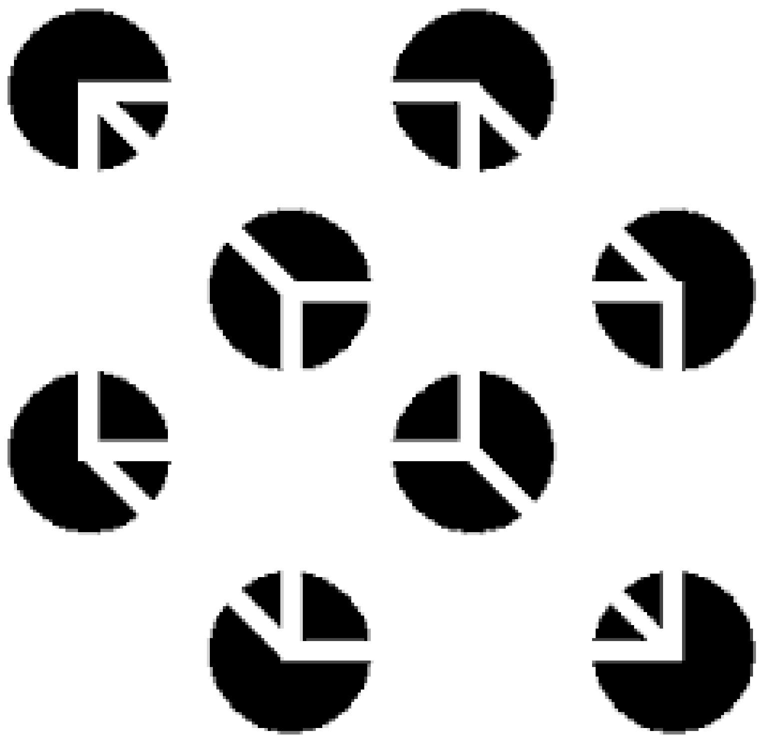
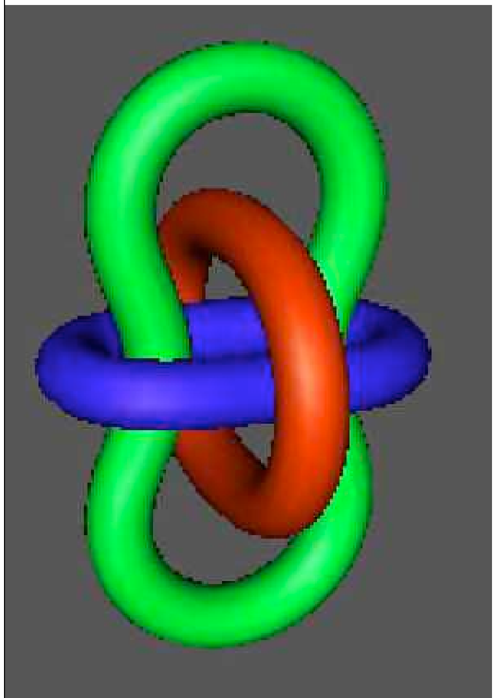
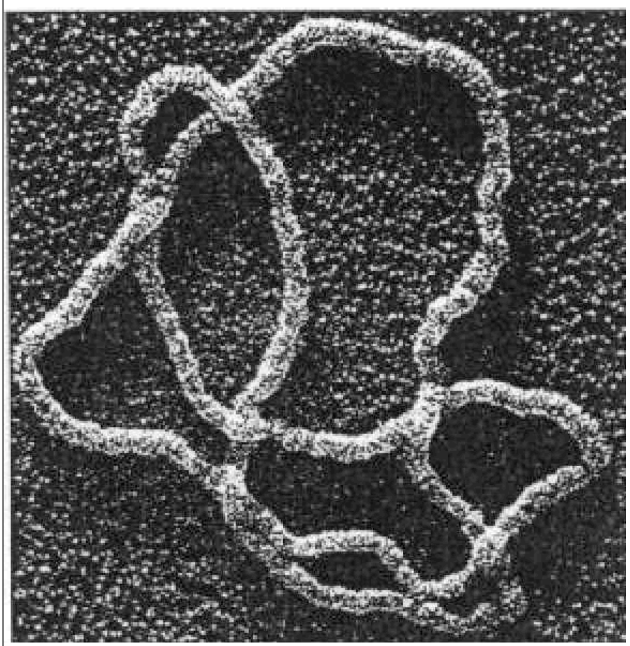
Virtual Knot Invariants and Virtual Knot Cobordism

Louis H. Kauffman, UIC







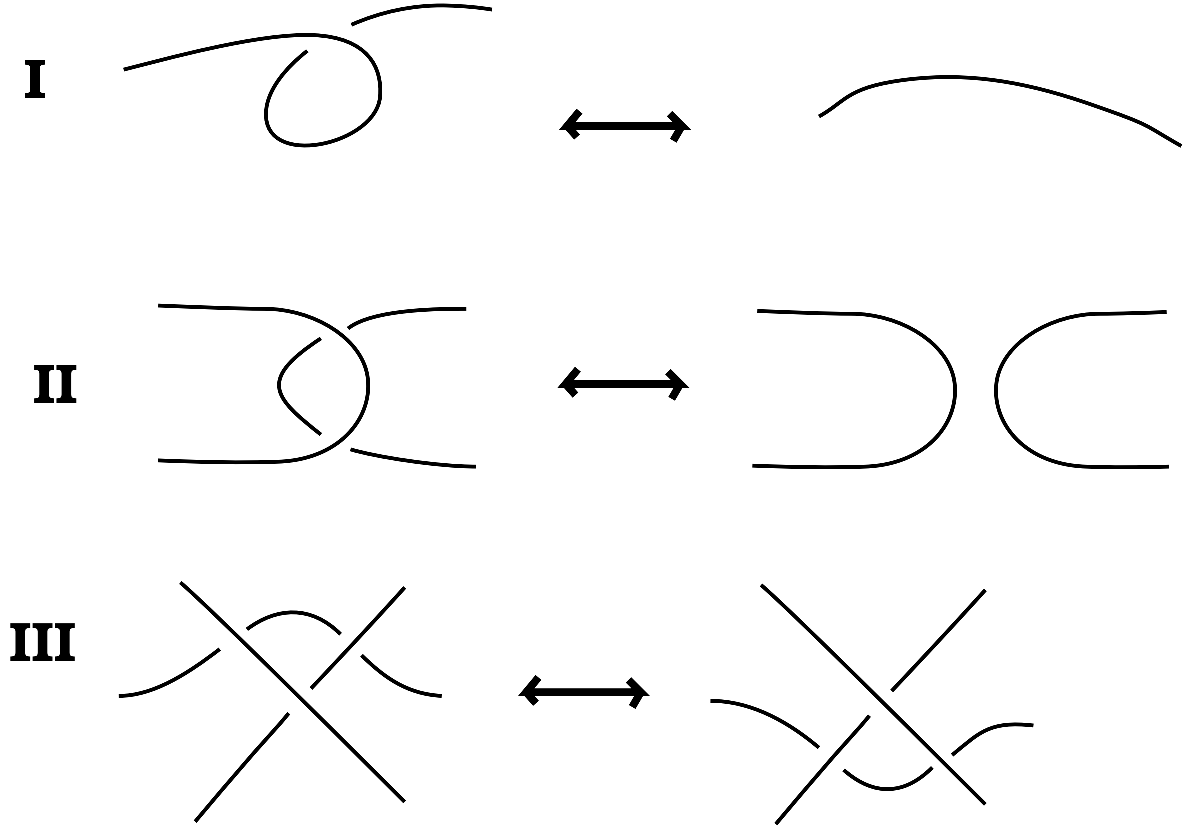


Figure 2 - The Reidemeister Moves.

Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

Virtual Knot Theory
studies stabilized knots in thickened surfaces.

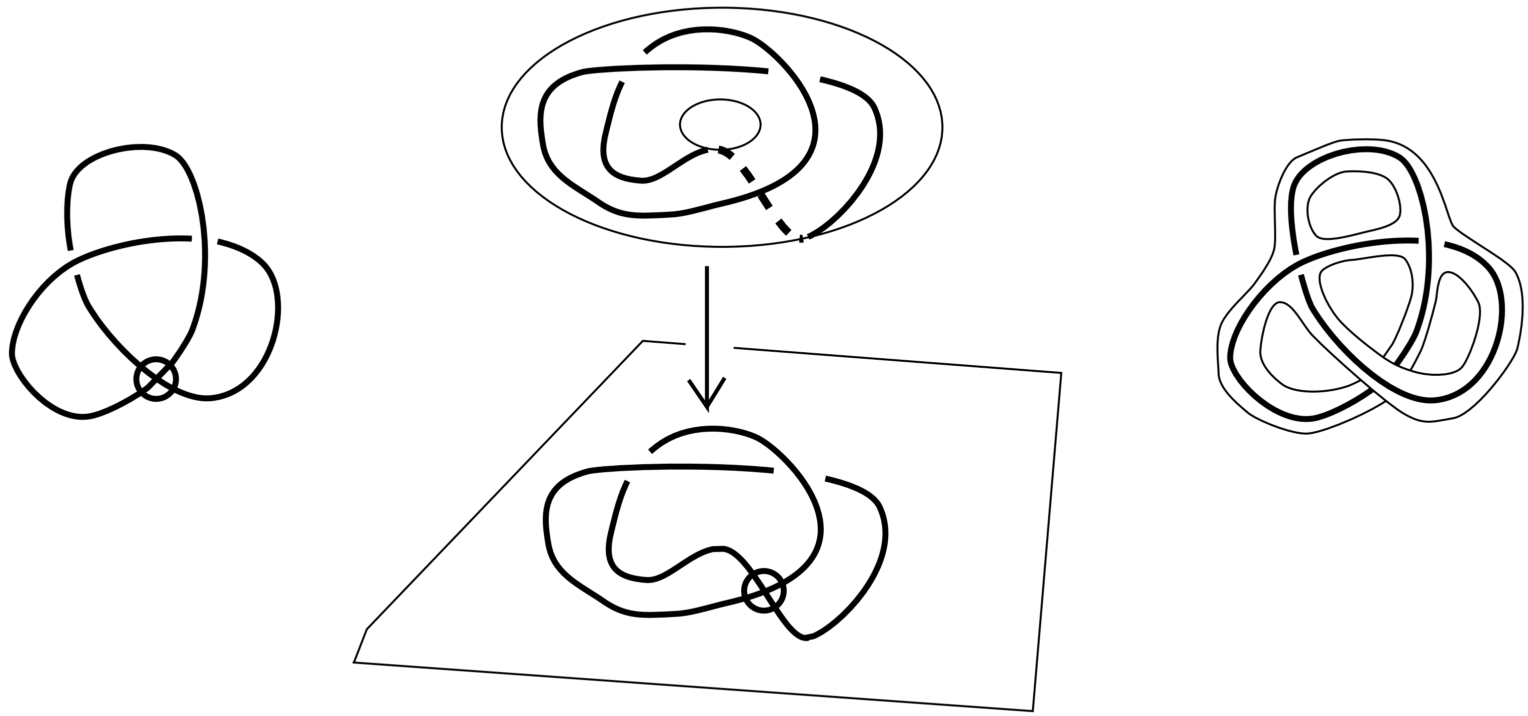
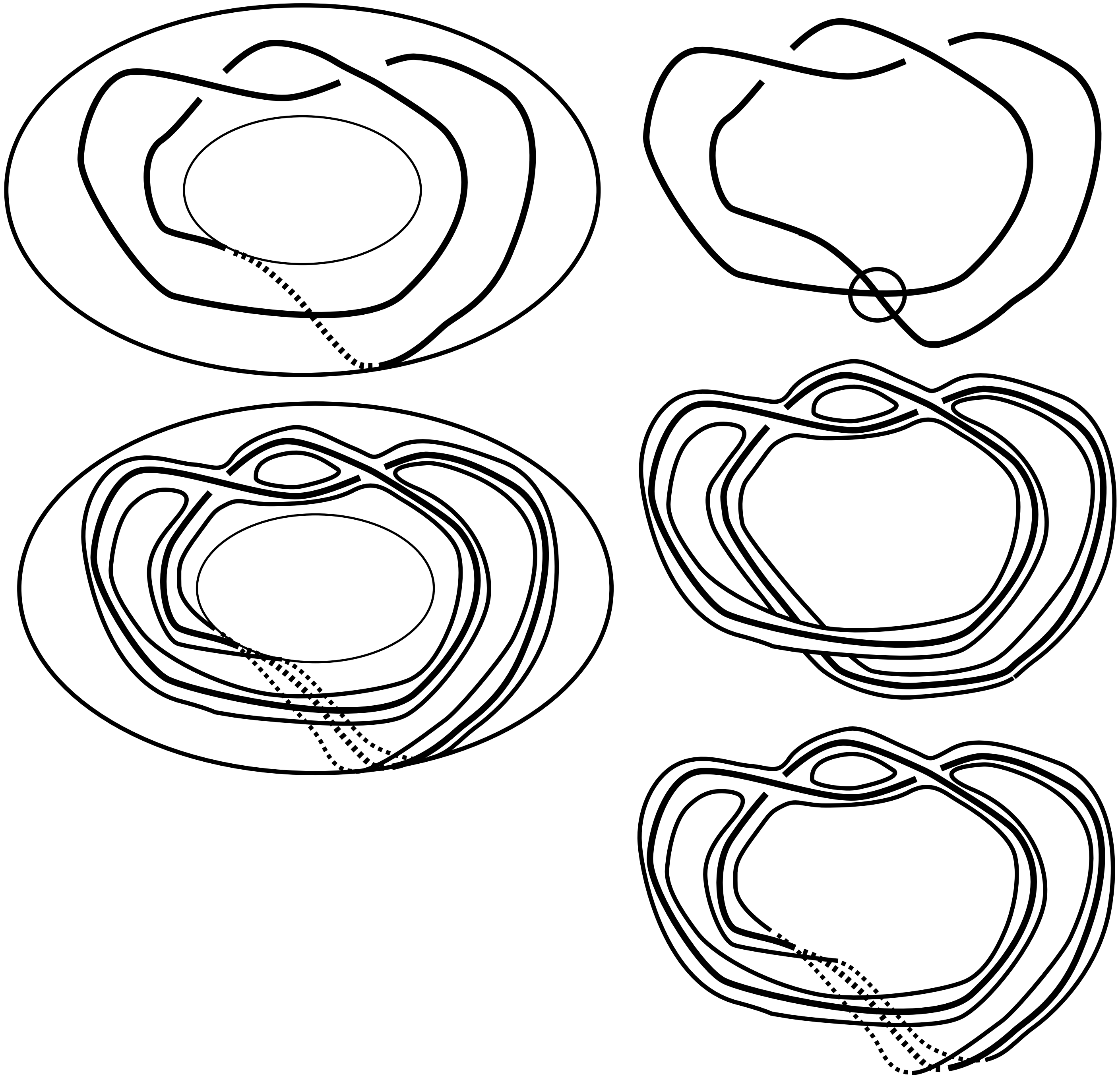
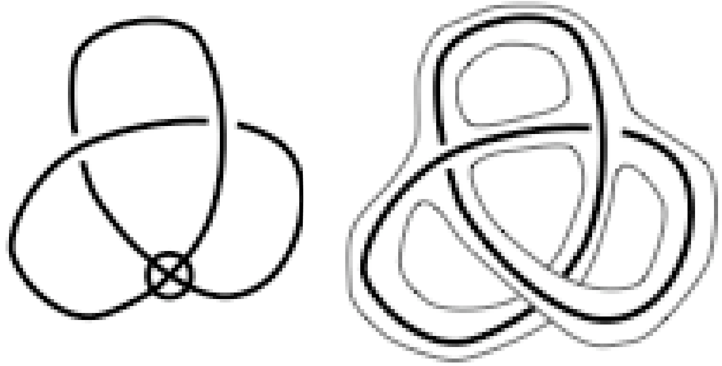


Figure 4: **Surfaces and Virtuals**





$v = 2$
$L = 2$
$g = 1$

v-e+L = 2-2g

4v=2e

Hence

g = 1+(v L)/2.

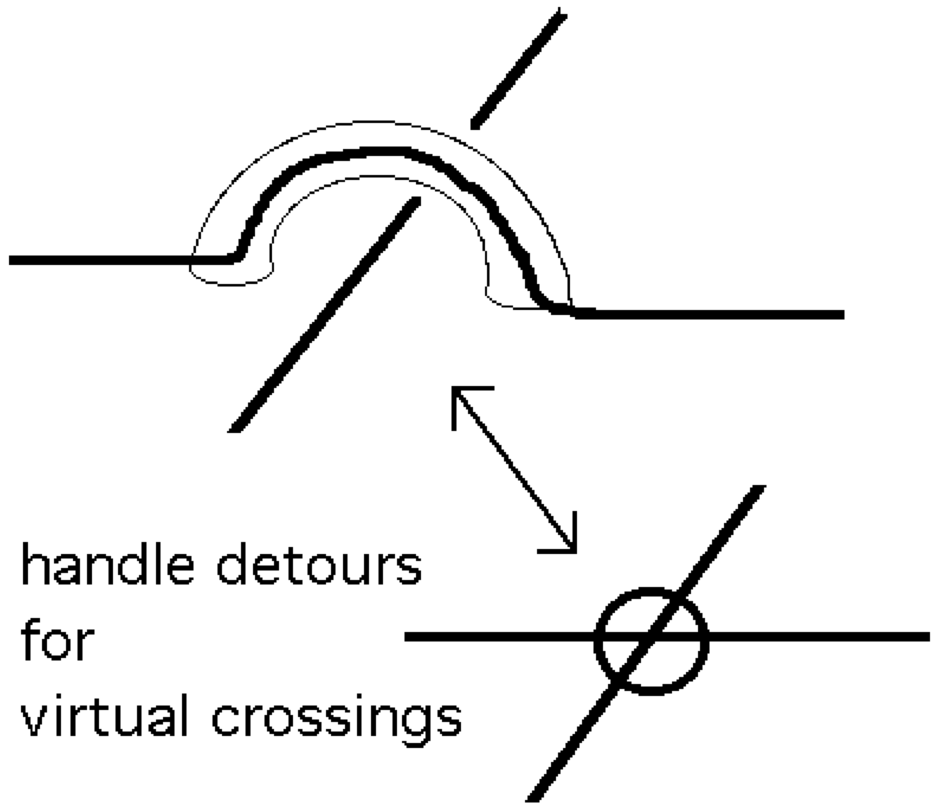
Euler ⟶ $g = 1 + (v - L)/2$

v = # classical crossings

L = # loops on boundary

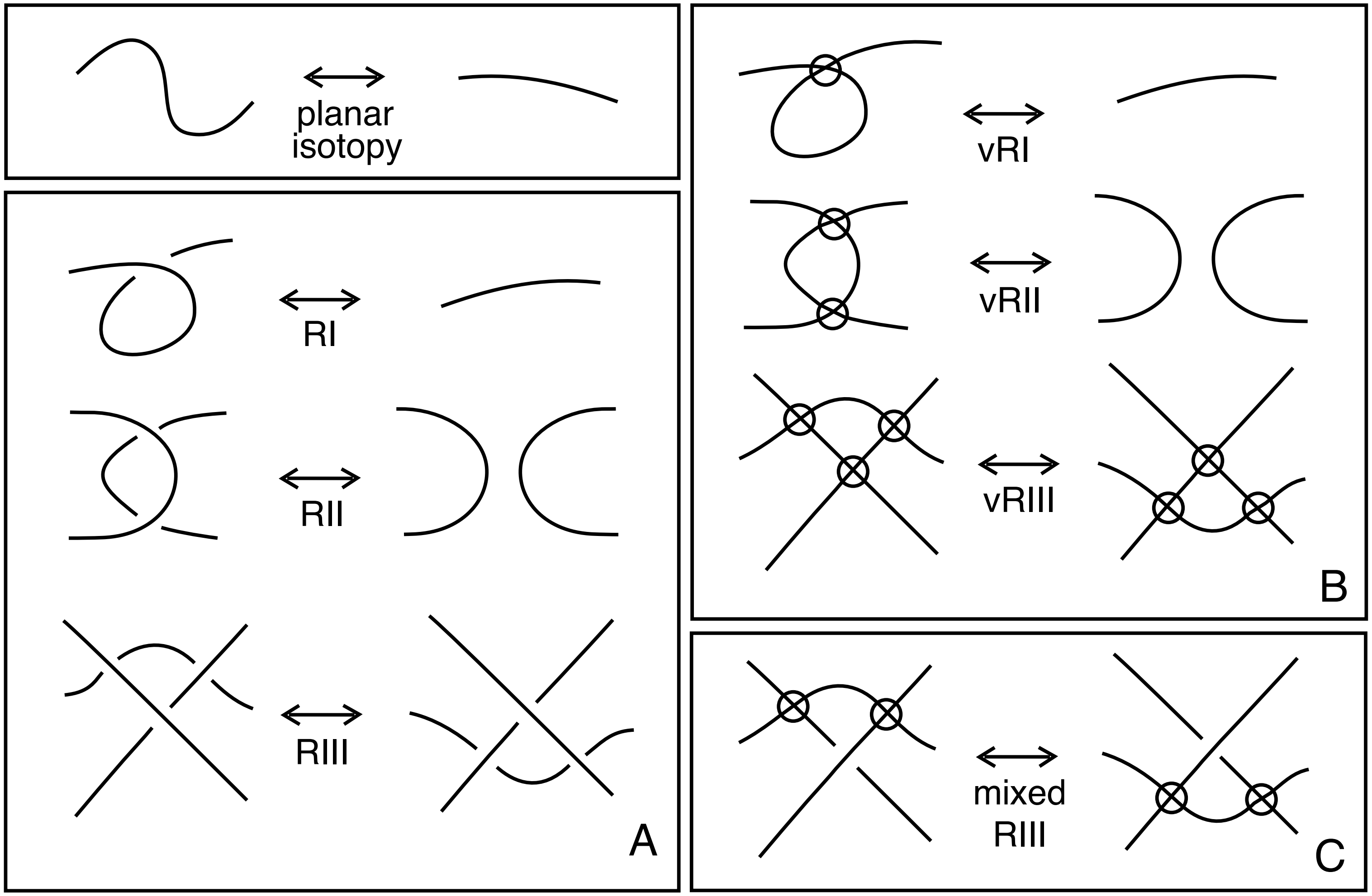
g = genus of surface obtained by attaching disks to the loops.
This surface is the least genus surface associated with the diagram, but not always with the virtual knot.

FACT: g is invariant under Reidemeister I and III moves.

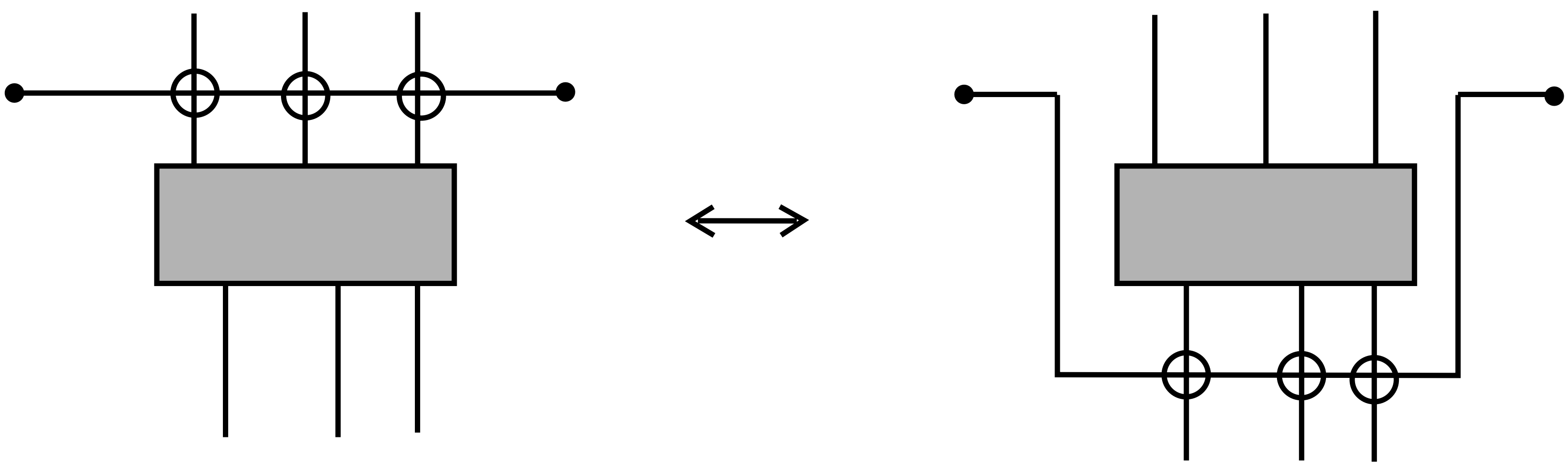

handle detours
for
virtual crossings


an empty
handle

Generalized Reidemeister Moves for Virtual Knots and Links



Detour Move



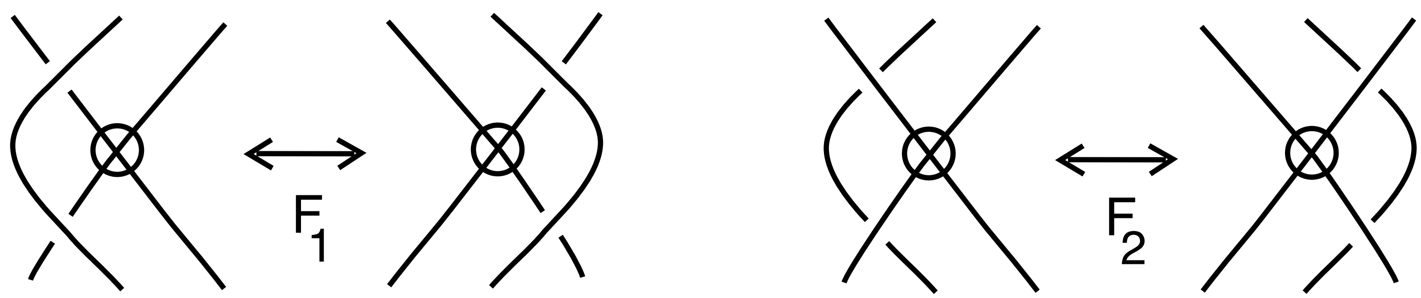


Figure 3. Forbidden Moves

VKT

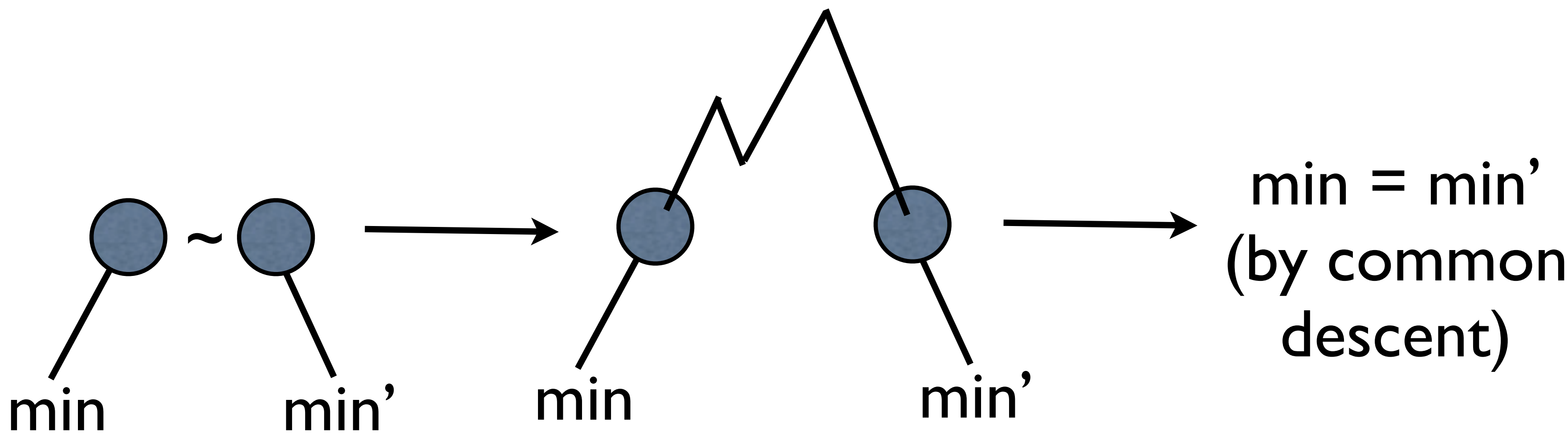
= Virtual Knot Theory

= Virtual Diagrams up to Virtual Equivalence

= Oriented Gauss Codes up to Reidemeister Moves

= Links in Thickened Surfaces up to 1-handle stabilization

Kuperberg showed that 1-handle surgery gives unique knot type in the minimal genus surface.



From Kuperberg it follows that one only need descend by surgery from any given surface to reach the minimal surface.

Combinatorial Descent to Minimal Surface

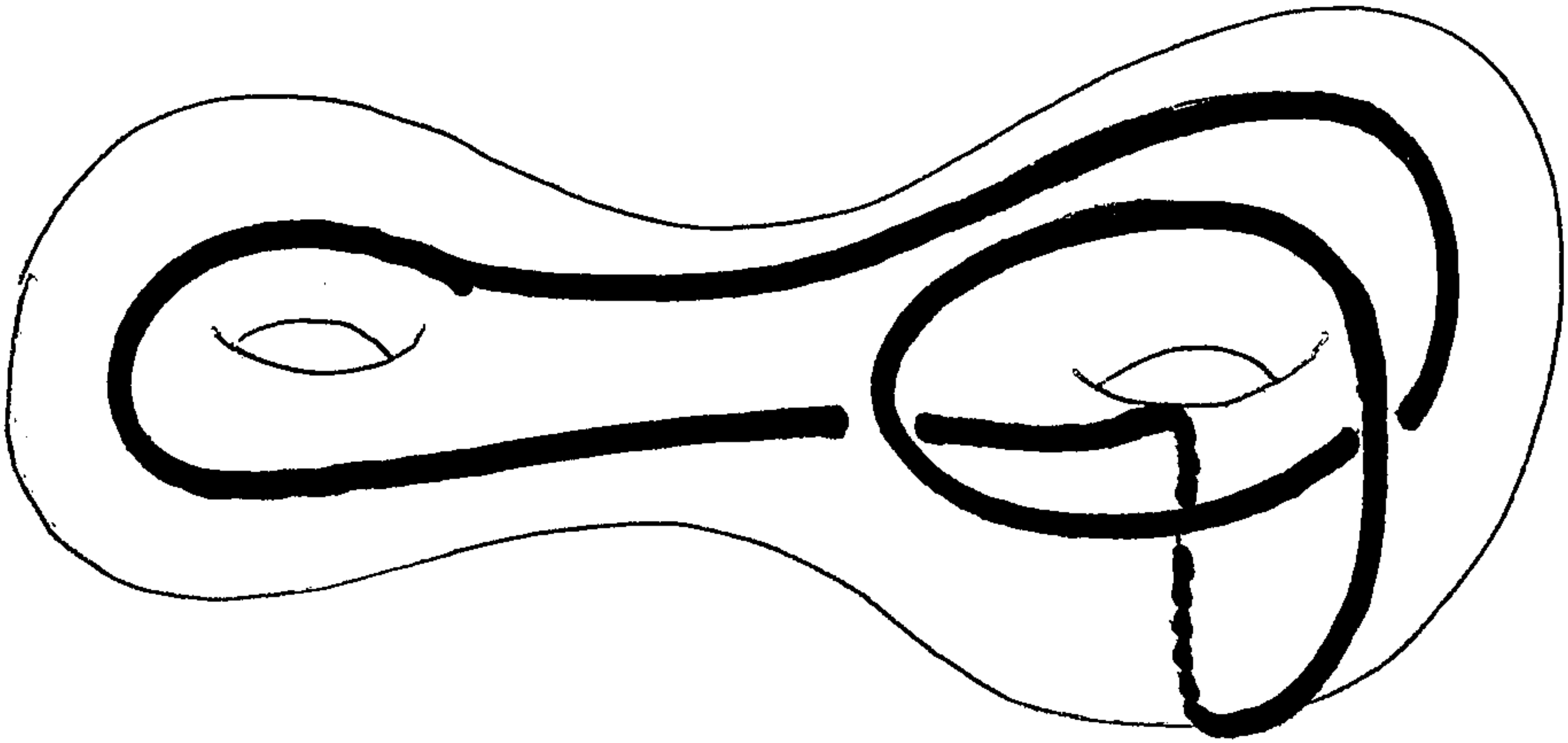
1. Given a virtual diagram, form the standard band surface.

2. Add 2-cells to the boundary.

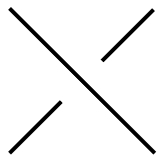
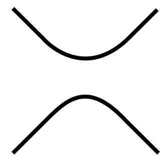
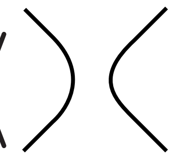
3. Allow Reidemeister moves on the diagram in the surface constructed in 2.

4. Cut out a band surface neighborhood of the link diagram in the surface.

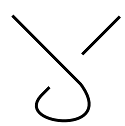
5. Go to 2.

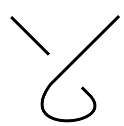


Bracket polynomial model for the Jones polynomial extends to virtuals by counting all loops the same way.

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

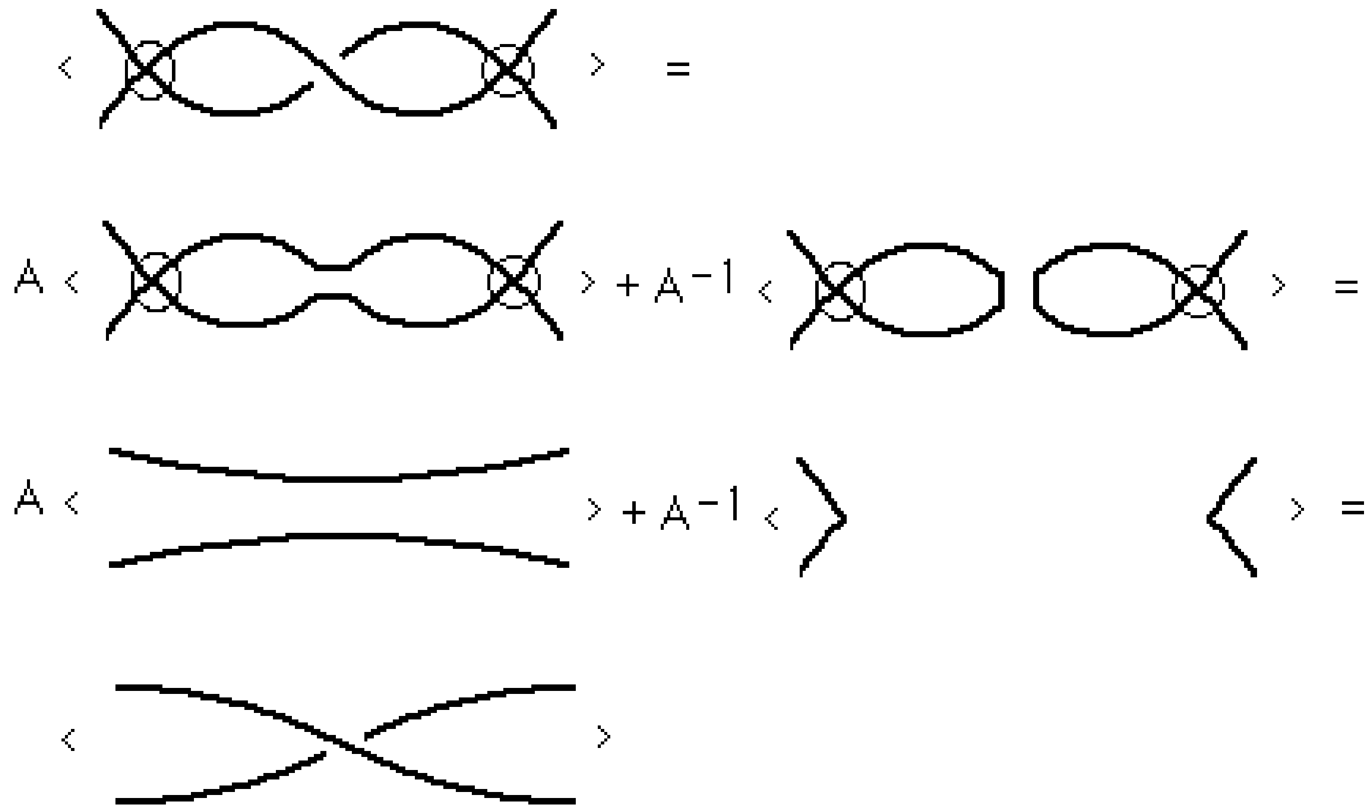
$$\langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$

$\langle$  $\rangle = (-A^3) \langle \smile \rangle$

$\langle$  $\rangle = (-A^{-3}) \langle \smile \rangle$

Conjecture: (Modification of a conjecture of Jozef Przytycki) If K in a surface S is in minimal genus, then this fact is detected by the surface bracket polynomial.

Bracket Polynomial is Unchanged when smoothing flanking virtuals.



Z - EQUIVALENCE

$\underset{Z}{\sim}$

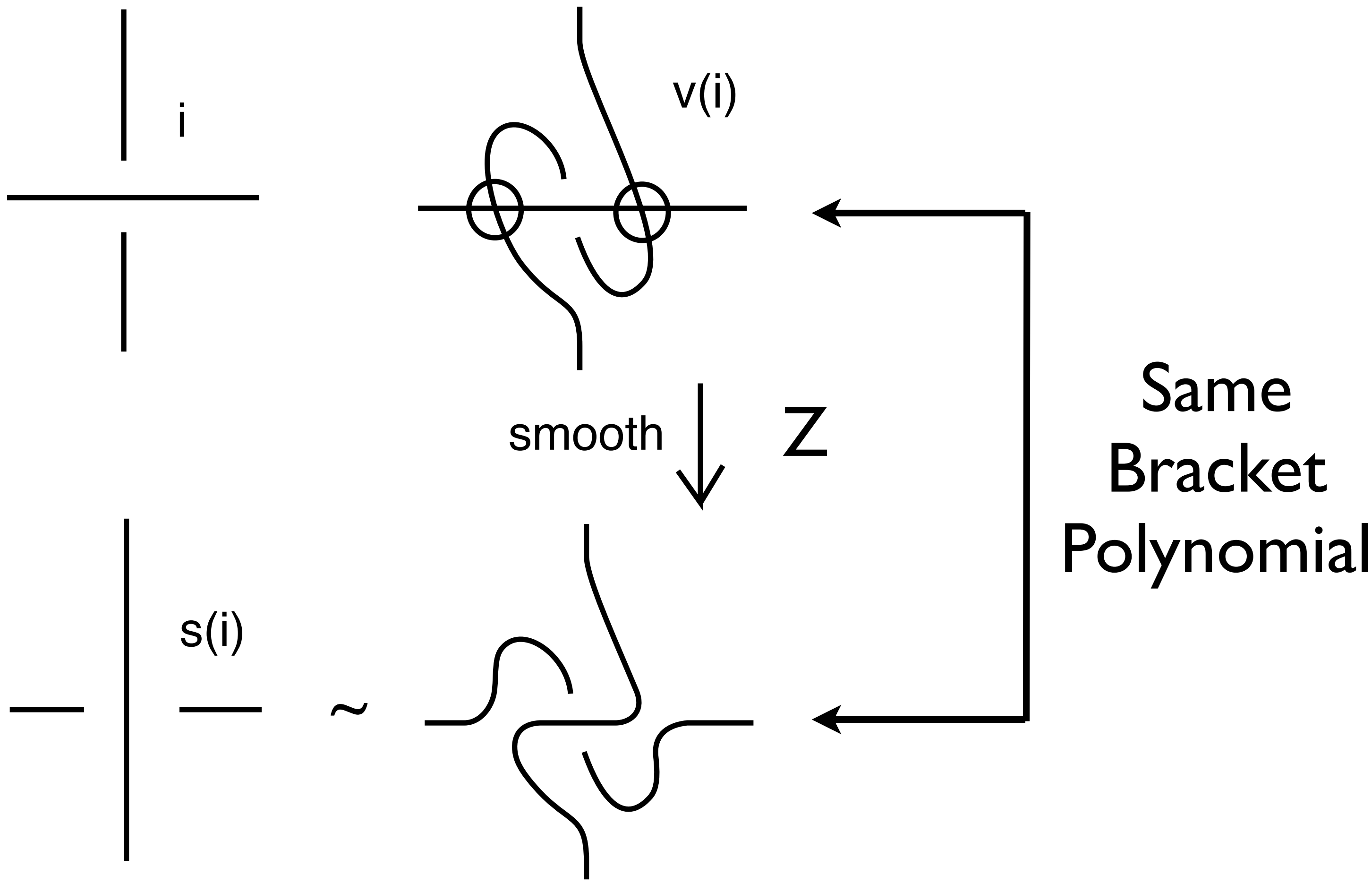


Figure 7. Switch and Virtualize

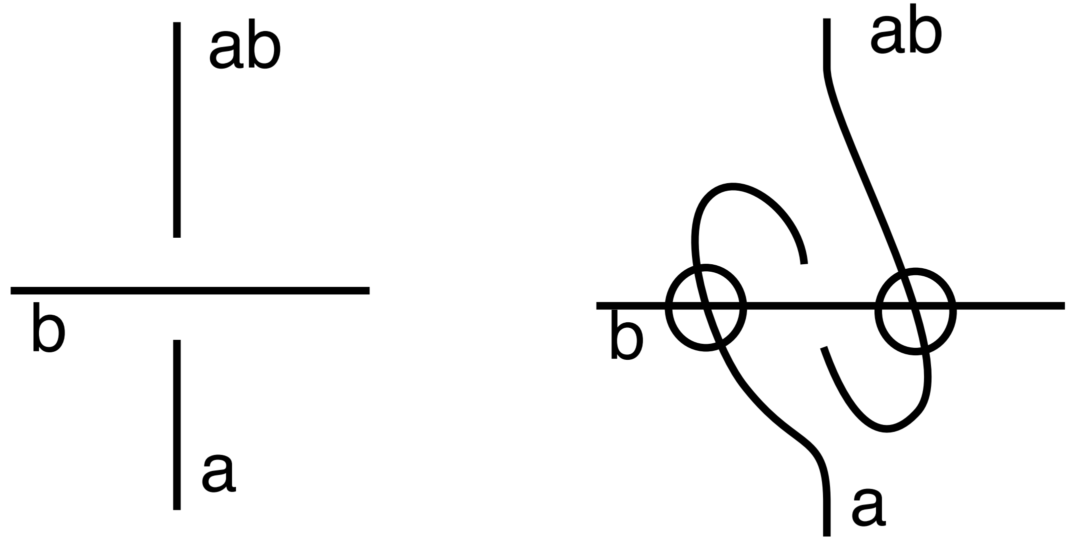
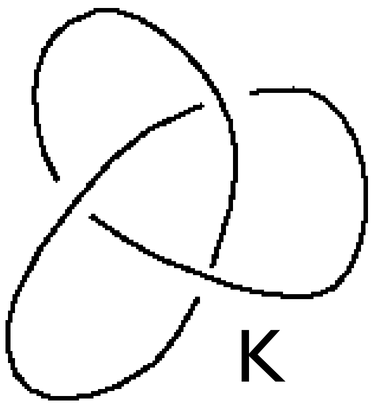


Figure 8. IQ(Virt)

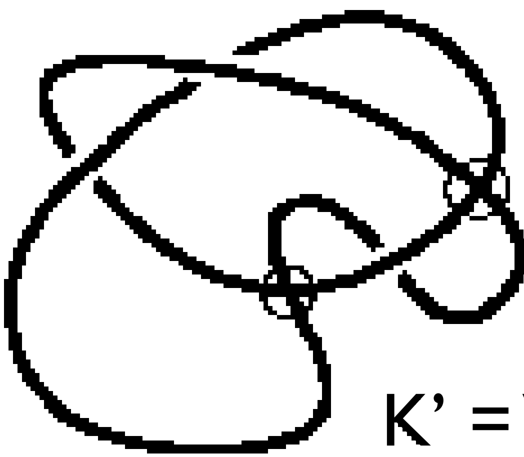


K

<Virt(K)> = <Switch(K)>
and

IQ(Virt(K)) = IQ(K).

There exist infinitely many non-trivial Virt(K) with unit Jones polynomial.



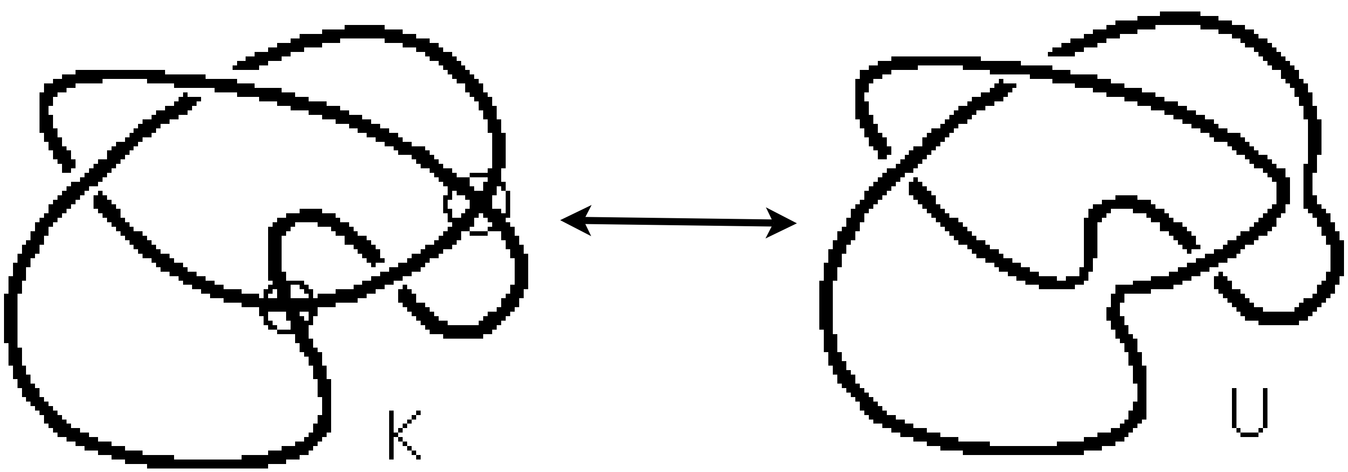
K' = Virt(K)

U

There exist infinitely many non-trivial K
with unit Jones polynomial.

Bracket Polynomial is Unchanged
when smoothing flanking virtuals.

Z-Equivalence



Conjecture:
If K is a classical knot (known to be knotted) and Virt(K) is a virtual knot obtained from K by virtualizing a set of crossings that unknot K, then the minimal surface genus of Virt(K) is > 0.

Approachable Conjecture:
A virtualization (corresponding to an unknotting choice) of a reduced alternating diagram has minimal surface genus > 0.

Classical knot theory embeds in virtual knot theory.

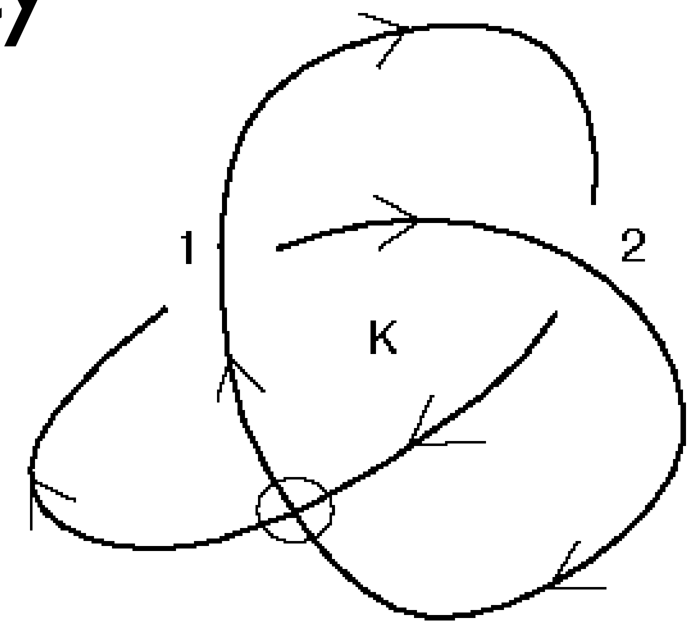
Open Question:
Does classical knot theory embed in virtual knot theory modulo Z-equivalence?

Z-Knot Theory

Open Question:
Are all the virtual knots with unit Jones polynomial made by the virtualization process non-classical?

Parity

The Odd Writhe



Bare Gauss Code
1212

Crossings 1 and 2 are odd.

A crossing is odd if it flanks an odd number of symbols in the Gauss code.

The odd writhe of K, J(K).

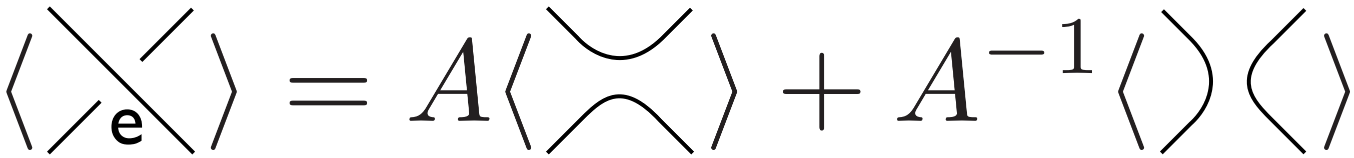
J(K) = Sum of signs of the odd crossings of K.

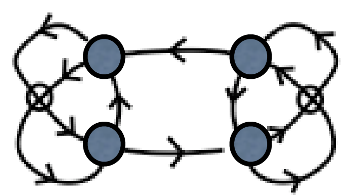
Here J(K) = -2.

Facts: J(K) is an invariant of virtual isotopy.
J(K) = 0 is K is classical.
J(Mirror Image of K) = -J(K).

Hence this example is not classical and is not isotopic to its mirror image.

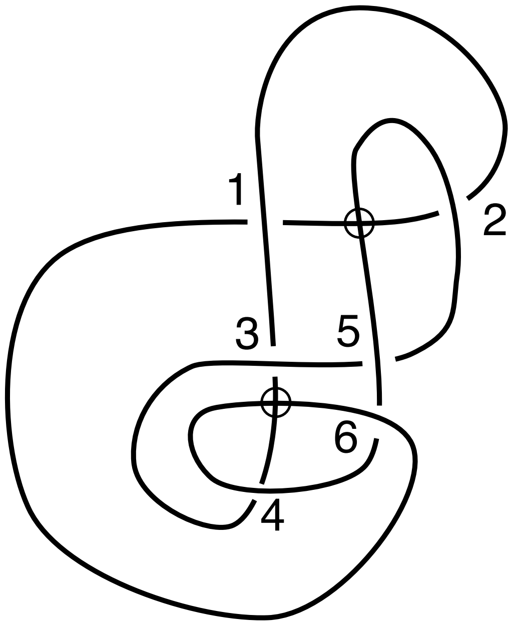
Parity
Manturov Parity Bracket

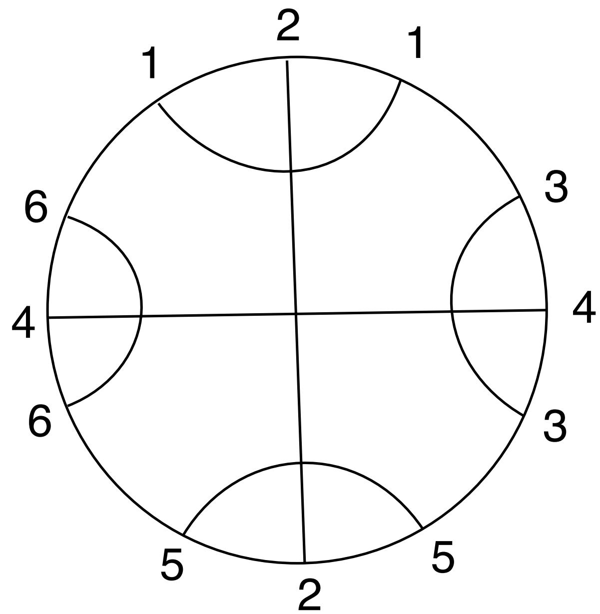


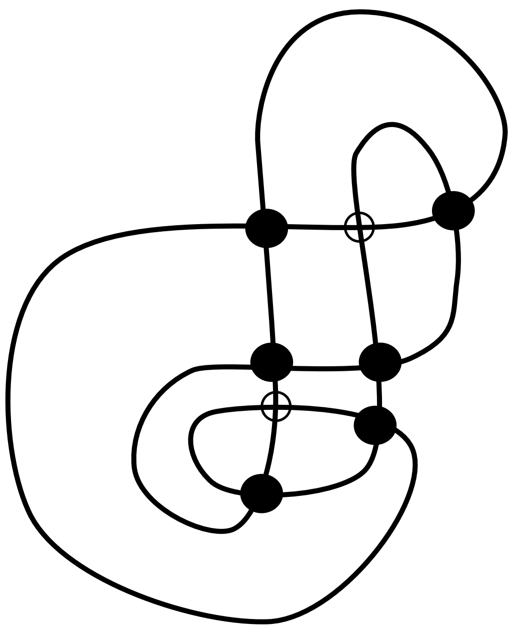


The Parity Bracket provides the simplest proof that the Kishino diagram is non-trivial.

Determining Genus for Odd Knots



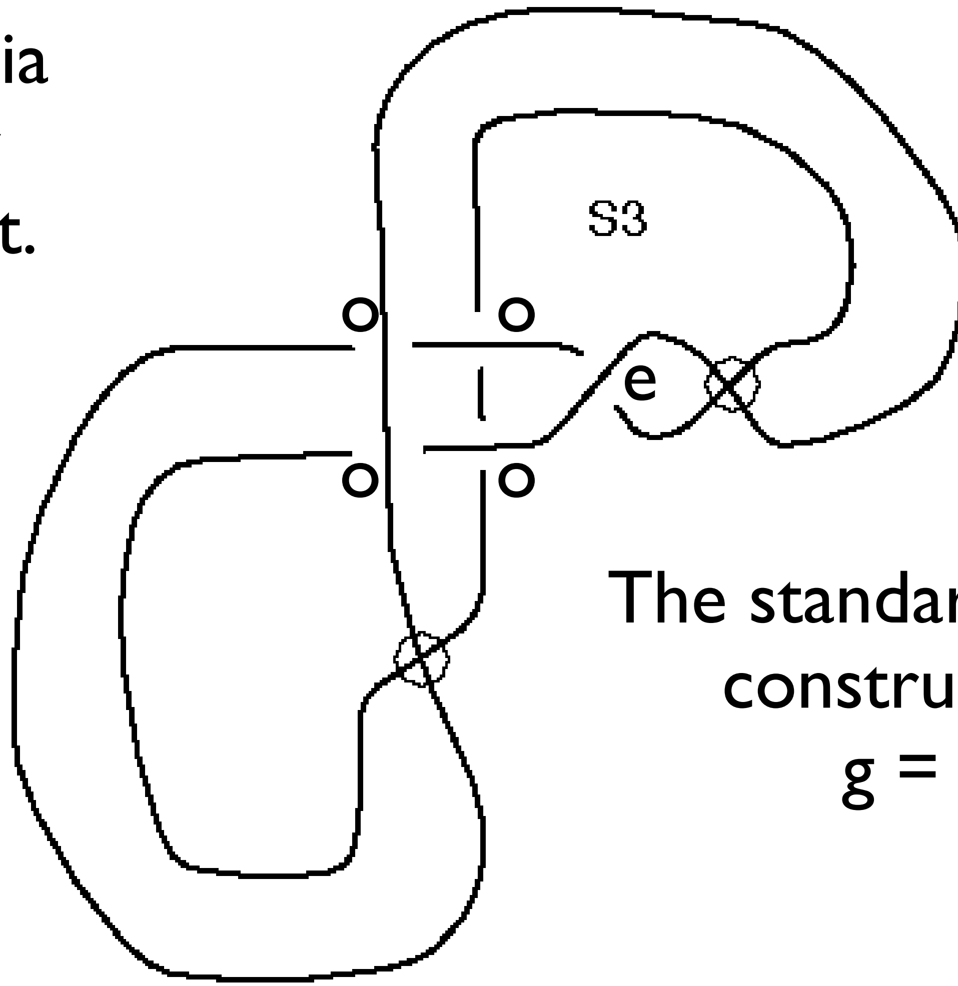




All classical nodes are odd.
Graph is irreducible.
One parity bracket state.
Genus g = 2.

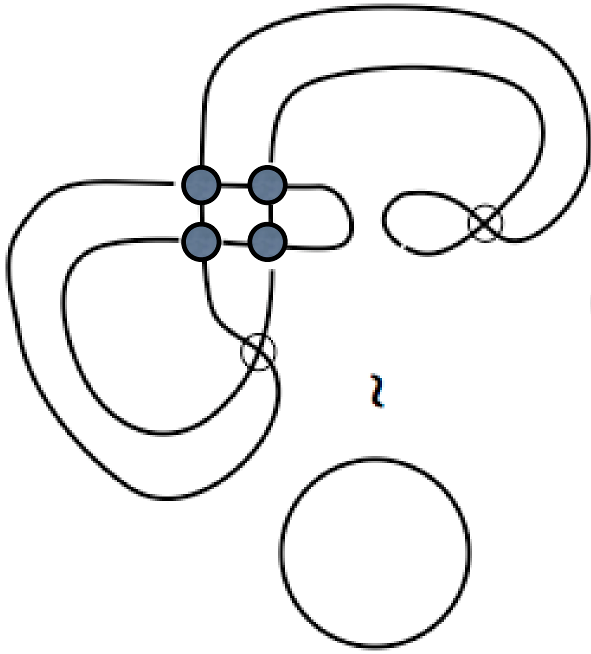
The Knot S3 (found with Slavik Jablan) has unit Jones polynomial. It is not Z-equivalent to a classical knot.

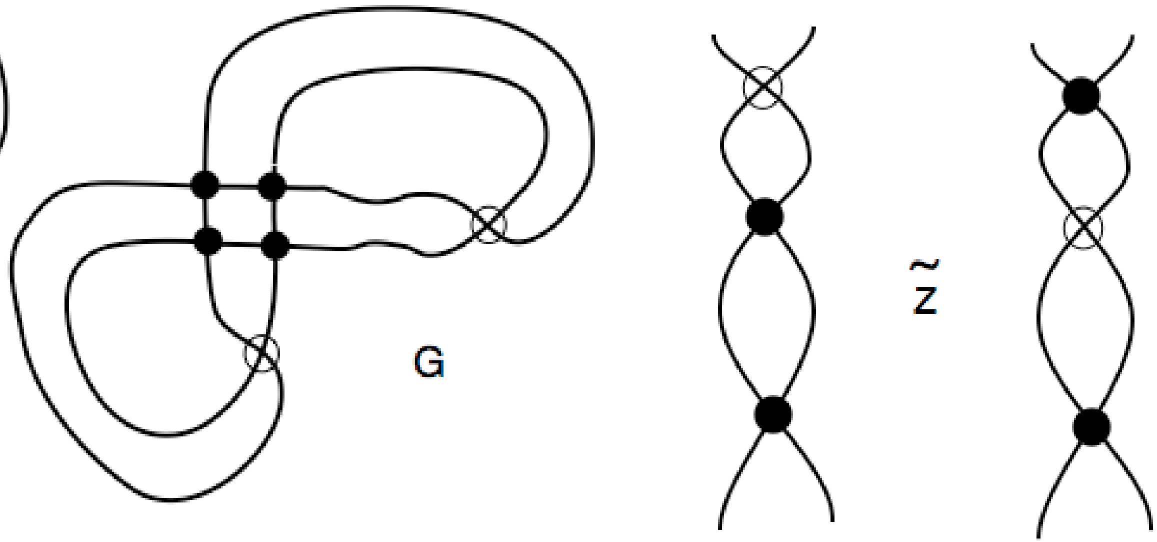
Proof via Parity Bracket.



The standard surface construct has g = 2.

A[S3] = -2K1^2 + K2 + A^4 (1 - 2K1^2 + K2)





This state graph G
has g = 2 and does not reduce under
graphical Z move.

The Parity bracket of S3 has only two terms and includes the graph G. The virtual graph G cannot be reduced by Reidemeister Two moves on its nodes.

Conclusion: The knot S3 has surface genus g = 2.

ARROW POLYNOMIAL

The arrow polynomial is a generalization of the Jones polynomial (bracket polynomial) that takes into account the state structure of oriented diagrams.

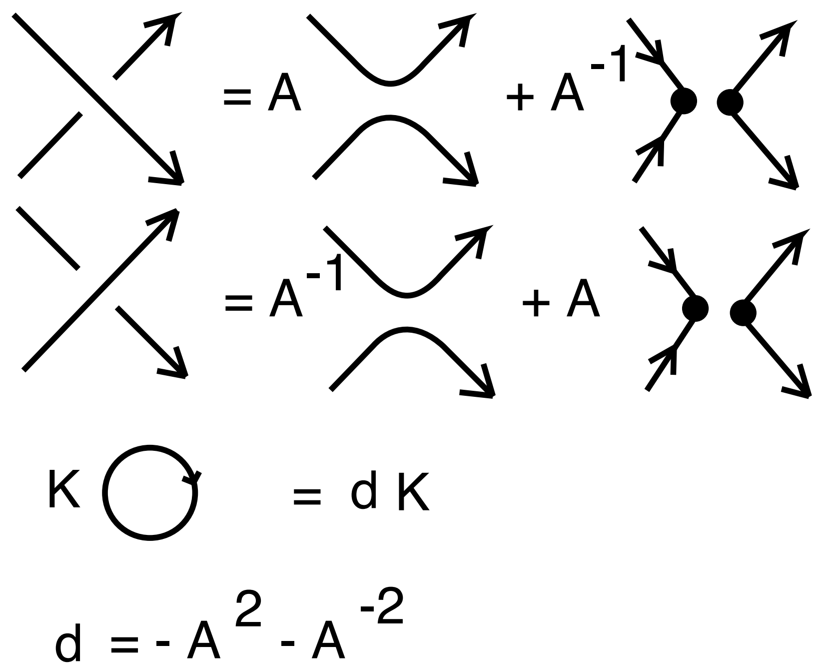
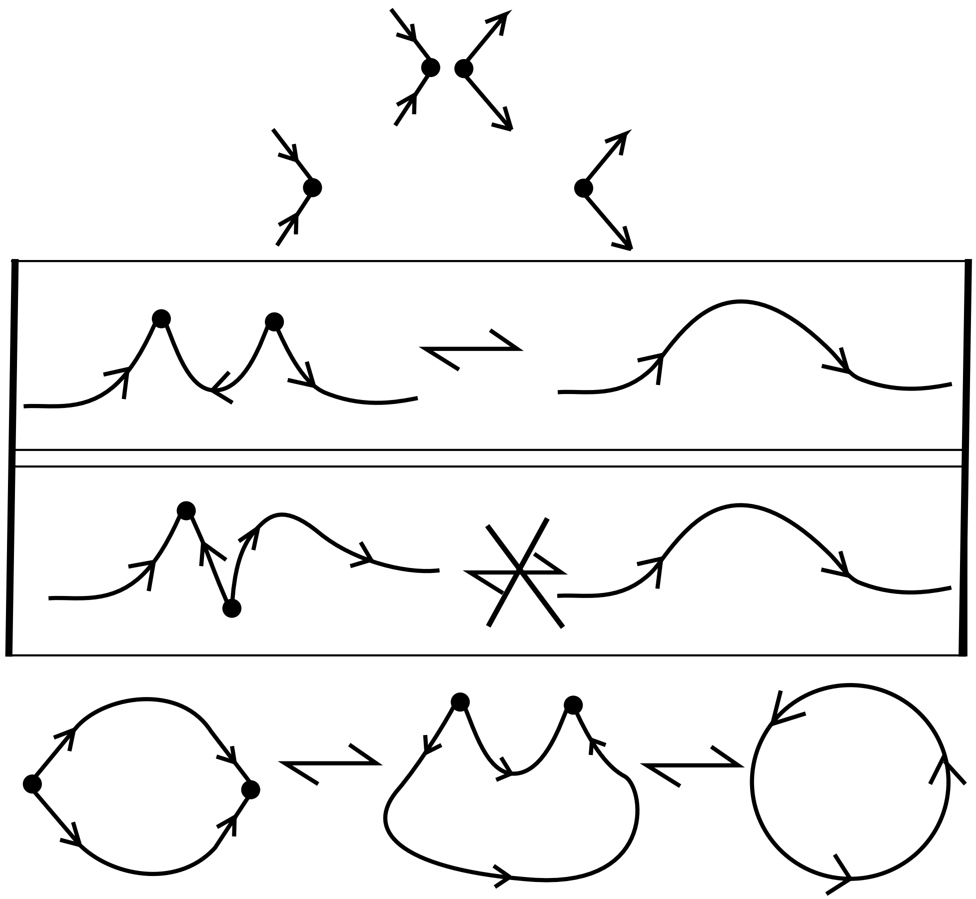
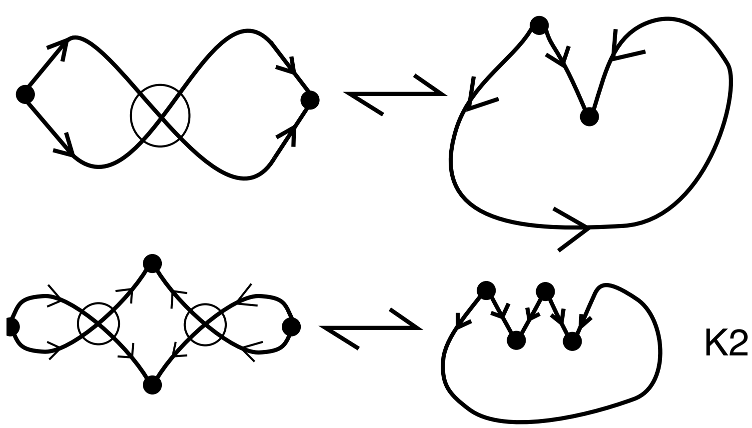


Figure 1: **Oriented Bracket Expansion.**

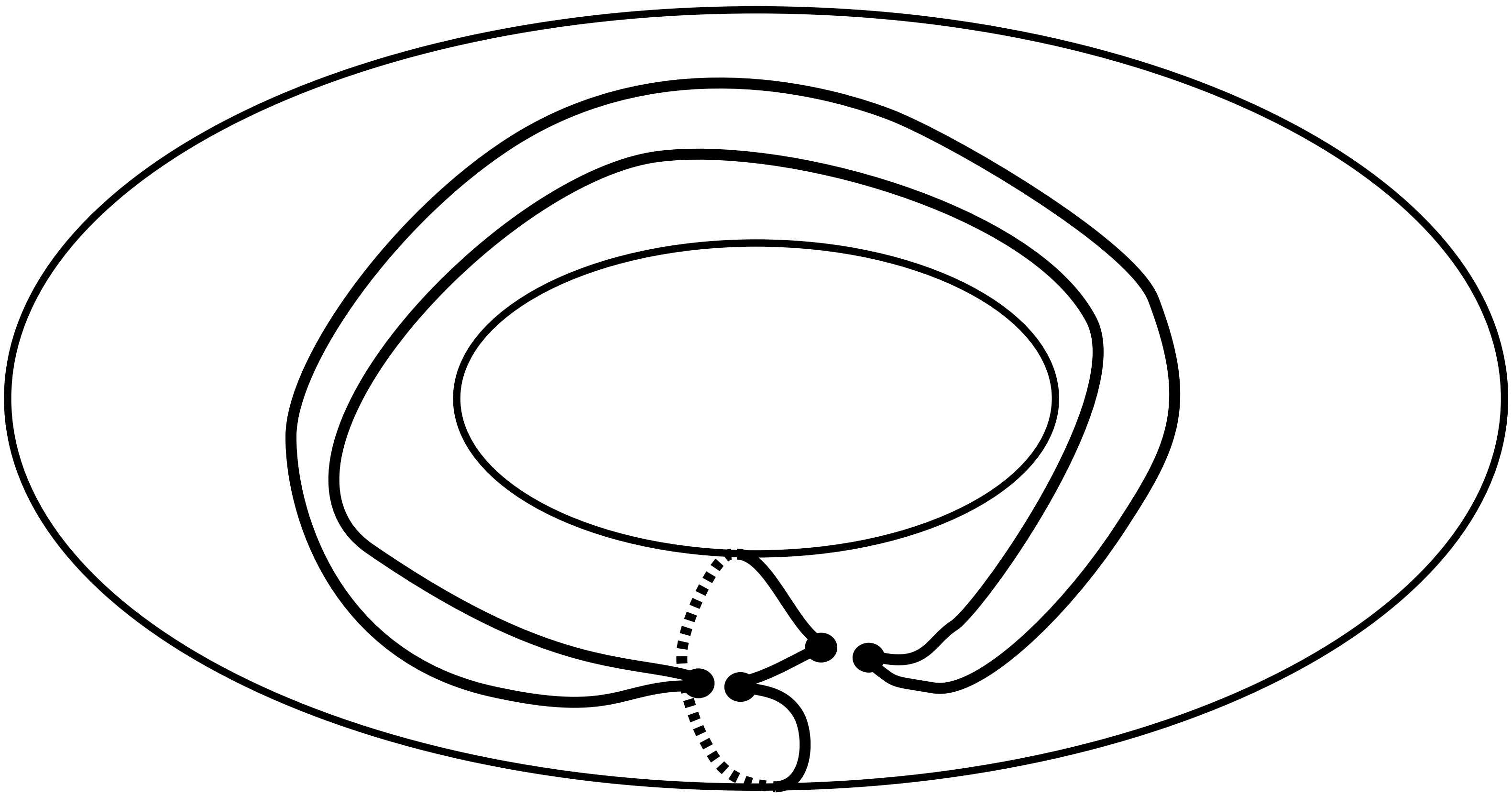


Sufficient for
invariance under
Reidemeister moves



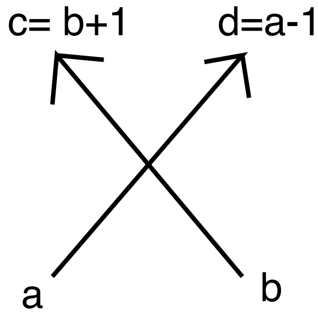
Non trivial reduced loops
do not occur
for classical knots.

Zig-zags survive in higher genus.

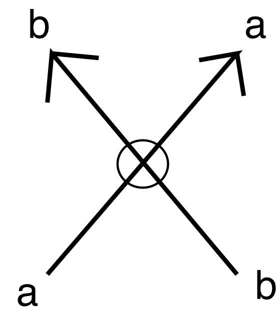


Affine Index Polynomial

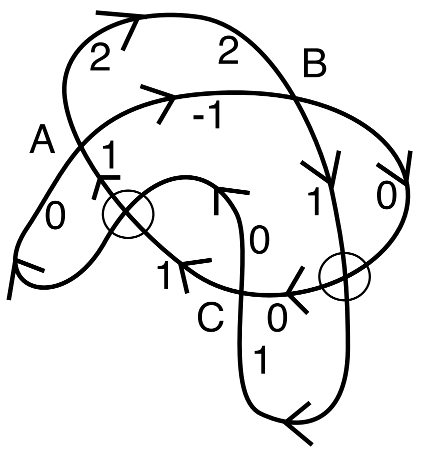
$$P_K(t) = \sum_c sgn(c)(t^{W_K(c)} - 1)$$

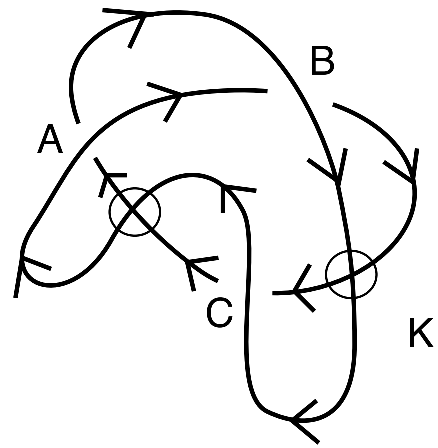


$W_+ = a - c$
$W_- = b - d$



No change at a virtual crossing.



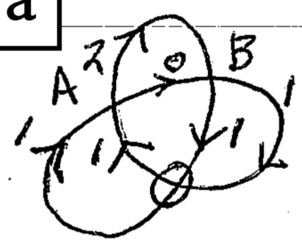


	W_+	W_-
A	-2	+2
B	+2	-2
C	0	0

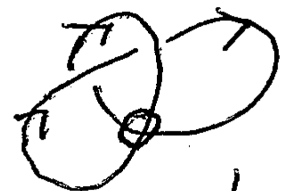
sgn(A) = sgn(B) = +1
sgn(C) = -1
wr(K) = 1

$P_K(t) = t^{-2} + t^{2} - 2$

a

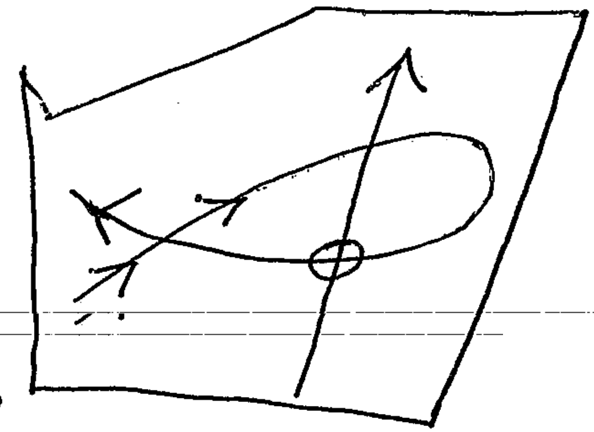


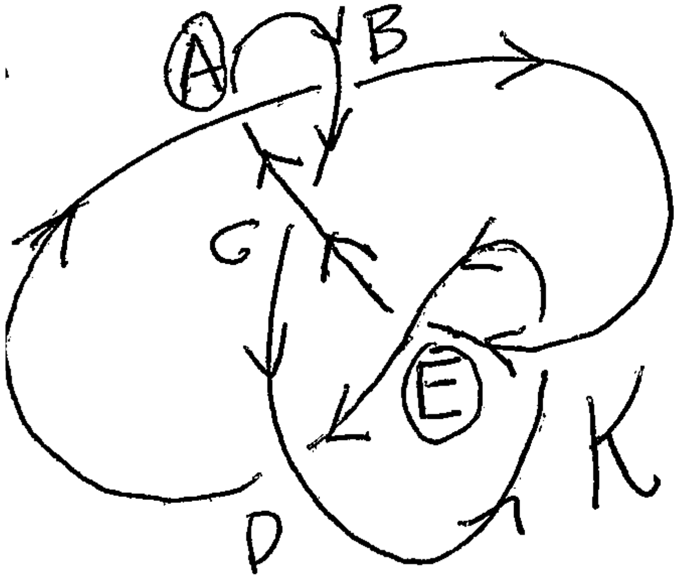
	W_+	W_-
A	−1	1
B	1	−1

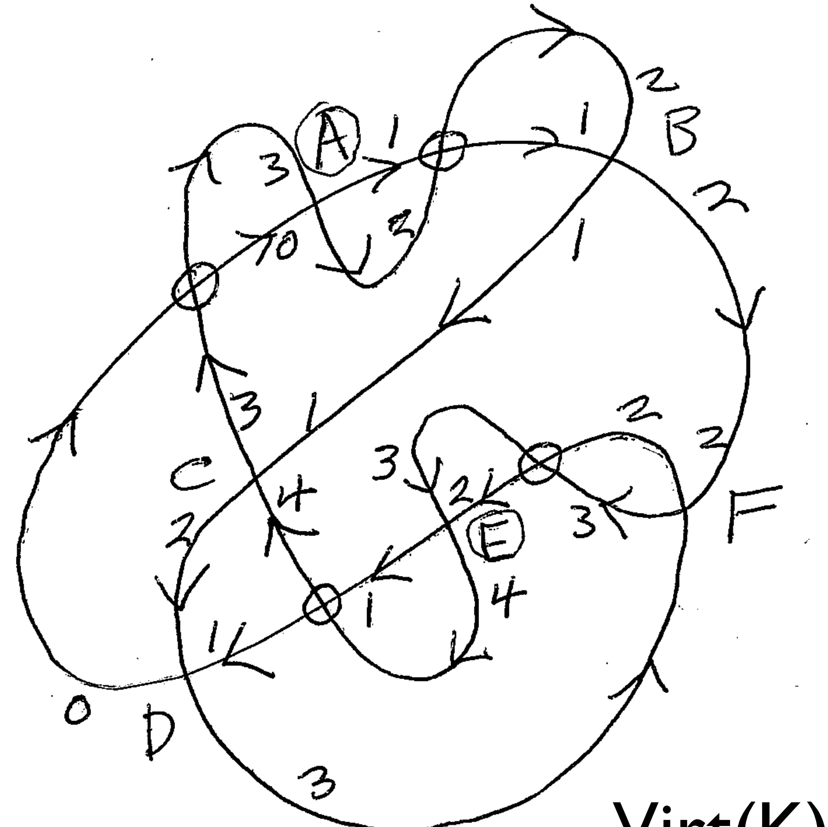


K

$$P_K = t^{-1} + t - 2$$





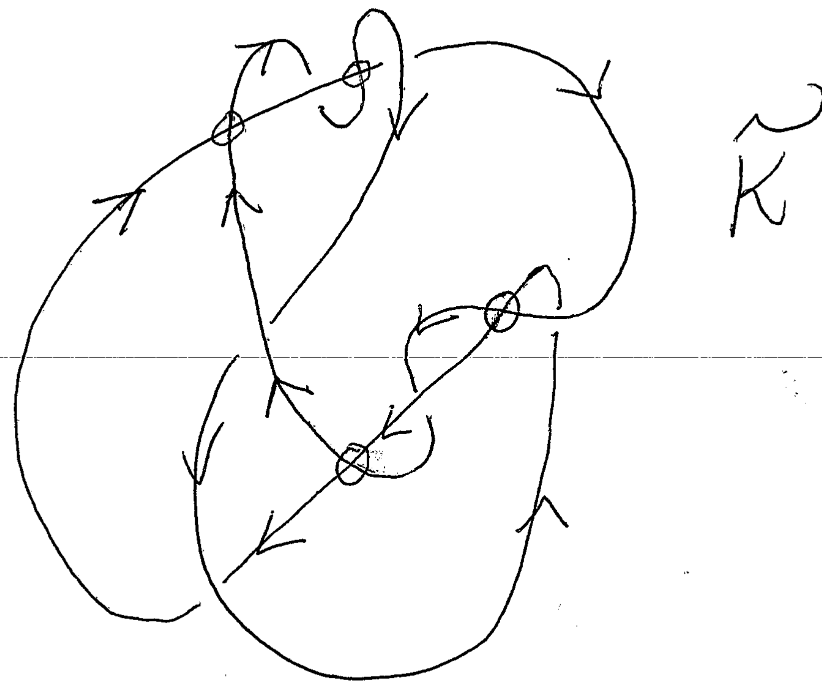


Virt(K)

		W_+	W_-
−	A	2	−2
+	B	0	0
+	C	2	−2
−	D	−2	2
+	E	−2	2
−	F	0	0

$P_{Virt(K)} = 0.$

This one is not detected by the Affine Index Poly.

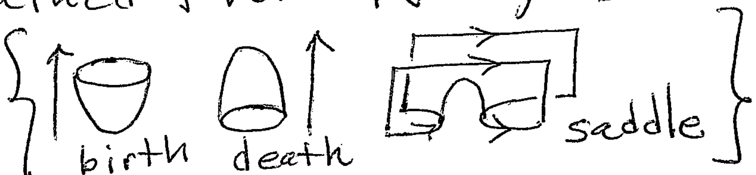


$$\begin{aligned}
\text{arrow}(\tilde{K}) = {} & A^{-8}(K_1^2 - K_1^4) \\
& + A^{8}(K_1^2 - K_1^4) \\
& + 2A^{-4}(K_1^2(1 - 2K_1^2 + K_2)) \\
& + 2A^{4}(K_1^2(1 - 2K_1^2 + K_2)) \\
& + (1 - 6K_1^4 + K_1^2(2 + 4K_2))
\end{aligned}$$

4° Virtual Knot Cobordism

(a) Two (oriented) virtual links K, K' are <u>cobordant</u> if K' can be obtained from K by <u>virtual isotopy</u>

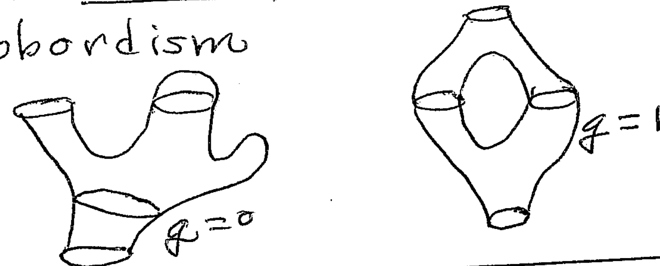
+ {



}

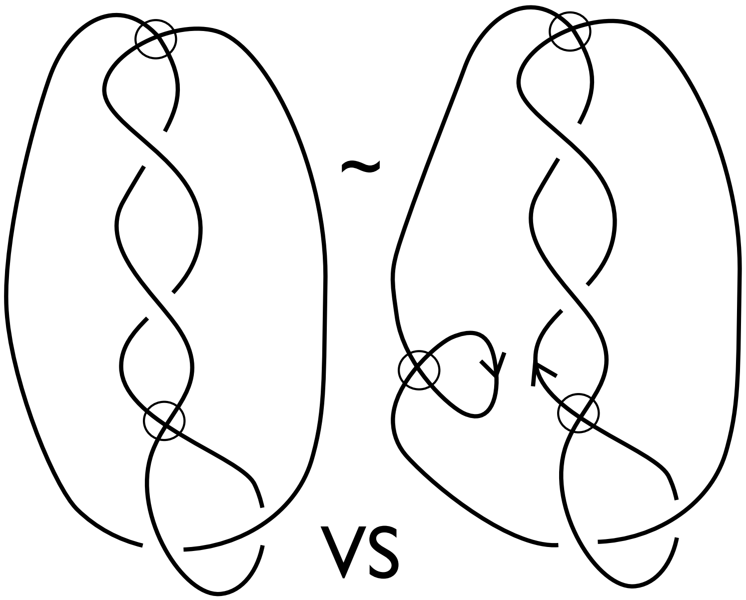
The <u>abstract</u> <u>schema</u> of such a cobordism has a <u>genus</u> g.

e.g.

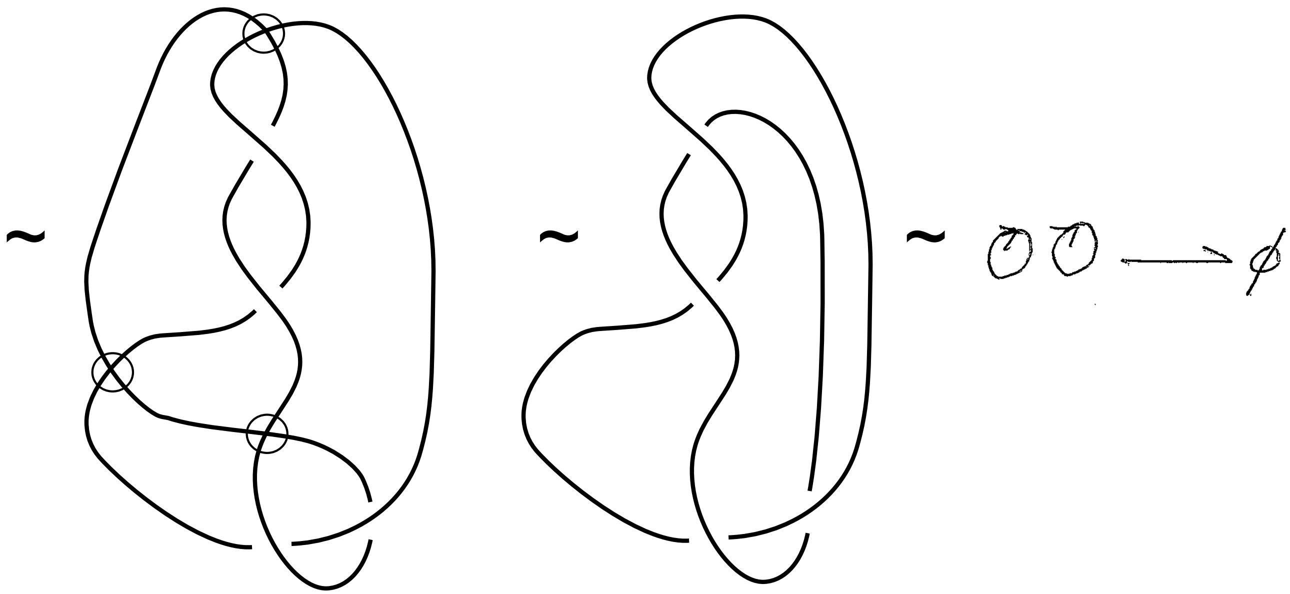


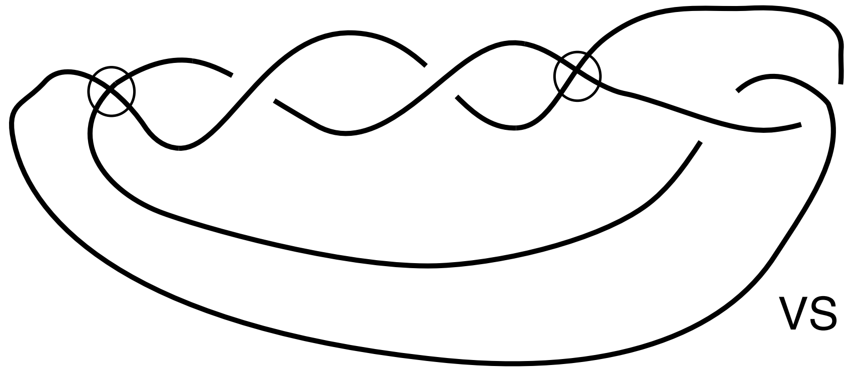
If $K \underset{cob}{\sim} K'$ with genus $g = 0$, we say that K and K' are <u>concordant</u>.

A virtual knot is <u>slice</u> if it is <u>concordant to O</u>.

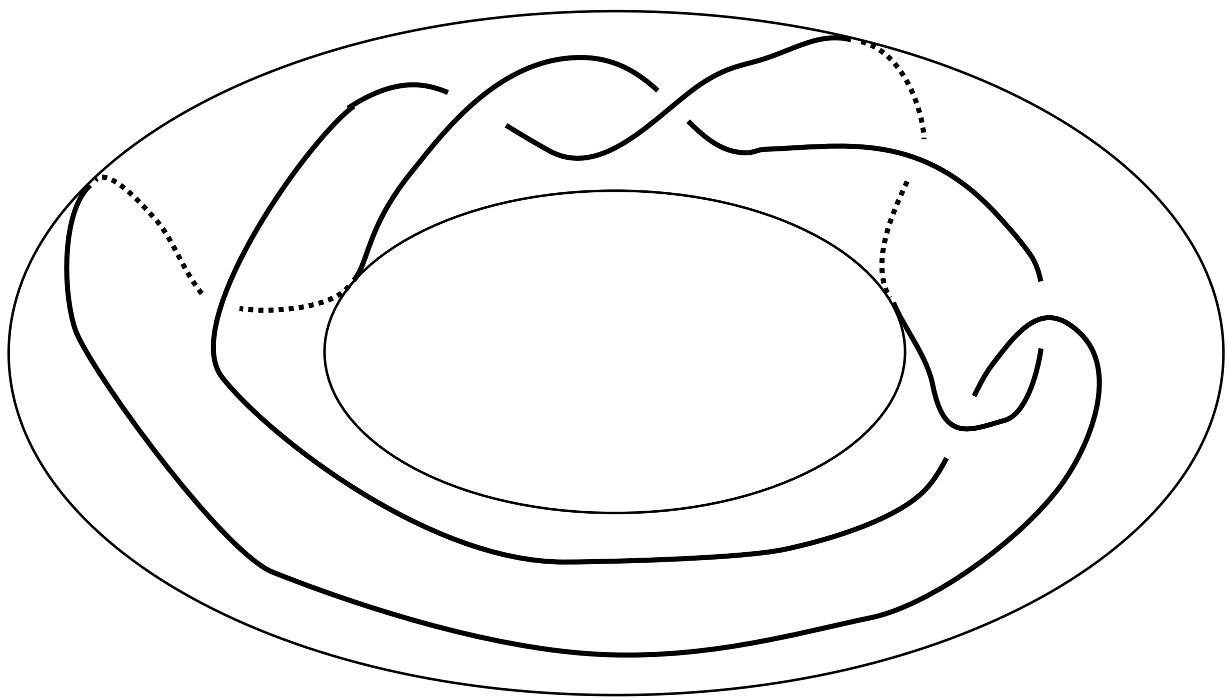


The virtual stevedore's knot VS is slice.

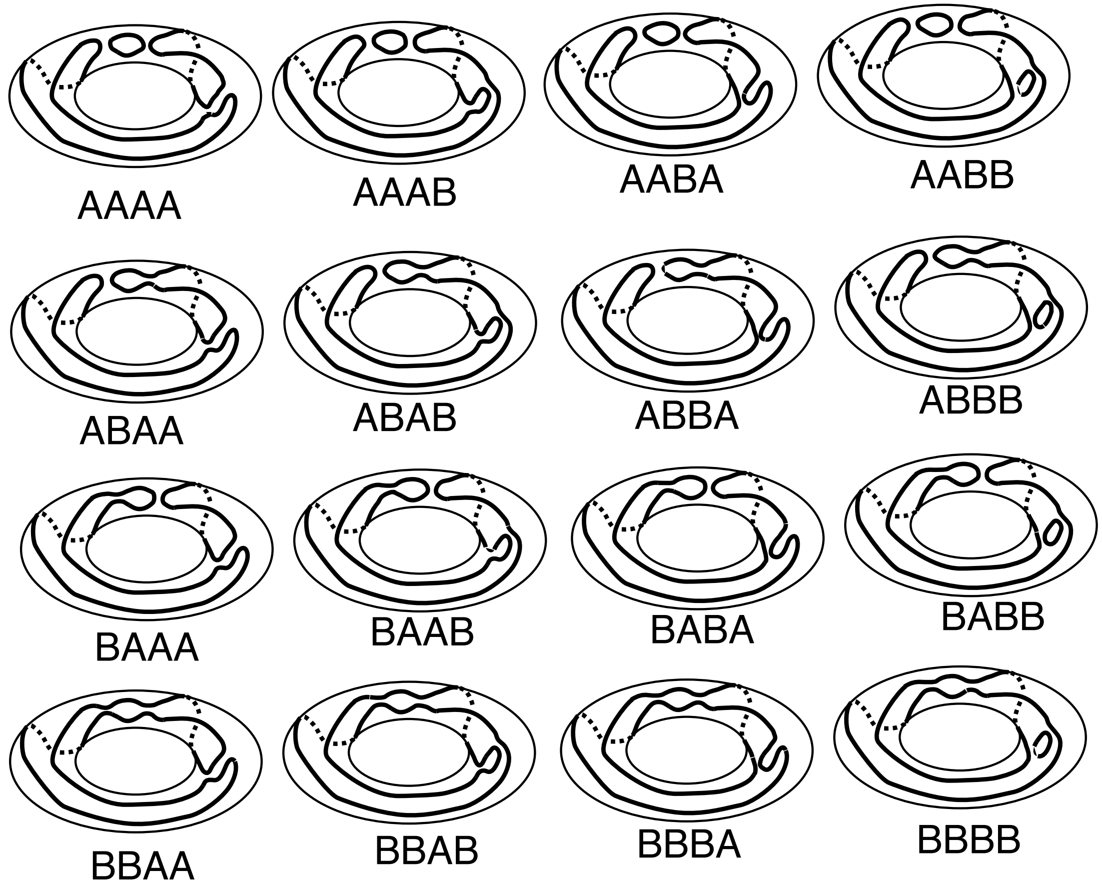




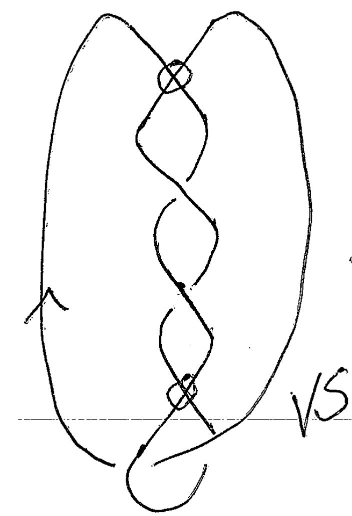
VS

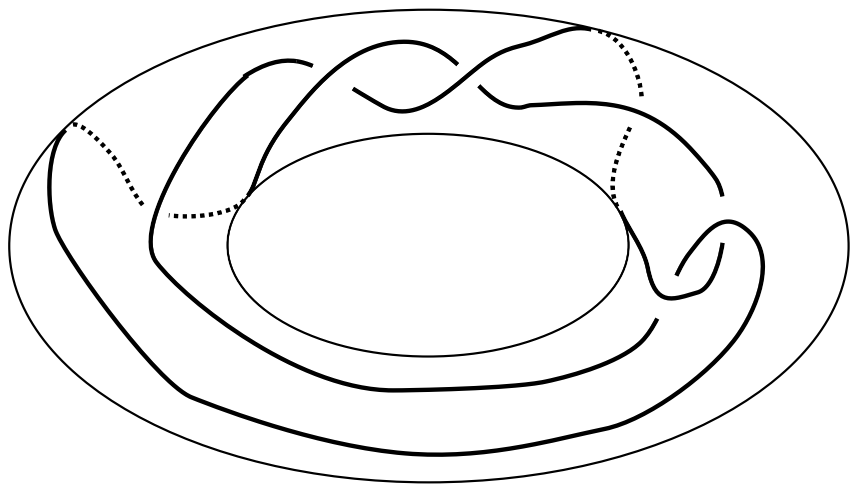


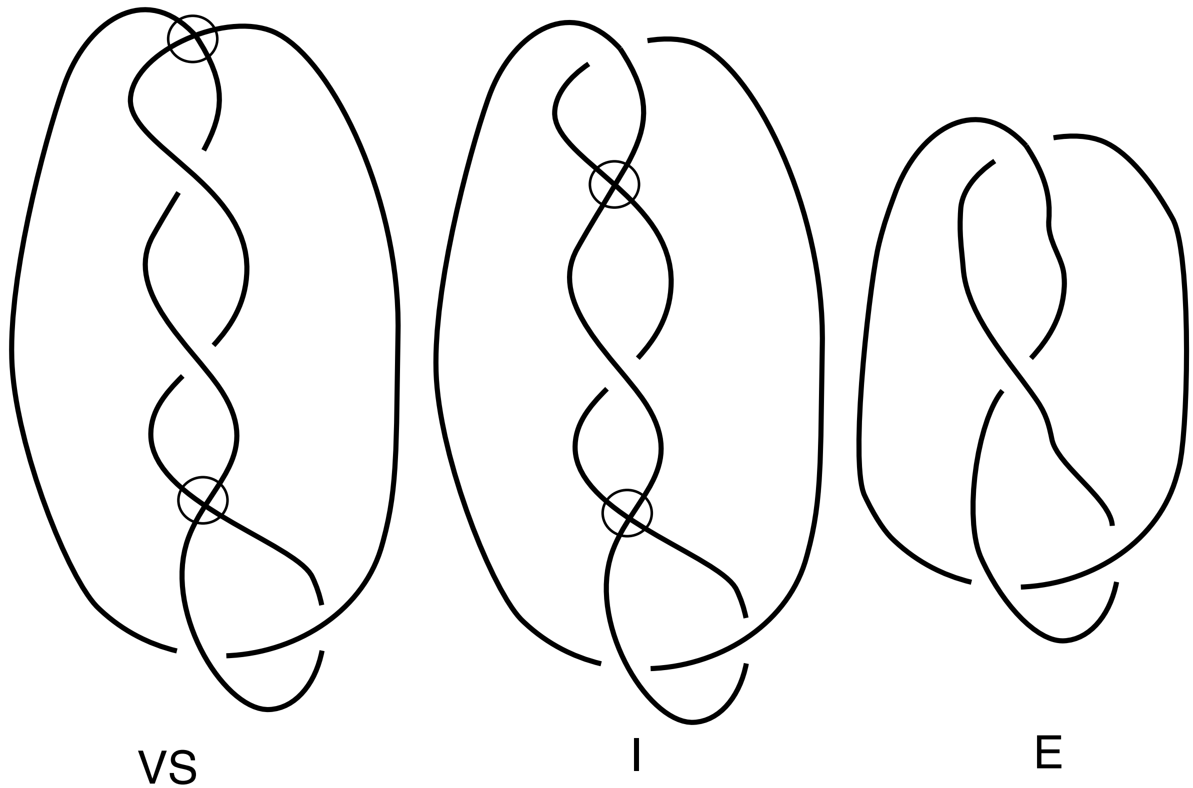
VS on a torus.



Virtual Stevedore
is not
classical.



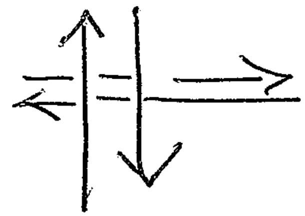
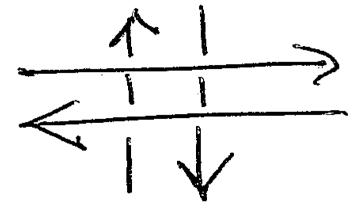


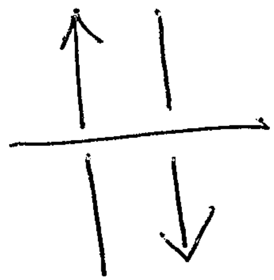
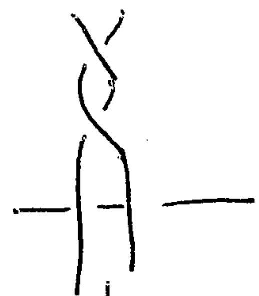


$$\langle VS \rangle = \langle I \rangle = \langle E \rangle = A^{-8} - A^{-4} + 1 - A^{4} + A^{8}$$

The knot VS has bracket polynomial equal to the bracket polynomial of the classical figure eight knot diagram E. This implies that VS is not a connected sum.

Virtual Band Passing

VKT +  $\underset{P}{\sim}$ 

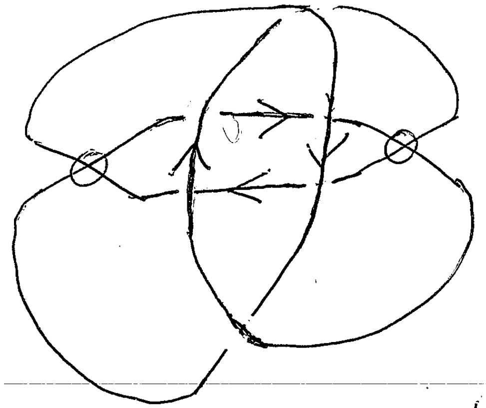
 $\underset{P}{\sim}$ 

Classically there are <u>two</u> pass classes for knots.

{} & {O}.

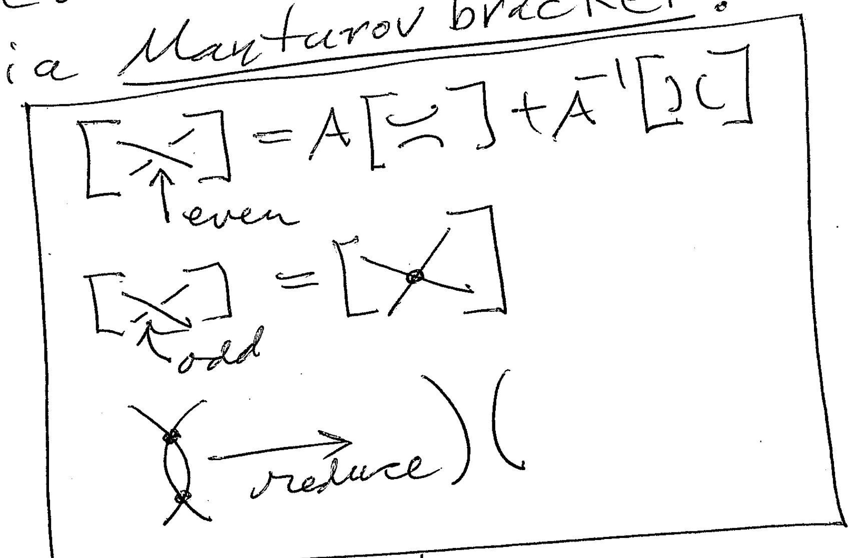
(a) What are the pass-classes for virtuals?

(b) Relate to cobordism.



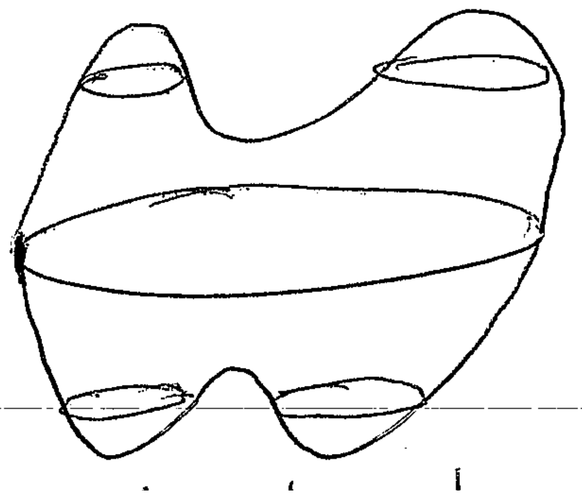
K

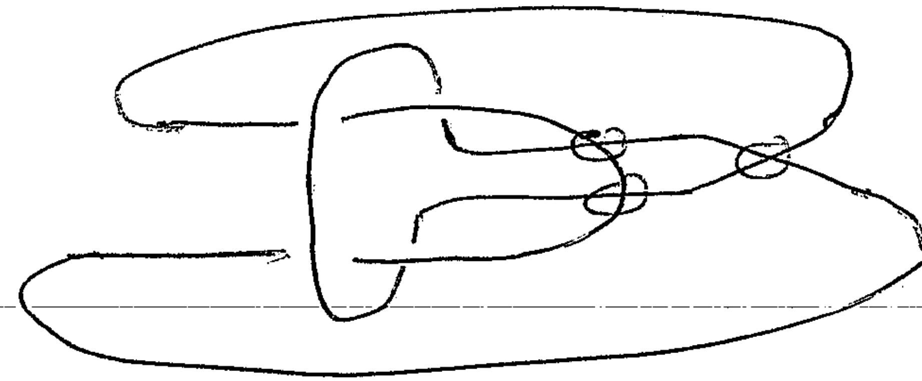
K has a non-trivial
<u>odd pass class</u>
(pass only if all 4 crossings
are odd crossings)
via <u>Manturov bracket</u>.

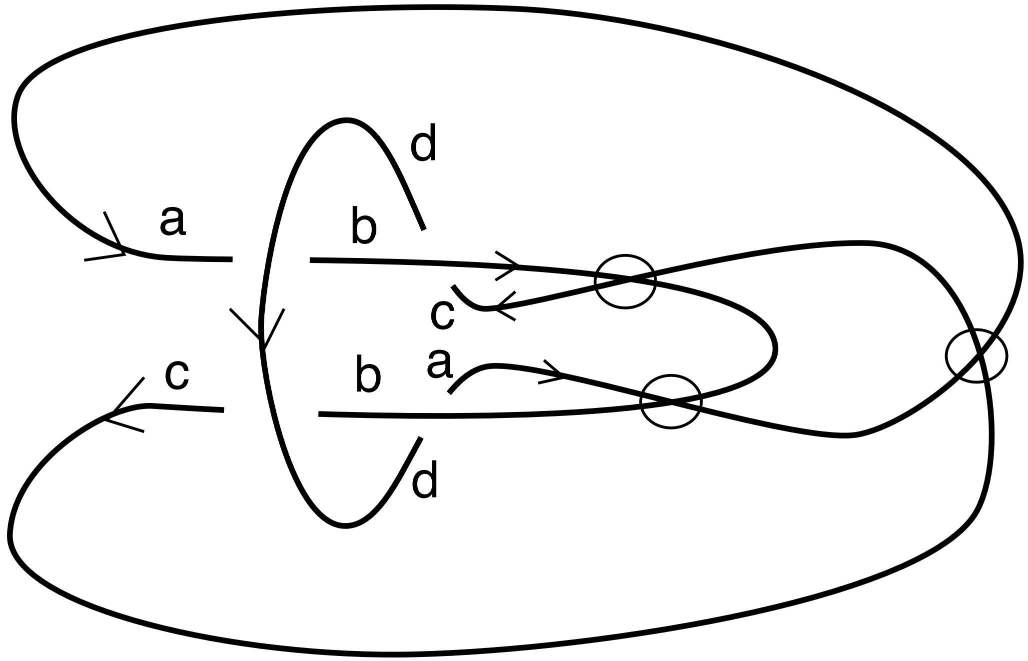


<u>Manturov Bracket is an
invariant of odd passing.</u>

Virtual Surfaces in $\mathbb{R}^4$ Via Movies







$$d^{-1}ad = b, dbd^{-1} = c, b^{-1}cb = d, bdb^{-1} = a$$

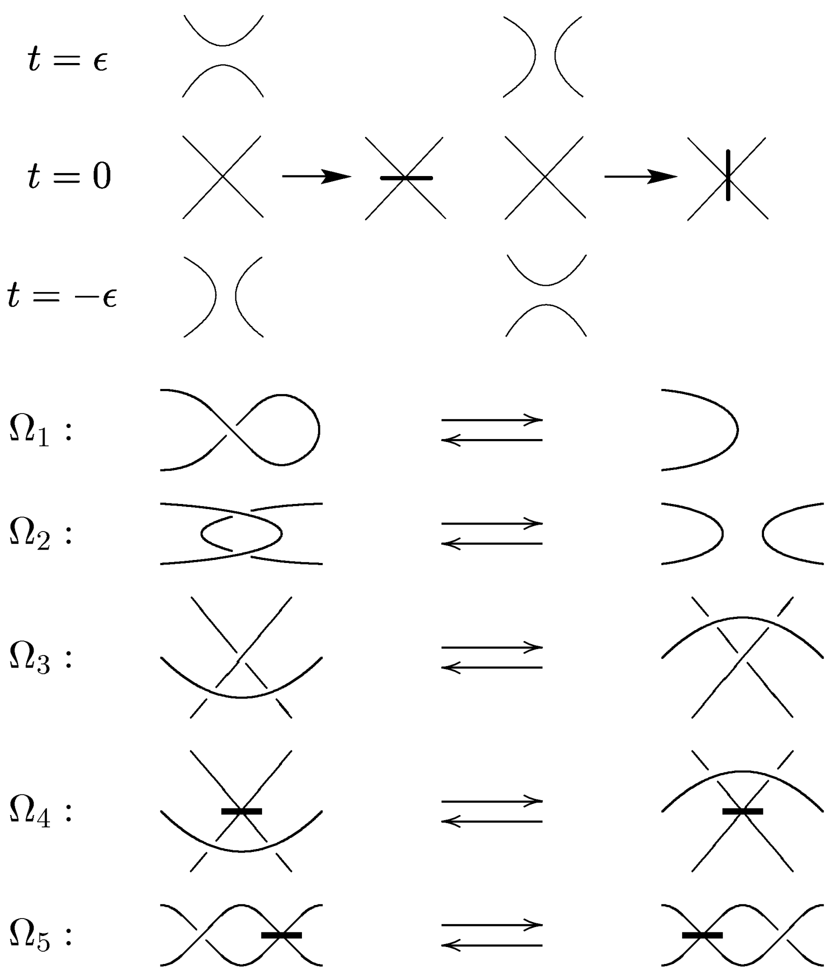
Whence, $a = c$. Thus

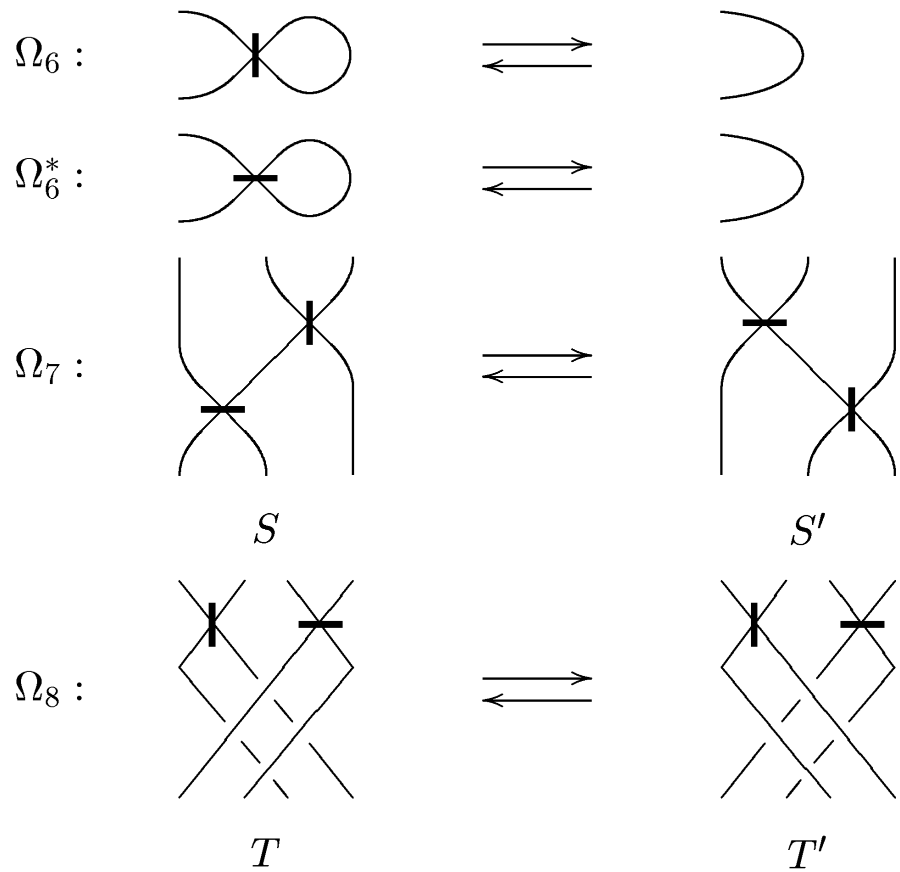
$$\pi(VS) = (a, d | d^{-1}ad = b, b^{-1}ab = d)$$

$$= (a, b | aba^{-1} = bab^{1}).$$

Since in the original presentation, $a = c$, we see that this is the group of the corresponding virtual $2 - sphere$ in four-space.

Equivalence of Virtual Surfaces via the Yoshikawa Moves



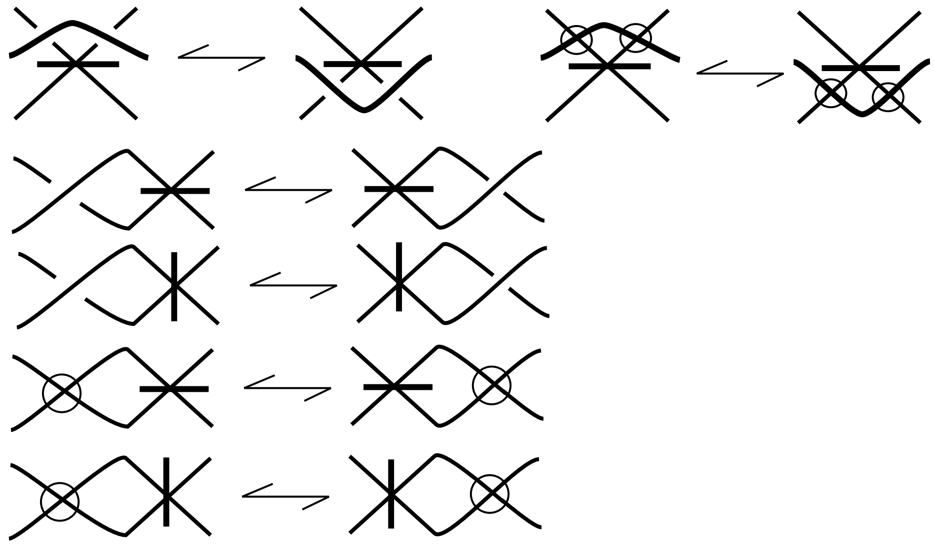


Plus Detour Moves for virtual crossings.

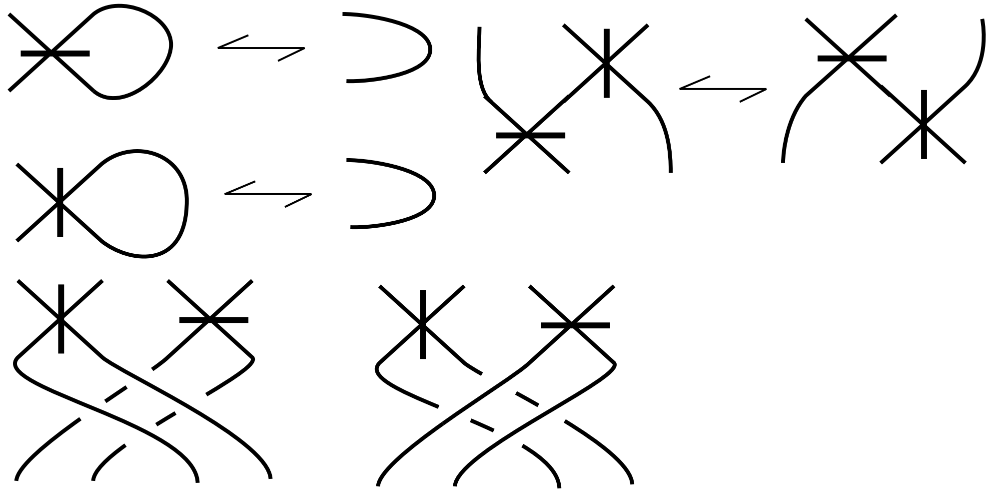
And Virtual Moves -- Next Slide

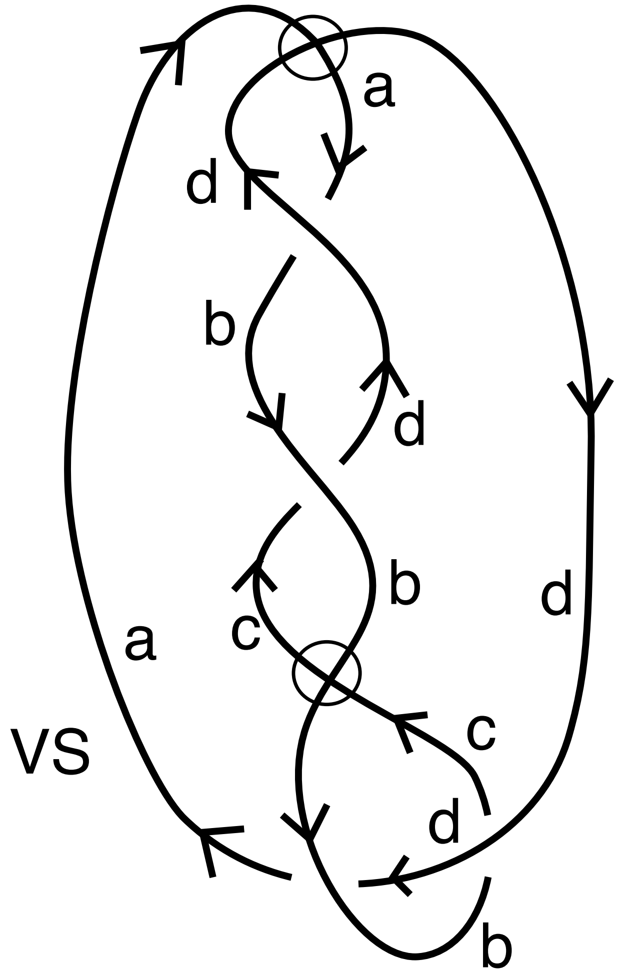
1. Reidemeister Moves and Virtual Moves (Detour).

2. Moves on Markers.



3. Yoshikawa Moves





$d^{-1} a\, d = b$

$d\, b\, d^{-1} = c$

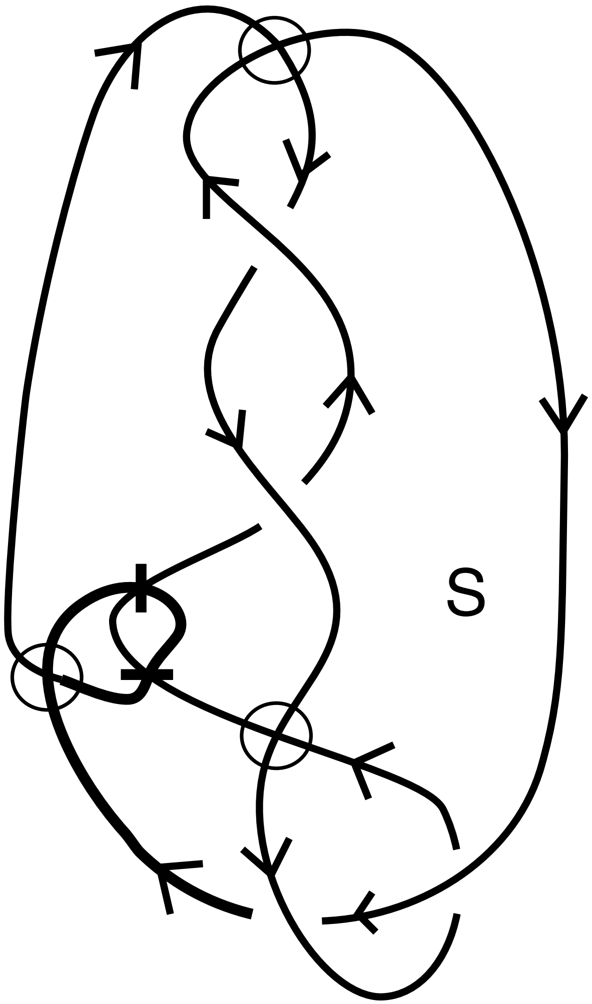
$b^{-1} c\, b = d$

$b\, d\, b^{-1} = a$

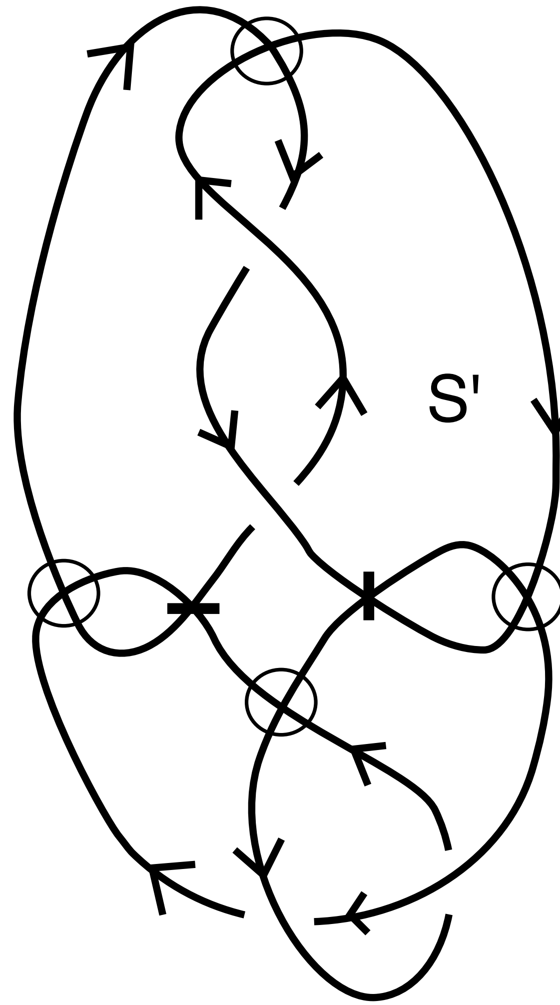
Therefore c = a and

$d\, b\, d^{-1} = b\, d\, b^{-1}$

Fundamental Group (VS) = (d,b| $d\, b\, d^{-1} = b\, d\, b^{-1}$) .



FundGrp(S) = FundGrp(VS)



FundGrp(S') = Z.

Advantage of using Yoshikawa moves is computability and formulation of invariants.

Fundamental group or quandle via movies is an invariant.

Bracket generalizations of S.Y. Lee will generalize to virtual surfaces.

Does the Yoshikawa move definition for virtual surfaces correspond to Jonathan Schneider and Yasushi Takeda definitions via generalizations of Roseman moves?

There is more to come.