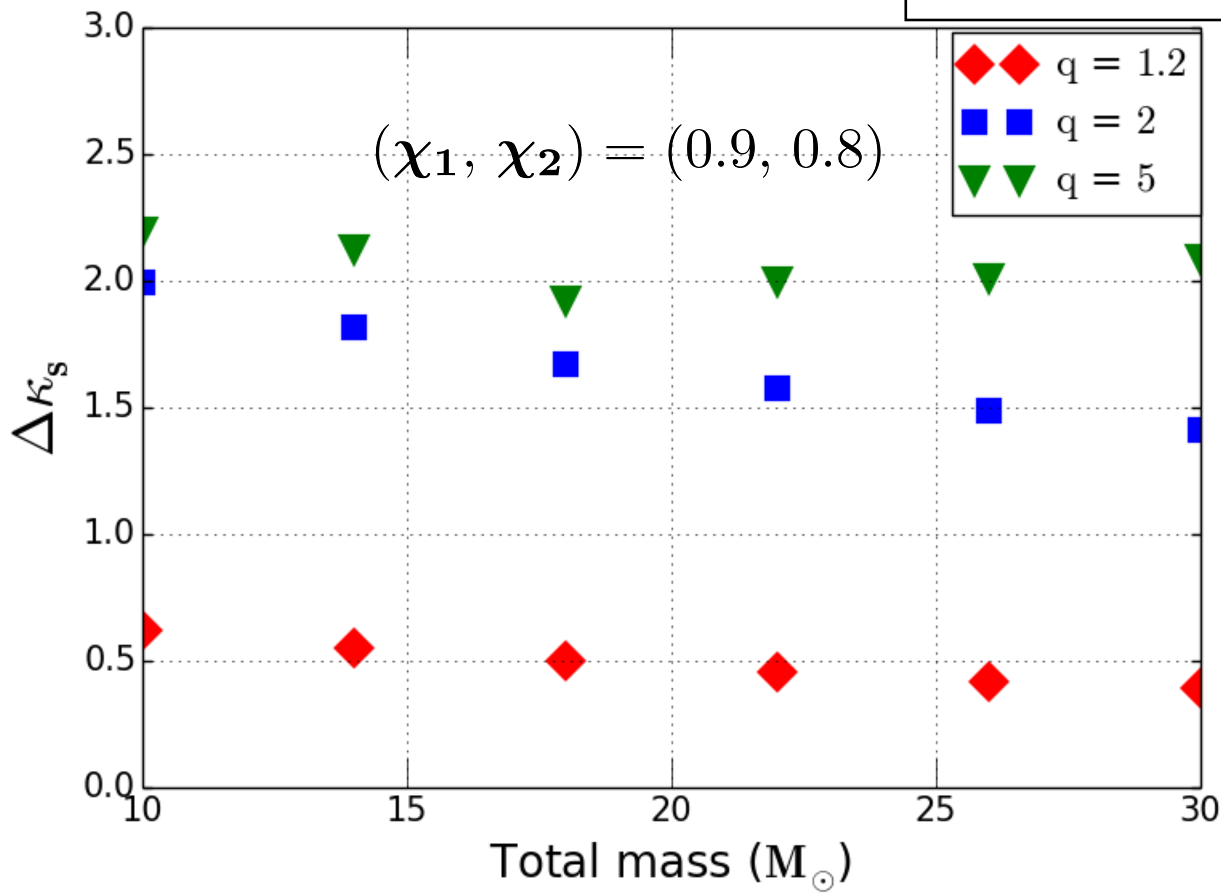
DETECTOR SENSITIVITIES

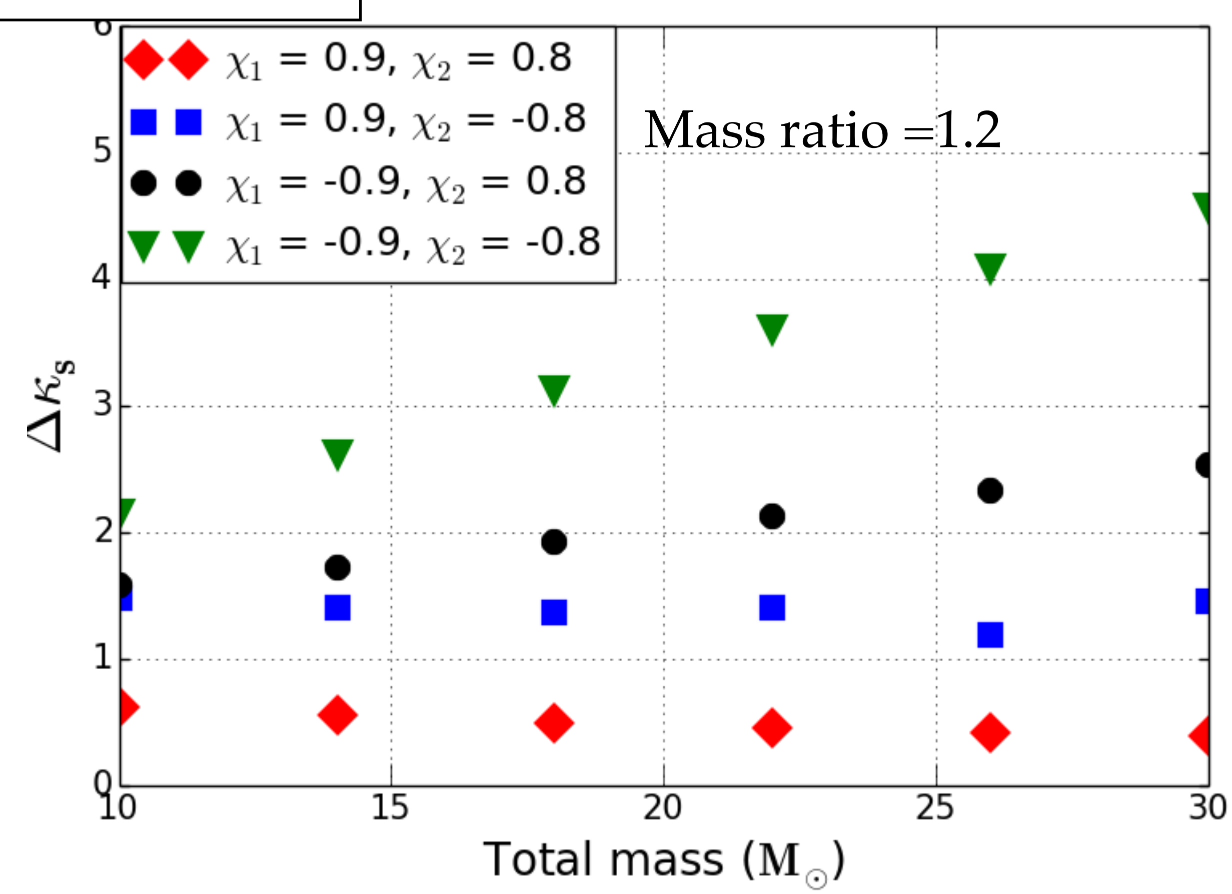


RESULTS

Advanced LIGO design sensitivity

Signal to noise ratio is 10





- Nearly equal mass systems with aligned-spin orientation give better constraints

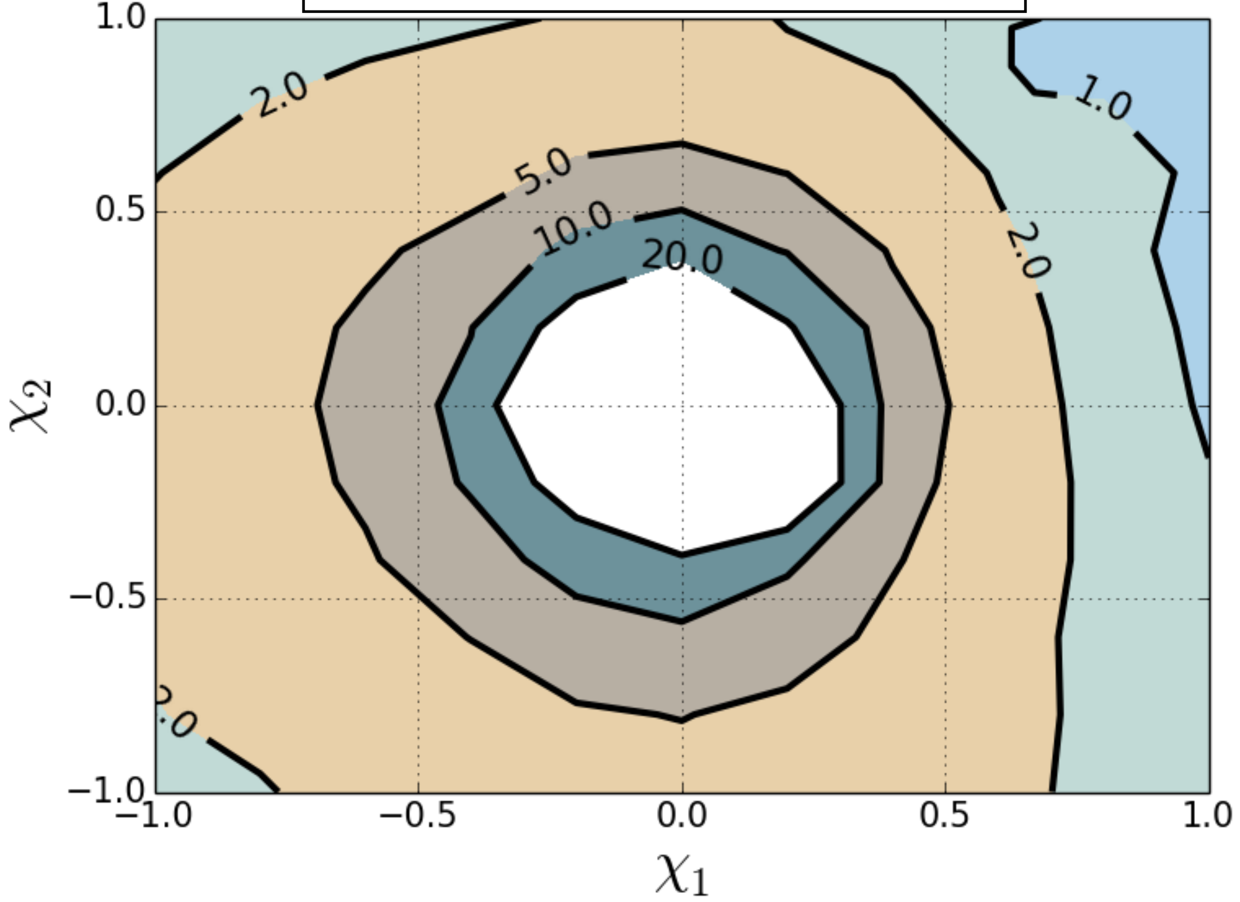
[N. V. Krishnendu, K. G. Arun, C. K. Mishra, PRL 119, 091101, 2017]

RESULTS

Advanced LIGO design sensitivity

- Iso-error contours in the (χ_1, χ_2) plane for binary with masses $(\mathbf{m_1}, \mathbf{m_2}) = (5, 4)\ \mathbf{M}_\odot$

Signal to noise ratio is 10

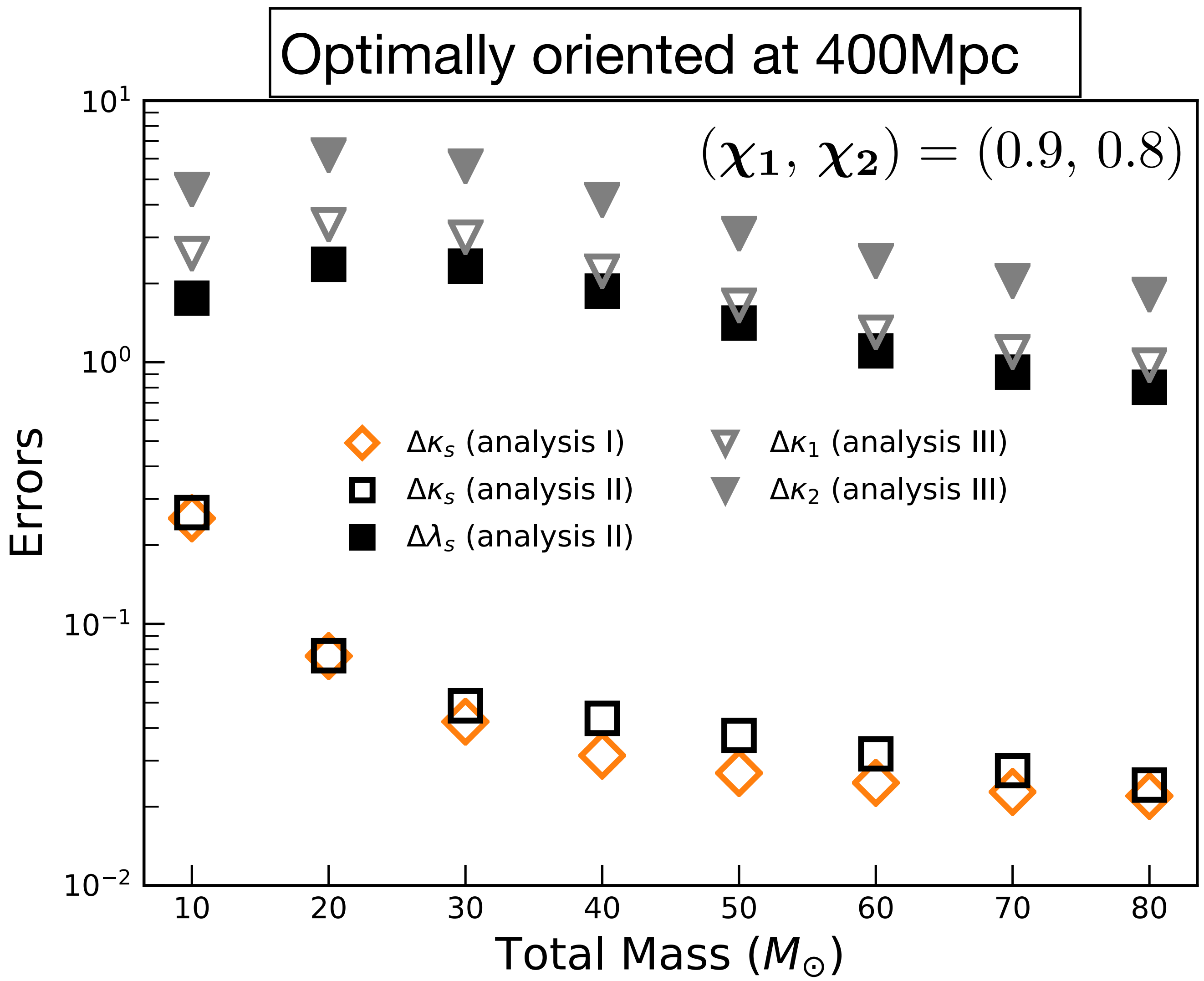


- Outermost contour corresponds to the least error region

[N. V. Krishnendu, K. G. Arun, C. K. Mishra, PRL 119, 091101, 2017]

RESULTS

Tests of binary black hole nature using **third-generation** detectors

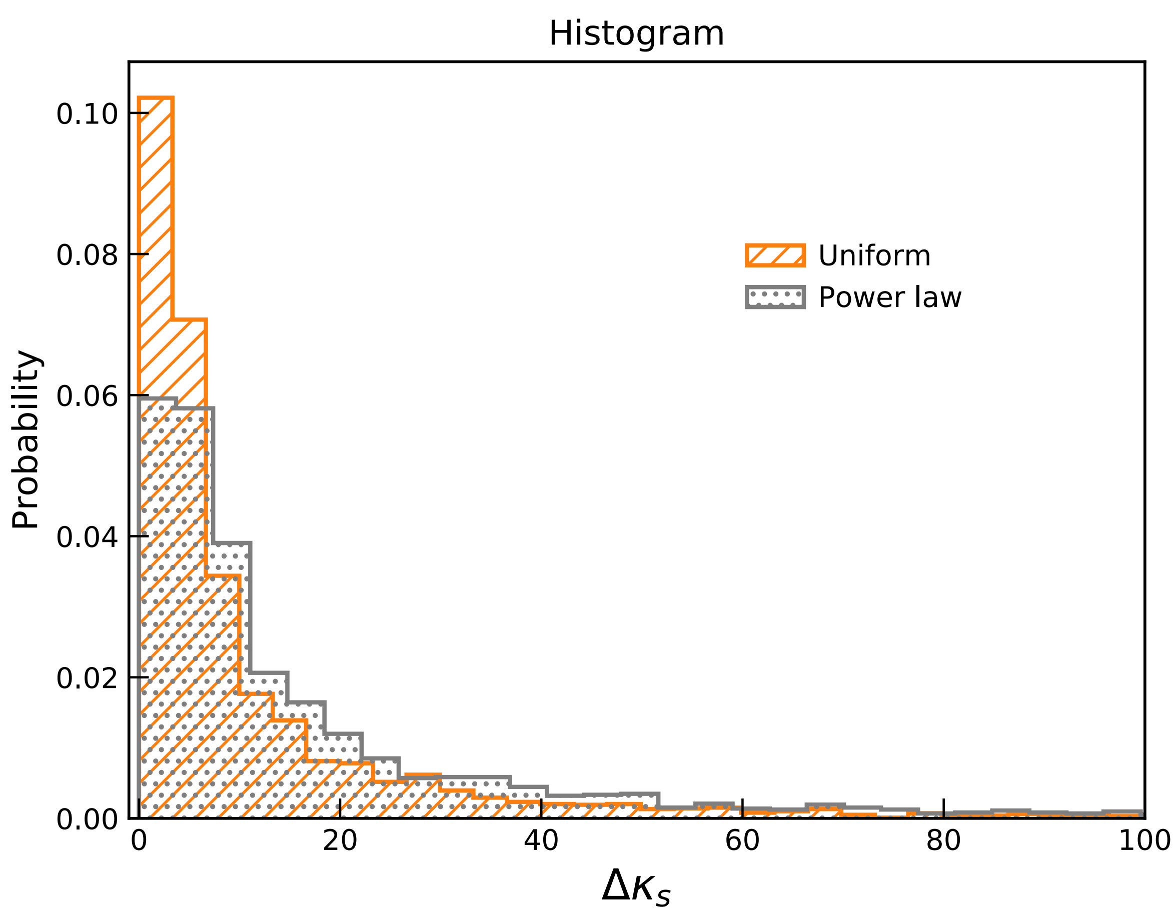


- Analysis I: Measure only κ_s
- Analysis II: Measure both κ_s and λ_s
- Analysis III: Measure κ_1 and κ_2

[Krishnendu, C. K. Mishra, K. G. Arun, arXiv:1811.00317 (under review, PRD)]

RESULTS

Astrophysical population of binary systems detectable by third-generation detectors



	M1	M2
$P(\Delta\kappa_s < 0.5)$	1.9%	1.3%
$P(\Delta\kappa_s < 5)$	23%	46%
$P(\Delta\kappa_s < 10)$	53%	68%

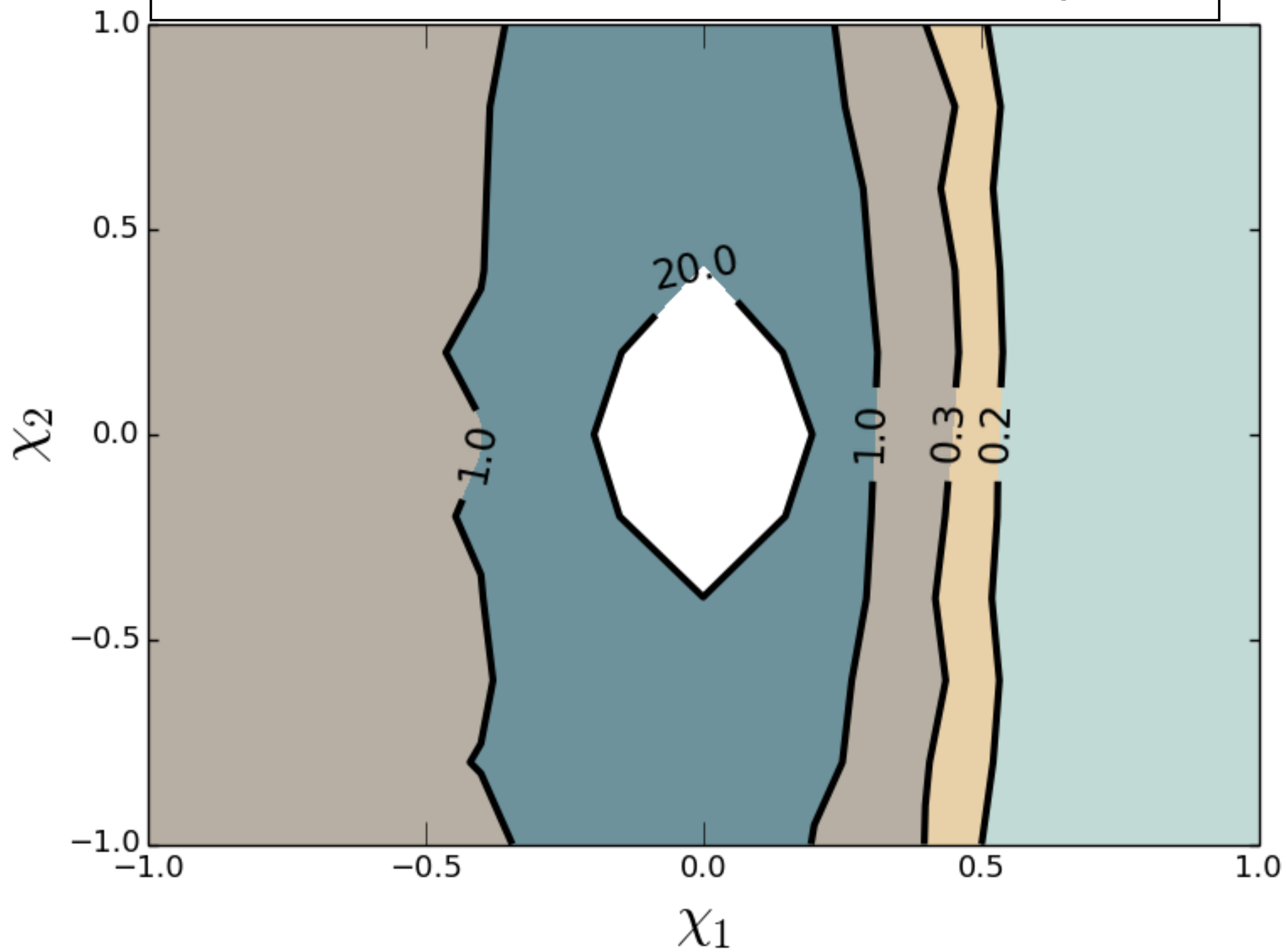
- Sources distributed with constant co-moving number density upto a redshift of 1 with isotropic orientations
- We consider two models:
  1. Component masses from uniform distribution (M1)
  2. Primary mass from power-law distribution, index $\alpha = 2.3$ (M2)

[Krishnendu, C. K. Mishra, K. G. Arun, arXiv:1811.00317]

RESULTS

Errors on spin-induced quadrupole moment parameter, using LISA

Source is assumed to be located at 3Gpc



$$m_1 = 5 \times 10^6 M_\odot,\ m_2 = 10^6 M_\odot$$

A detailed study using space-based detectors is in progress

[N. V. Krishnendu, K. G. Arun, C. K. Mishra, PRL 119, 091101, 2017]

Bayesian implementation is needed for a more robust analysis and the method can be applied to real gravitational wave signals

TESTING THE BINARY BLACKHOLE NATURE OF A COMPACT BINARY SYSTEM - BAYESIAN FRAME WORK

Bayesian implementation: using LALInference

- **Parameter estimation** using Bayesian inference

Posterior probability distribution

Likelihood distribution

Prior distribution

$$P(\theta^j|\mathcal{H},d) = \frac{P(d|\mathcal{H},\theta^j)\,P(\theta^j|\mathcal{H})}{P(d|\mathcal{H})}$$

Evidence

- Compare the evidence between two models: **Bayes factor**

$$\mathcal{B}_2^1 = \frac{P(d|\mathcal{H}_1)}{P(d|\mathcal{H}_2)}$$

- **LALInference** is a toolkit which performs parameter estimation and model selection using Bayesian inference

LALInference studies: Simulation details

- We define the deviation parameter as,

$$\kappa_{\mathrm{A}} = 1 + \delta\kappa_{\mathrm{A}}$$

For black holes, $\delta\kappa_{\mathrm{A}} = 0$ and for binary black holes, $\delta\kappa_{\mathrm{s}} = 0$

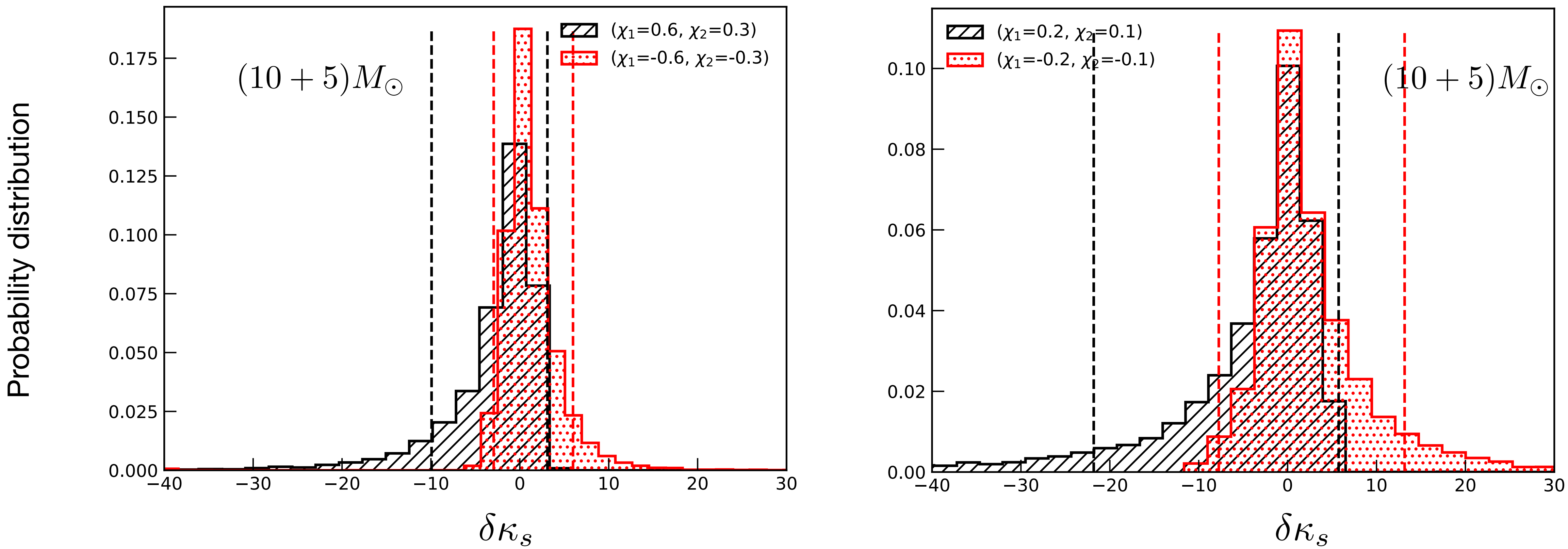
- Posterior distributions on $\delta\kappa_{\mathrm{s}}$ is obtained for simulated signals of binary black holes of different masses and spins

- The waveform model used for this study is IMRPhenomPv2 which incorporates spin-induced quadrupole moment terms upto 3PN order in the phase

[P. Ajith+, S. Khan+]

RESULTS

Posterior distributions on the spin-induced quadrupole moment coefficient

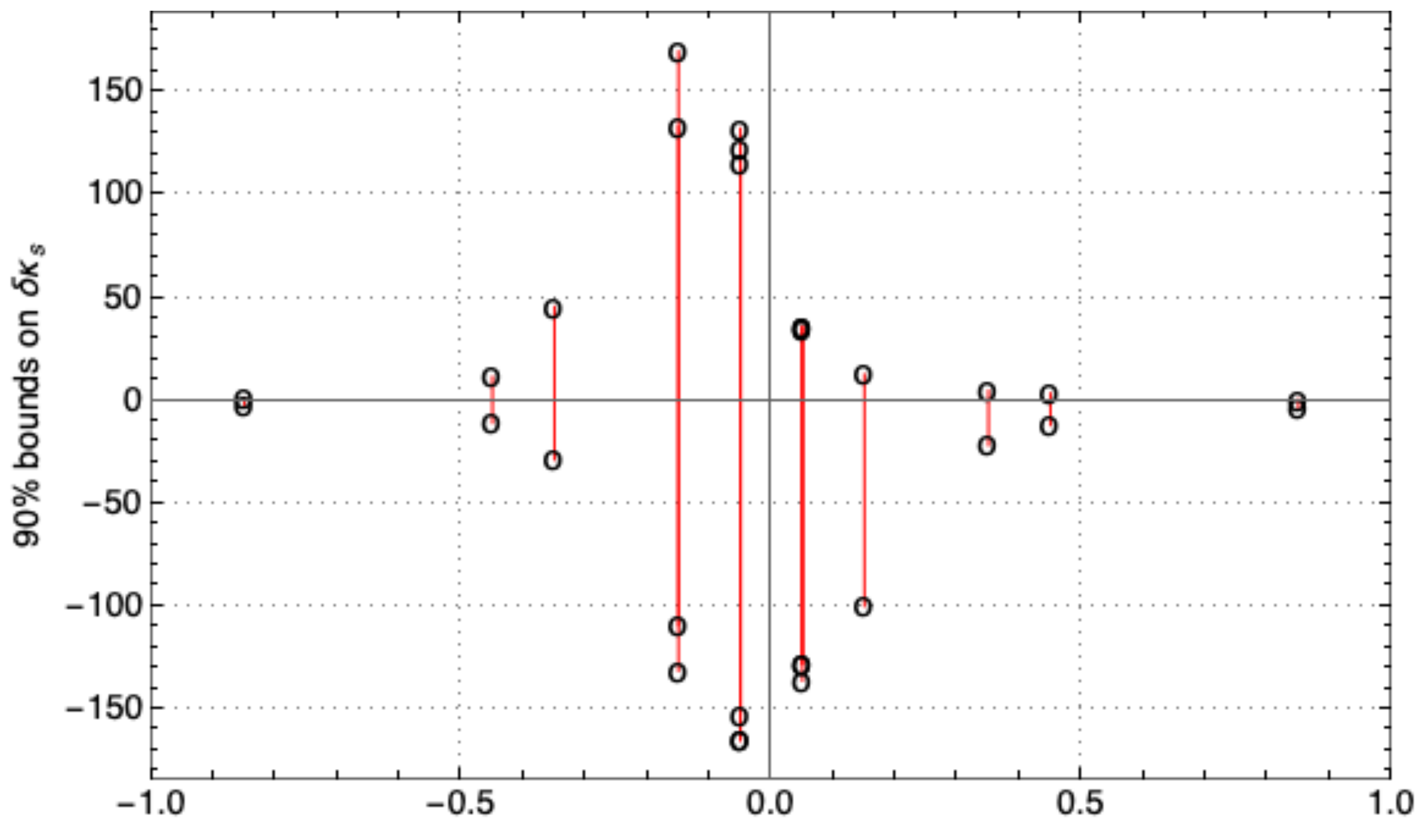


- The constraints we get are largely correlated with the effective spin parameter, which is a combination of masses and spins of the system
- Positive (negative) prior region is well constrained if both black holes are aligned (anti-aligned) to the orbital angular momentum axis

[N. V. Krishnendu +, in preparation]

RESULTS

Bounds as a function of effective spin parameter



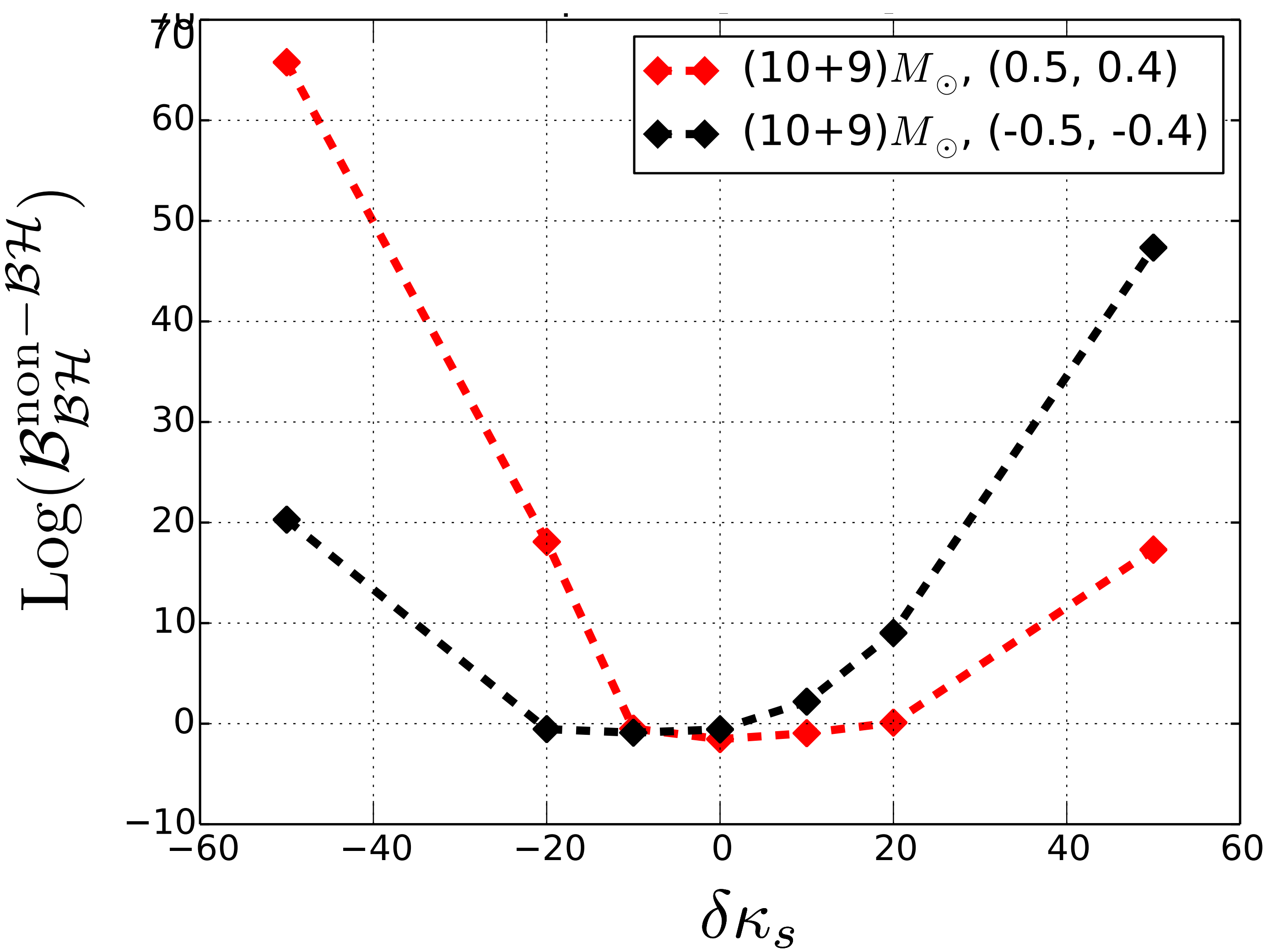
$$\chi_{\rm eff} = \frac{m_1\,\chi_1 + m_2\,\chi_2}{m_1 + m_2}$$

- Bounds appear to be dependent on effective spin parameter rather than masses and spins
- Bounds improve with respect to effective spin parameter
- Positive (negative) prior region is well constrained if $\chi_{\rm eff} > 0 (\chi_{\rm eff} < 0)$

[N. V. Krishnendu +, in preparation]

RESULTS

Detectability of BH mimickers



- There is an asymmetry with respect to $\delta\kappa_s = 0$ line. This reverses as the spin orientation reverses
- This asymmetry is caused by the degeneracy between spin parameters and the spin-induced quadrupole moment parameters.
- Bayes factor shows a preference towards non-black hole model as the injected value of $\delta\kappa_s$ increases

CONCLUSIONS

- We propose a new way to test the black hole nature of a compact object by measuring the spin-induced multipole moment coefficients.
- Using Fisher matrix we demonstrated that this method is a promising one for a wide range binary black hole systems.
- We find that the mass, spin, spin-induced quadrupole and octupole moment coefficients can be simultaneously measured with third-generation gravitational wave detectors.
- This method has been implemented in a more general framework using Bayesian inference, thus it can be used infer black hole nature of real gravitational wave signals.

FUTURE PLANS AND OTHER ONGOING STUDIES

- Ongoing works:
    - Study and quantify the systematic bias due to the ignorance of spin-induced quadrupole moment parameters in parameter estimation.
    - Apply the framework on real events and deduce constraints on black-hole mimicker models from the observed signals.
- Obtain possible constraints on model parameters of certain classes of boson stars, for example Ryan et. al. 1997, using gravitational wave measurements of spin-induced multipole moment parameters.
- Develop a method to distinguish low mass binary black holes from binaries composed of neutron stars and black holes

FUTURE PLANS AND OTHER ONGOING STUDIES

- We aim to systematically study systems such as Kerr black holes with scalar hair and identify the limits where such models show significant deviations from Kerr black holes, especially the values of spin-induced quadrupole moment parameter.

- Other interests:
  - I plan to be involved in the parameter estimation studies of gravitational-wave data from future observing runs.

ADDITIONAL SLIDES

Other ways of probing the BH nature

- By comparing with numerical relativity waveforms
    - Complexity increases for objects having detailed structure
- Measure the tidal deformability parameter and ensure that the value is zero
    - Relatively higher PN order (5 PN) effect [Cardoso et. al., 2016]
- Methods based on No Hair Conjecture
    - Post merger ringdown analysis [Berti et. al., 2009]
    - Inspiral-merger-ringdown consistency test [Abhirup et. al., 2016]
    - Electromagnetic observations
- Dissipation of energy and angular momentum at BH event horizon: Tidal heating

Post-Newtonian waveform with spin-induced quadrupole moment terms (for slide 7)

$$V_k = (2\,\pi\,M\,\frac{f_{gw}}{k})^{\frac{1}{3}}$$

$$\tilde{h}(f) = \frac{M^2}{D_L}\sqrt{\frac{5\,\pi\,\nu}{48}}\sum_{n=0}^{4}\sum_{k=1}^{6} V_k^{n-\frac{7}{2}}\mathcal{C}_k^{(n)}\mathrm{Exp}[i\,\psi(f)]$$

$$M = m_1 + m_2$$

$$\nu = \frac{m_1\,m_2}{(m_1+m_2)^2}$$

$$\mathcal{C}_2^{(0)} = \frac{1}{2}[-(1+cos_\iota^2)F_+ - 2\,i\,cos_\iota F_\times]$$

- Spin-Induced quadrupole moment terms at **2 PN, 3 PN, 3.5 PN** terms are available in the literature. In addition to that, **2 PN amplitude corrections** to the spin-induced coefficients are also been calculated. [C. K. Mishra, A. Kela, K. G. Arun]

- The leading order **(3.5PN) spin-induced octupole moment coefficient** in the phasing *also calculated*

Post-Newtonian waveform with spin-induced quadrupole moment terms (for slide 7)

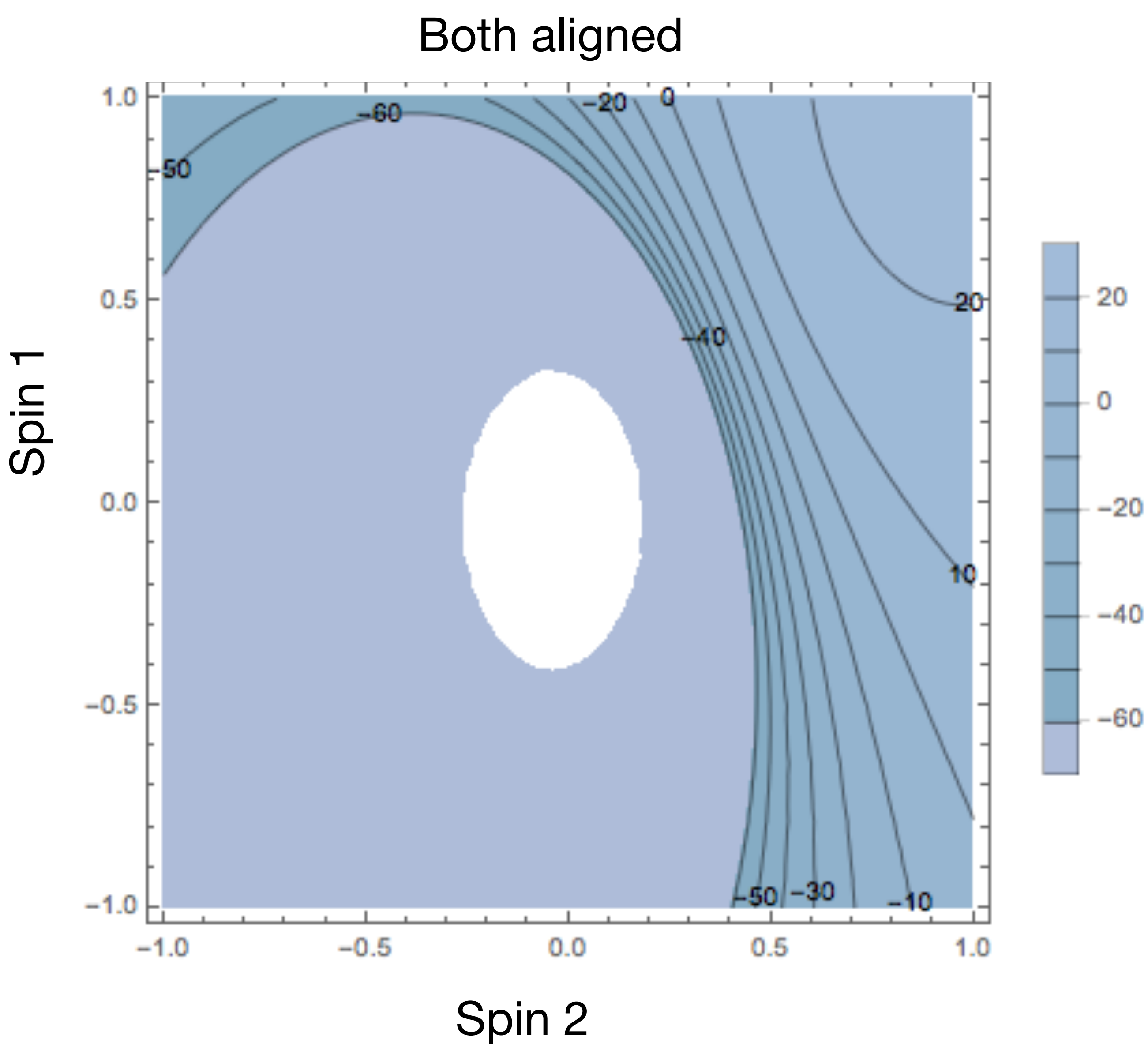
$$\tilde{h}(f) = \frac{M^2}{D_L}\sqrt{\frac{5\,\pi\,\nu}{48}}\sum_{n=0}^{4}\sum_{k=1}^{6} V_k^{n-\frac{7}{2}}\mathcal{C}_k^{(n)}\mathrm{Exp}[i\,\psi(f)]$$

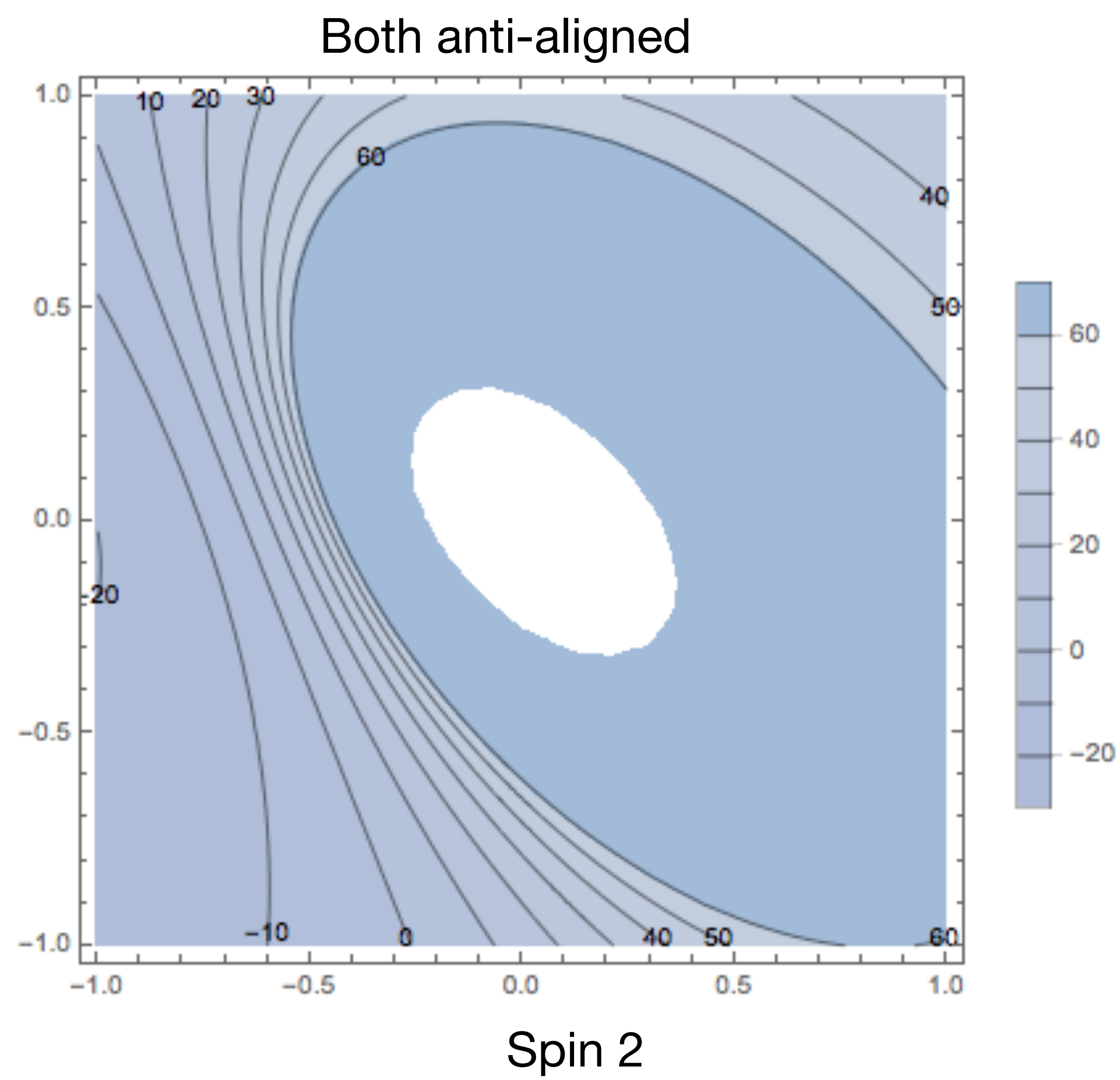
$$\psi(f) = \frac{3}{128\,\eta\,v^5}\sum_{\alpha=0}^{N}\Psi_\alpha\,v^\alpha$$

$$\begin{aligned}\Psi_{\alpha=4} = &-\frac{5}{8}(\boldsymbol{\chi}_{\rm s}\cdot\hat{\boldsymbol{L}}_{\rm N})^2\left[1+156\,\eta+80\,\delta\,\kappa_a+80(1-2\,\eta)\kappa_s\right]\\ &+(\boldsymbol{\chi}_{\rm a}\cdot\hat{\boldsymbol{L}}_{\rm N})^2\left[-\frac{5}{8}-50\,\delta\,\kappa_a-50\kappa_s+100\,\eta\,(1+\kappa_s)\right]\\ &+\frac{5}{4}(\boldsymbol{\chi}_{\rm a}\cdot\hat{\boldsymbol{L}}_{\rm N})(\boldsymbol{\chi}_{\rm s}\cdot\hat{\boldsymbol{L}}_{\rm N})\left[\delta+80\,(1-2\,\eta)\,\kappa_a+80\,\delta\,\kappa_s\right]\end{aligned}$$

RESULTS

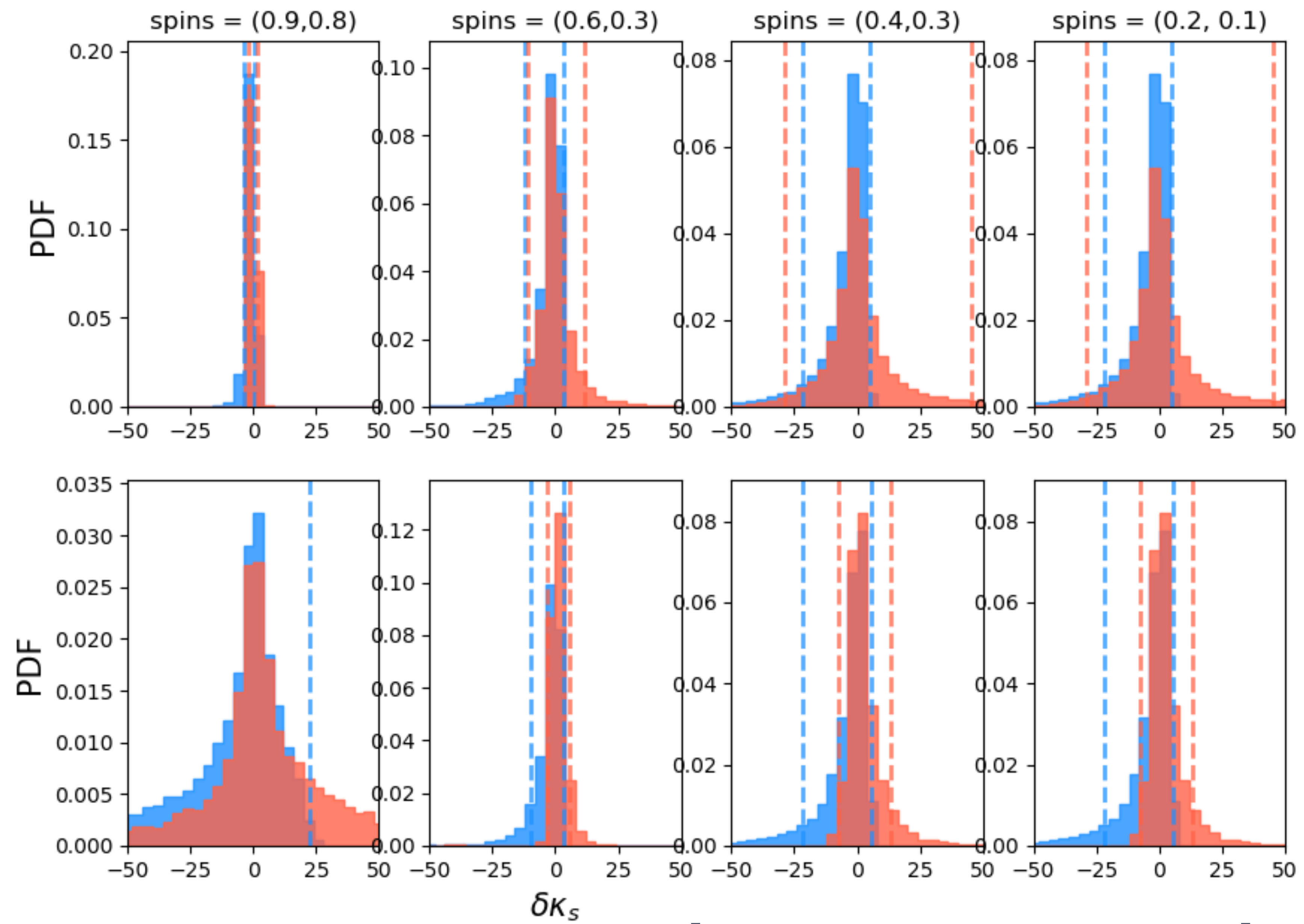
To explain the asymmetries in the posterior distribution





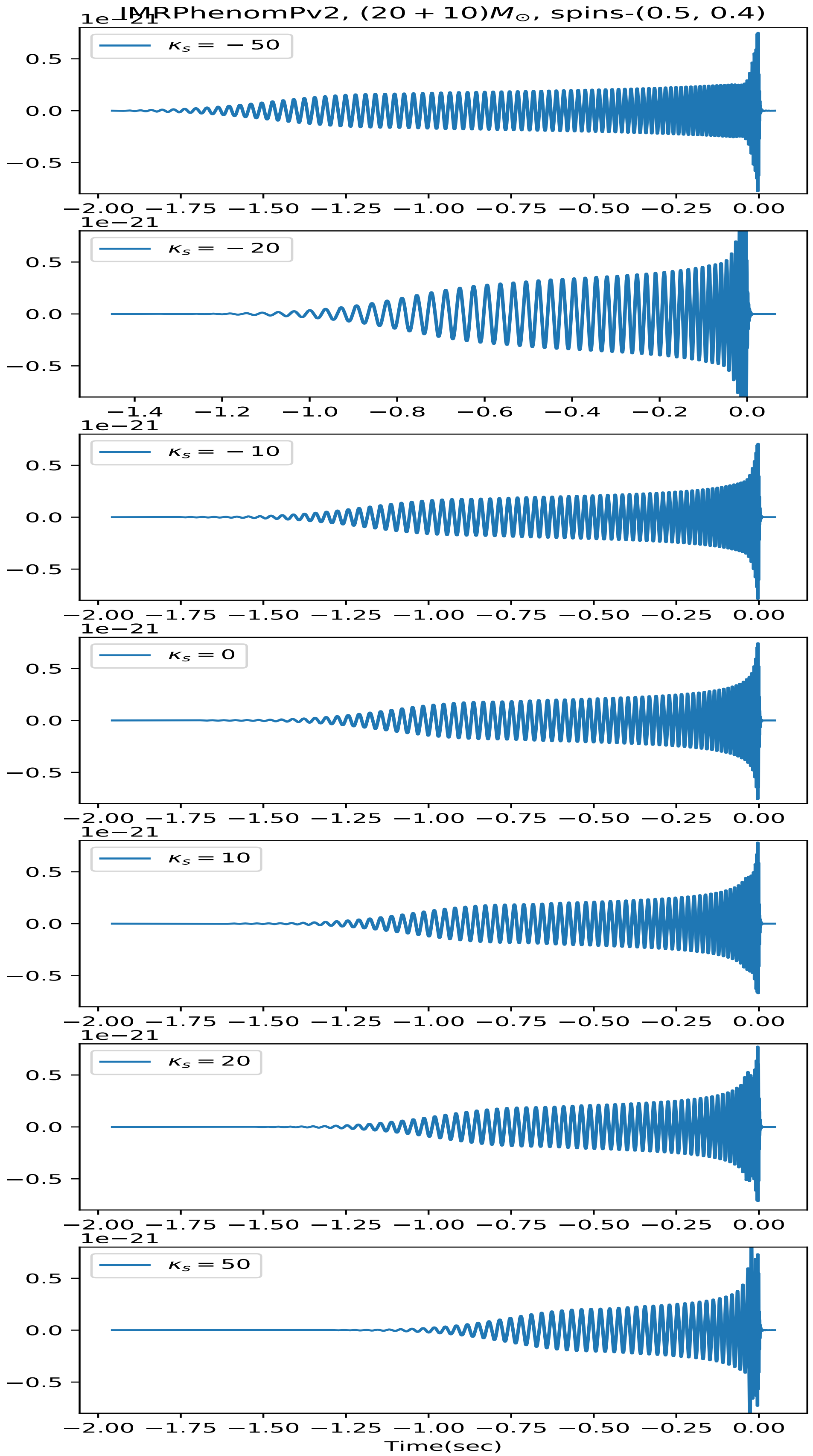
RESULTS

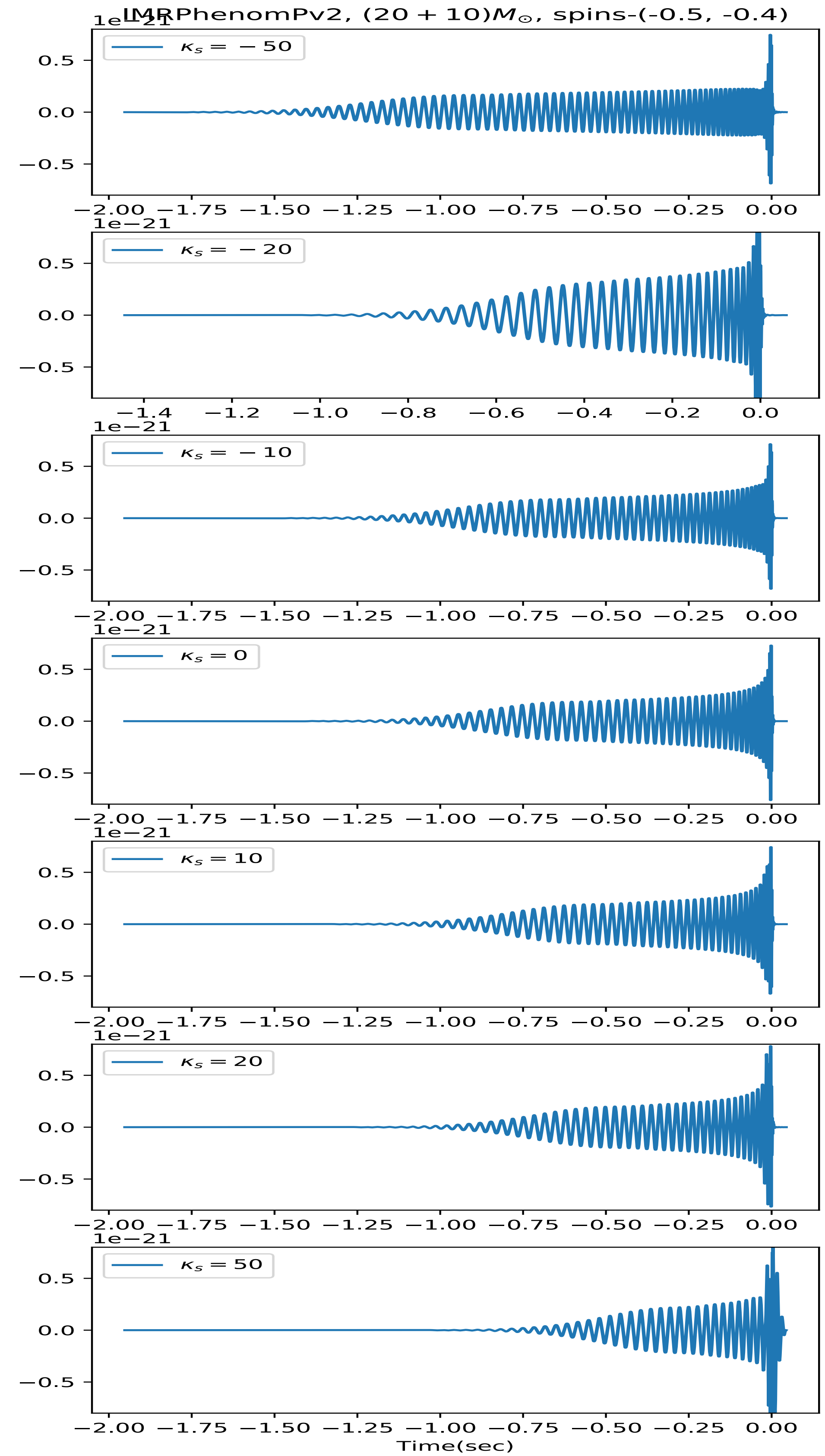
Posterior distributions on the spin-spin-induced quadrupole moment coefficient



[N. V. Krishnendu +, in preparation]

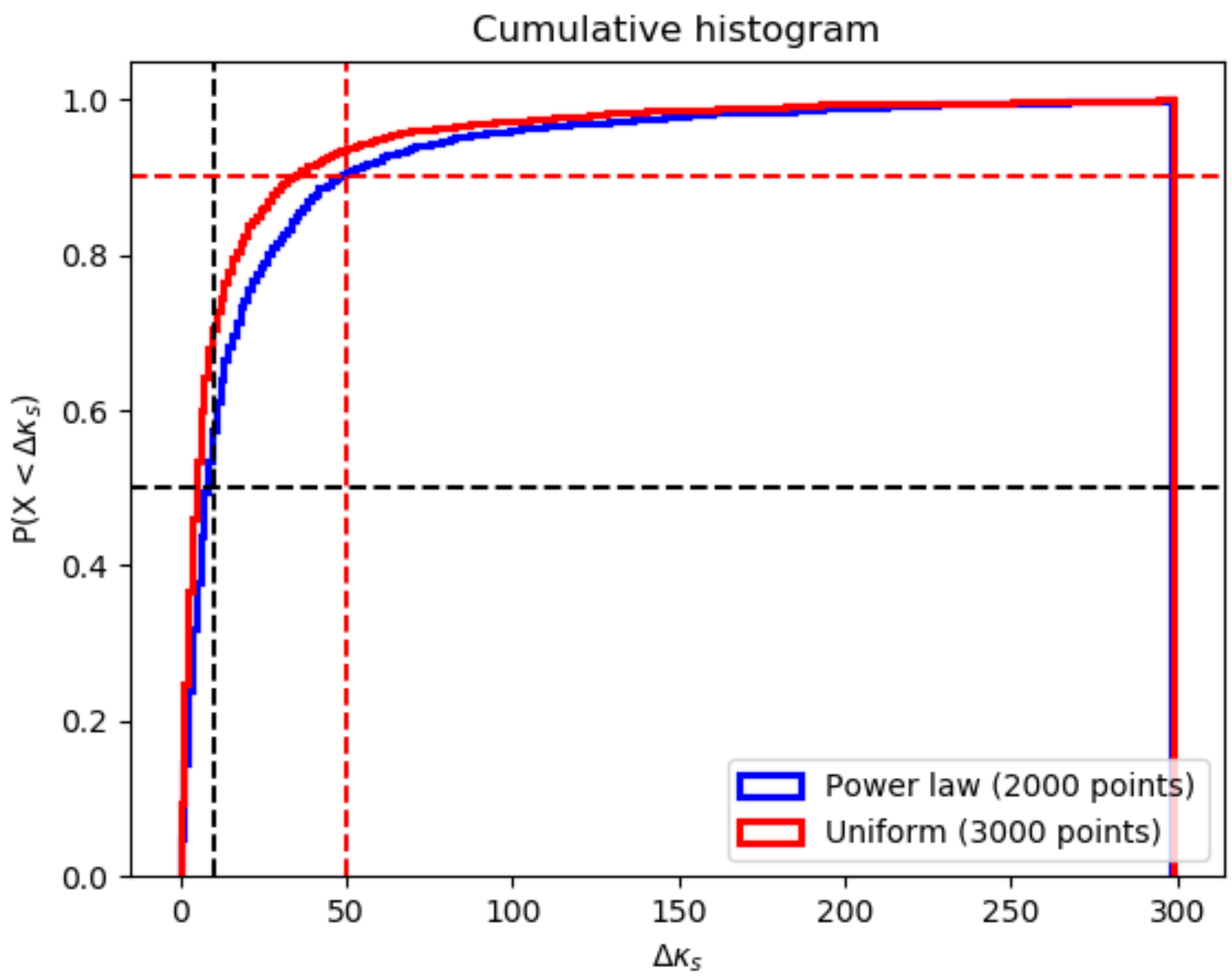
Waveform plots





RESULTS

For a population of binary systems detectable by third-generation detectors



[N. V. Krishnendu, C. K. Mishra, K. G. Arun, arXiv:1811.00317]

CBC Catalog

Event	$m_1/M_\odot$	$m_2/M_\odot$	$\mathcal{M}/M_\odot$	χ_{eff}	$M_f/M_\odot$	a_f	$E_{rad}/(M_\odot c^2)$	$\ell_{peak}/(\mathrm{erg\,s^{-1}})$	d_L/Mpc	z	$\Delta\Omega/\mathrm{deg}^2$
GW150914	$35.6^{+4.8}_{-3.0}$	$30.6^{+3.0}_{-4.4}$	$28.6^{+1.6}_{-1.5}$	$-0.01^{+0.12}_{-0.13}$	$63.1^{+3.3}_{-3.0}$	$0.69^{+0.05}_{-0.04}$	$3.1^{+0.4}_{-0.4}$	$3.6^{+0.4}_{-0.4} \times 10^{56}$	430^{+150}_{-170}	$0.09^{+0.03}_{-0.03}$	179
GW151012	$23.3^{+14.0}_{-5.5}$	$13.6^{+4.1}_{-4.8}$	$15.2^{+2.0}_{-1.1}$	$0.04^{+0.28}_{-0.19}$	$35.7^{+9.9}_{-3.8}$	$0.67^{+0.13}_{-0.11}$	$1.5^{+0.5}_{-0.5}$	$3.2^{+0.8}_{-1.7} \times 10^{56}$	1060^{+540}_{-480}	$0.21^{+0.09}_{-0.09}$	1555
GW151226	$13.7^{+8.8}_{-3.2}$	$7.7^{+2.2}_{-2.6}$	$8.9^{+0.3}_{-0.3}$	$0.18^{+0.20}_{-0.12}$	$20.5^{+6.4}_{-1.5}$	$0.74^{+0.07}_{-0.05}$	$1.0^{+0.1}_{-0.2}$	$3.4^{+0.7}_{-1.7} \times 10^{56}$	440^{+180}_{-190}	$0.09^{+0.04}_{-0.04}$	1033
GW170104	$31.0^{+7.2}_{-5.6}$	$20.1^{+4.9}_{-4.5}$	$21.5^{+2.1}_{-1.7}$	$-0.04^{+0.17}_{-0.20}$	$49.1^{+5.2}_{-3.9}$	$0.66^{+0.08}_{-0.10}$	$2.2^{+0.5}_{-0.5}$	$3.3^{+0.6}_{-0.9} \times 10^{56}$	960^{+430}_{-410}	$0.19^{+0.07}_{-0.08}$	924
GW170608	$10.9^{+5.3}_{-1.7}$	$7.6^{+1.3}_{-2.1}$	$7.9^{+0.2}_{-0.2}$	$0.03^{+0.19}_{-0.07}$	$17.8^{+3.2}_{-0.7}$	$0.69^{+0.04}_{-0.04}$	$0.9^{+0.0}_{-0.1}$	$3.5^{+0.4}_{-1.3} \times 10^{56}$	320^{+120}_{-110}	$0.07^{+0.02}_{-0.02}$	396
GW170729	$50.6^{+16.6}_{-10.2}$	$34.3^{+9.1}_{-10.1}$	$35.7^{+6.5}_{-4.7}$	$0.36^{+0.21}_{-0.25}$	$80.3^{+14.6}_{-10.2}$	$0.81^{+0.07}_{-0.13}$	$4.8^{+1.7}_{-1.7}$	$4.2^{+0.9}_{-1.5} \times 10^{56}$	2750^{+1350}_{-1320}	$0.48^{+0.19}_{-0.20}$	1033
GW170809	$35.2^{+8.3}_{-6.0}$	$23.8^{+5.2}_{-5.1}$	$25.0^{+2.1}_{-1.6}$	$0.07^{+0.16}_{-0.16}$	$56.4^{+5.2}_{-3.7}$	$0.70^{+0.08}_{-0.09}$	$2.7^{+0.6}_{-0.6}$	$3.5^{+0.6}_{-0.9} \times 10^{56}$	990^{+320}_{-380}	$0.20^{+0.05}_{-0.07}$	340
GW170814	$30.7^{+5.7}_{-3.0}$	$25.3^{+2.9}_{-4.1}$	$24.2^{+1.4}_{-1.1}$	$0.07^{+0.12}_{-0.11}$	$53.4^{+3.2}_{-2.4}$	$0.72^{+0.07}_{-0.05}$	$2.7^{+0.4}_{-0.3}$	$3.7^{+0.4}_{-0.5} \times 10^{56}$	580^{+160}_{-210}	$0.12^{+0.03}_{-0.04}$	87
GW170817	$1.46^{+0.12}_{-0.10}$	$1.27^{+0.09}_{-0.09}$	$1.186^{+0.001}_{-0.001}$	$0.00^{+0.02}_{-0.01}$	≤ 2.8	≤ 0.89	≥ 0.04	$\geq 0.1 \times 10^{56}$	40^{+10}_{-10}	$0.01^{+0.00}_{-0.00}$	16
GW170818	$35.5^{+7.5}_{-4.7}$	$26.8^{+4.3}_{-5.2}$	$26.7^{+2.1}_{-1.7}$	$-0.09^{+0.18}_{-0.21}$	$59.8^{+4.8}_{-3.8}$	$0.67^{+0.07}_{-0.08}$	$2.7^{+0.5}_{-0.5}$	$3.4^{+0.5}_{-0.7} \times 10^{56}$	1020^{+430}_{-360}	$0.20^{+0.07}_{-0.07}$	39
GW170823	$39.6^{+10.0}_{-6.6}$	$29.4^{+6.3}_{-7.1}$	$29.3^{+4.2}_{-3.2}$	$0.08^{+0.20}_{-0.22}$	$65.6^{+9.4}_{-6.6}$	$0.71^{+0.08}_{-0.10}$	$3.3^{+0.9}_{-0.8}$	$3.6^{+0.6}_{-0.9} \times 10^{56}$	1850^{+840}_{-840}	$0.34^{+0.13}_{-0.14}$	1651

$$\mathcal{C}_{2}^{(0)}=\frac{1}{2}[-(1+cos_{\iota}^2)F_{+}-2\,i\,cos_{\iota}F_{\times}]$$

$$\mathcal{A}({\mathcal {\theta}}^{j},\,f)\propto\,\frac{\mathcal{M}_{c}^{5/6}}{D_{L}}$$

$$\boldsymbol{\Gamma}_{ij}=2 \int^{f_{up}}_{f_{low}} df \frac{\tilde{h_{i}}\tilde{h_{j}^{\ast}}+ \tilde{h_{j}}\tilde{h_{i}^{\ast}}}{S_{n}(f)}, \, h^{i}=\frac{\partial}{\partial\theta^{i}}\tilde{h}(f,\,\theta^{i})$$

$$\mathrm{Log}(\mathcal{B^{{\rm{non}}-BH}_{BH}})$$

$$\mathrm{P}(m_{1})=m_{1}^{-1.6},\,5M_{\odot}\leq m_{1}\leq 20 M_{\odot},\\$$

$$\mathrm{P}(m_{2})=\mathrm{U}(5 M_{\odot}, m_{1})$$

$$P(\theta^{j}|\mathcal{H},d)=\frac{P(d|\mathcal{H},\theta^{\it{j}})\,P(\theta^{\it{j}}|\mathcal{H})}{P(d|\mathcal{H})}$$

$$\mathrm{P}(\chi_{1,\,2})=\mathrm{U}(-1,\,1)$$

$$\chi_{\rm{eff}}=\frac{m_{1}\,\chi_{1}+m_{2}\,\chi_{2}}{m_{1$$

$$\sigma^{j}=\sqrt{(\Gamma^{-1})^{jj}}$$

$$\Psi_{4}^{\rm{qm}}=-\frac{5}{2}\sum_{A=1,2}\frac{Q_{A}}{m_{A}M^2}\left(3\left(\hat{L}.s_{A}^2\right)^2-1\right)$$

$$\mathrm{P}(m_{1})=m_{1}^{-1.6},\,5M_{\odot}\leq m_{1}\leq 20 M_{\odot}$$

$$\theta^{j}=\lbrace \kappa_s,\,\mathcal{M}_{c},\,\delta,\,\mathcal{\chi}_{1},\,\mathcal{\chi}_{2},\,t_{c},\,\phi_{c},\,\rm{cos}\iota,\, \mathcal{D_{L}}\rbrace$$

$$\psi(\theta^{j},\,f)=\frac{3}{128\,\eta\,v^5}\sum^{N}_{\alpha=0}\Psi_{\alpha}\,v^{\alpha}$$

$$\tilde{h_{\rm}}({\mathcal {\theta}}^{j},\,f)=\mathcal{A}({\mathcal {\theta}}^{j},\,f)\,{\rm{Exp}}[i\,\psi({\mathcal {\theta}}^{j},\,f)]$$

$$\tilde{h}(f)=\frac{M^2}{D_{L}}\sqrt{\frac{5\,\pi\,\nu}{48}}\sum_{n=0}^{4}\sum_{k=1}^{6}\,V_{k}^{n-\frac{7}{2}}\mathcal{C}_{k}^{(n)}{\rm{Exp}}[i\,\psi(f)]$$