Applications of M-theory Uplifted Desingularized Conifold Geometries Relevant to Holographic Thermal QCD at Finite Coupling

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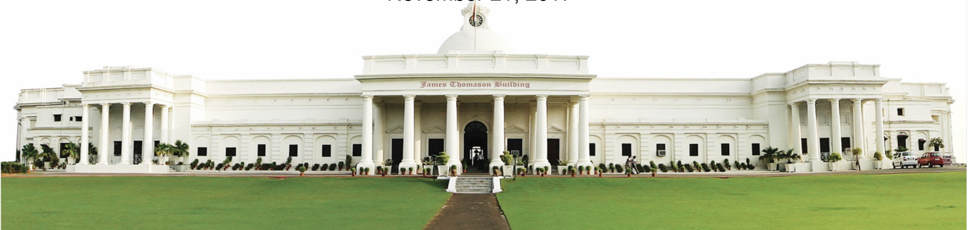


Outline

- Introduction leading up to the ‘MQGP Limit’ and M-theory uplift of the UV-complete holographic type IIB dual of large-N thermal QCD of K. Dasgupta et al [2009]
- Lattice-compatible Deconfinement Temperature, hints of a $D = 1 + 1$ Luttinger Liquid via the temperature dependence of electrical conductivity σ, Einstein’s relation and temperature dependence of Ouyang embedding paramter
- Speed of Sound via scalar metric perturbation modes, Diffusion Coefficient up to NLO in N via vector metric perturbations, NLO Corrections in N to η and $\frac{\eta}{s}$, thermal/electrical conductivity and Wiedemann-Franz law and $D = 1 + 1$ Luttinger Liquid up to LO in N
- Lattice-Compatible Glueball $(0^{++}, 0^{--}, 1^{++}, 2^{++})$ type IIB/IIA/M-theory Spectroscopy
- Summary

Part I

The Background

Aim

- Explore:
    1. Holographic condensed matter physics related issues: Thermodynamics and Transport properties,
    2. Holographic phenomenology related issues: Glueball spectroscopy,

  of the strongly coupled QGP phase of thermal QCD, using a top-down holographic model.
- In usual AdS/CFT correspondence, the gauge theory is maximally supersymmetric and also conformal
- QCD is neither supersymmetric nor conformal: generalization of AdS/CFT correspondence is required.

Generalization AdS-CFT correspondence

- A particular choice of *D*-brane configuration in specific target spacetime: both supersymmetry and conformal invariance are broken at the very beginning.
- *D*-branes placed in (singular/resolved/deformed/resolved warped deformed) conifold backgrounds; for example I. R. Klebanov, E. Witten [2000], I. R. Klebanov, M. J. Strassler [2000], K. Dasgupta et al[2009]etc.
- Here we have considered the set up by K. Dasgupta et al [2009] which is UV complete and to the best our knowledge is closest to thermal QCD.

Quick Review of Conifolds

- A conifold is a six dimensional cone embedded in $\mathbb{C}^4$.
- The base of a conifold has the topology of $S^2 \times S^3$.
- There is a singularity at the tip of a conifold, which can be removed by
  - a blown up S^3 (Deformed conifold)
  - a blown up S^2 (Resolved conifold).
- Moreover there can be both resolution and deformation at the tip which is particularly important in our case, known as resolved and deformed conifold geometry. The most general metric is given as:

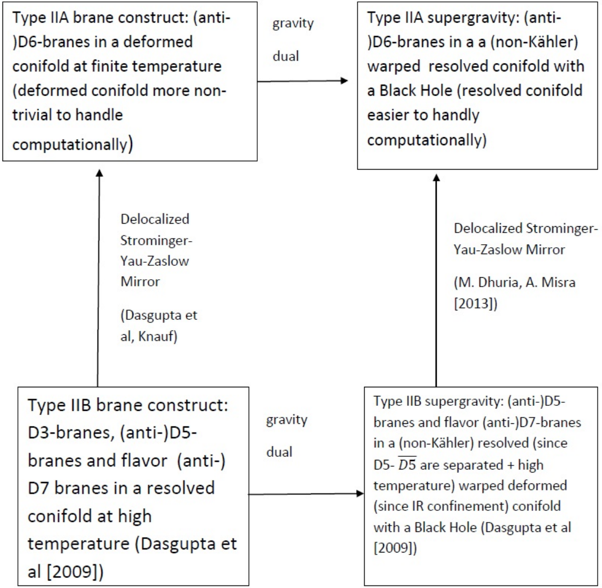
$$
\begin{aligned}
ds^2 &= F_3(r)dr^2 + F_4(r)\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \\
&\quad + F_1(r)\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + F_2(r)\left(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) \\
&\quad + 2b(r)\Big[\cos\psi\left(d\theta_1 d\theta_2 + \sin\theta_1 \sin\theta_2 d\phi_1 d\phi_2\right) - \sin\psi\left(\sin\theta_2\, d\theta_1 d\phi_2 - \sin\theta_1\, d\theta_2 d\phi_1\right)\Big]
\end{aligned}
$$

- Singular conifold: $F_1(r) = F_2(r)$, $b = 0$
- Deformed conifold: $F_1(r) = F_2(r)$, $b \neq 0$
- Resolved conifold: $F_1(r) \sim F_2(r) = a^2 (a \equiv \text{resolution parameter})$, $b = 0$.

K. Dasgupta et al [2009]'s set up

Brane Construction

1. A stack of N $D3$ branes were considered at the tip of a conifold along with a stack of M $D5$ branes wrapping vanishing two cycle of the conifold. To ensure the cancelation of UV logarithmic divergence by turning off all the three form fluxes, a stack of M anti-$D5$ branes were also placed around the antipodal point relative to the location of regular $D5$ branes on the blown-up S^2 of the cone.
   - 1.1 For very small value of r, the gauge theory is confining: (Deep) IR region
   - 1.2 anti-$D5$ branes are added at the boundary of IR-UV interpolating region/UV
   - 1.3 the three-form fluxes decay to zero in the UV region
2. To include fundamental quarks, a stack of N_f $D7$ branes were introduced via Ouyang embedding P.Ouyang [2003] in the UV dipping all the way into the IR. Again to cancel diverging fluxes, N_f anti-$D7$ branes were also placed in the UV dipping into the UV-IR interpolating region
3. To introduce temperature a black hole is also placed in the (super)gravity dual



Dasgupta et al's [2009] Type IIB brane construct and the gravity dual of large-N QCD at high temperature, and their (Delocalized [SYZ]) mirrors

- The whole set up results into a resolved warped deformed conifold. The supergravity limit in this case is given as,

$$(N, M, N_f, g_s N, g_s M) \sim \mathrm{Large}$$
$$(g_s, g_s M/N, g_s N_f, M/N) \sim \mathrm{Small.}$$

- The type IIB supergravity solution on this background is given as:

$$ds^2 = \frac{1}{\sqrt{h}}\left(-g_1 dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{h}\left[g_2^{-1} dr^2 + r^2 d\mathcal{M}_5^2\right],$$

where the black hole functions g_i in the above limit are of the form:

$$g_{1,2}(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}\left(\frac{g_s M^2}{N}\right).$$

- The warp factor that includes the back-reaction is given in IR as:

$$h = \frac{L^4}{r^4}\left[1 + \frac{3 g_s M^2}{2\pi N}\log r\left\{1 + \frac{3 g_s N_f}{2\pi}\left(\log r + \frac{1}{2}\right) + \frac{g_s N_f}{4\pi}\log\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\right\}\right],$$

The 'MQGP Limit' and M-theory uplift

- In M. Dhuria, A. Misra[2013], the authors had considered the following limit:

$$\text{MQGP limit}: \frac{g_s M^2}{N} \ll 1, g_s N \gg 1, \text{finite } g_s, M$$
$$\text{effected by}: g_s \sim \epsilon^d, M \sim (\mathcal{O}(1)\epsilon)^{-\frac{3d}{2}}, N \sim (\mathcal{O}(1)\epsilon)^{-39d}, \epsilon \lesssim 1, d > 0.$$

- This is due to the fact that here we will be dealing with the strongly coupled thermal system such as sQGP. Hence we not only have to consider a large t'Hooft coupling but also a finite string coupling. The finiteness of the string coupling necessitates addressing the same from an M theory perspective. This was the reason for coining the name: 'MQGP limit'.

- In M. Dhuria, A. Misra[2013], the authors first constructed the type IIA mirror of the above type IIB background via 'delocalized' (i.e. around some fixed values of the angular coordinates as non-zero deformation ['$b(r)$'] does not permit a T^3-worth of isometries [$\exists\phi_{1,2} \to \phi_{1,2}$ + constant, $\nexists\psi \to \psi$ + constant]) 'Strominger-Yau-Zaslow' prescription of constructing a mirror via three T dualities and its 11-dimensional M-theory uplift. The uplift, strictly speaking involves a seven-fold of 'G_2-structure' (A.Misra in A.Misra, KS [2015]); can be thought of as black $M3$-brane metric (reminiscent of the $M3$-brane solution of Cvetic et al [2001]) obtained from a black $M5$-brane wrapping a two cycle M.Dhuria, A.Misra [2014]. In the IR-UV interpolating region/UV, the $D = 11$ space-time is a warped product of $AdS_5(\mathbb{R}^{1,3} \times \mathbb{R}_{>0})$ and $\mathcal{M}_6(\theta_{1,2}, \phi_{1,2}, \psi, x_{10})$
- In the K. Sil, A. Misra[2015] we chose to work around particular values of θ_1 and θ_2, given by:

$$\theta_1 \sim \frac{\alpha_{\theta_1}}{N^{1/5}} \qquad \theta_2 \sim \frac{\alpha_{\theta_2}}{N^{3/10}},$$

and $\psi \sim 2n\pi, n = 0, 1, 2$ whereat the five dimensional spacetime defined by $\{t, x_{1,2,3}, r\}$ decouples from the six dimensional internal space defined by $\{\theta_{1,2}, \phi_{1,2}, \psi, x_{11}\}$.

- The five dimensional black *M*3 brane metric is given as,

$$ds_5^2 = e^{-\frac{2\Phi_{IIA}}{3}}\left(-g_{tt}dt^2 + g_{\mathbb{R}^3}\sum_{i=1}^{3}dx_i^2 + g_{rr}dr^2\right),$$

where at the above mentioned values of θ_1 and θ_2, the metric components and the dilaton factor are given as,

$$g_{tt} = \frac{(r^4 - r_h^4)}{r^2\sqrt{4\pi g_s N}}\left\{1 + \frac{3g_s M^2}{4\pi N}\left[1 + \frac{3g_s N_f}{2\pi}\left(\log\ r + \frac{1}{2}\right) + \frac{g_s N_f}{4\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right)\right]\log\ r\right\}$$

$$g_{\mathbb{R}^3} = \frac{r^2}{\sqrt{4\pi g_s N}}\left\{1 - \frac{3g_s M^2}{4\pi N}\left[1 + \frac{3g_s N_f}{2\pi}\left(\log\ r + \frac{1}{2}\right) + \frac{g_s N_f}{4\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right)\right]\log\ r\right\}$$

$$g_{rr} = \frac{r^2\sqrt{4\pi g_s N}}{r^4 - r_h^4}\left(\frac{6a^2 + r^2}{9a^2 + r^2}\right)\left\{1 - \frac{3g_s M^2}{4\pi N}\left[1 + \frac{3g_s N_f}{2\pi}\left(\log\ r + \frac{1}{2}\right) + \frac{g_s N_f}{4\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right)\right]\log\ r\right\}$$

and Φ_{IIA} is the triple $T - dual$ of the type IIB Φ_{IIB}:

$$e^{-\Phi_{IIB}} = \frac{1}{g_s} - \frac{N_f}{8\pi}\log(r^6 + a^2 r^4) - \frac{N_f}{2\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right).$$

Part 2

Deconfinement Temperature and hints of a $D = 1 + 1$ Luttinger Liquid

KS, A.Misra [2016]

Calculation of Deconfinement temperature T_c

- QCD describes the interaction between Quarks and Gluons. It has an interesting phase structure. The temperature at which the quarks and the gluons are screened from each other is known as the deconfinement temperature. This deconfinement temperature can be calculated using dual gravitational background.
- A thermal state in the field theory side corresponds to two different solutions in the gravity picture:
  - The thermal background, where temperature is introduced as the inverse of the periodicity of the Euclidian time.
  - The Black hole background, where the temperature is proportional to the horizon radius.

- The metric for the black hole background in our set up is given as

$$ds^2_{BH} = G^{\mathcal{M}}_{tt} dt^2 + G^{\mathcal{M}}_{\mathbb{R}^3} \sum_{i=1}^{3} dx_i^2 + G^{\mathcal{M}}_{rr} dr^2,$$

  where we have defined $G^{\mathcal{M}}_{tt} = e^{-\frac{2\Phi}{3}} g_{tt}$, $G^{\mathcal{M}}_{\mathbb{R}^3} = e^{-\frac{2\Phi}{3}} g_{\mathbb{R}^3}$ and $G^{\mathcal{M}}_{rr} = e^{-\frac{2\Phi}{3}} g_{rr}$ with g_{tt}, $g_{\mathbb{R}^3}$ and g_{rr} as discussed quoted earlier.

- The thermal background metric in our set up can be read off from the BH gravity dual by setting $r_h = 0$; we further assume $a(r_h = 0) = 0$.

- The singularity at $r = 0$ of the above metric is fixed by a thermal IR cut-off provided by the Ouyang embedding parameter μ. More specifically the IR cut-off r_0 is related to the embedding parameter μ as $|\mu|^{2/3} = \delta r_0$, where δ is a positive constant and is greater than one

- The non-compact radial direction can be partitioned for thermal and black hole background as follows:
  (a) For the thermal background
  • $r = r_0$ is the thermal IR cut-off - this is the point from where the redial direction starts
  • $r = \mu^{2/3} = \delta r_0$ is the end point of the IR-UV interpolating region or this is where the UV region begins
  • $r \to \infty$ is the far UV region
  (b) For the black hole background
  • $r = r_h$ is the horizon and it is the starting point of the radial direction.
  • $r = \mathcal{R}_{D5/\overline{D5}}$(average $D5 - \overline{D5}$ separation)$= \sqrt{3}a$ is the point where IR-UV interpolating region ends or the UV region starts.

  • $r \to \infty$ is again the far UV region.
- The dilaton profile for the two backgrounds is given as,

$$
\begin{aligned}
(a) \quad & \text{Black hole background :} \\
& e^{-\Phi} = \frac{1}{g_s} - \frac{N_f}{8\pi}\log(r^6 + a^2 r^4) - \frac{N_f}{2\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right), \ r < \mathcal{R}_{D5/\overline{D5}}, \\
& e^{-\Phi} = \frac{1}{g_s}, \ r > \mathcal{R}_{D5/\overline{D5}}; \\
(b) \quad & \text{Thermal background :} \\
& e^{-\Phi} = \frac{1}{g_s} - \frac{3N_f}{4\pi}\log r - \frac{N_f}{2\pi}\log\left(\frac{\alpha_{\theta_1}\alpha_{\theta_2}}{4\sqrt{N}}\right), \ r < |\mu|^{\frac{2}{3}}, \\
& e^{-\Phi} = \frac{1}{g_s}, \ r > |\mu|^{\frac{2}{3}},
\end{aligned}
$$

where the resolution parameter a is defined as before.

- For the calculation of T_c, we consider the Einstein-Hilbert (EH) action along with the Gibbons-Hawking York surface term, per unit $M_6(\phi_{1,2}, \theta_{1,2}, \psi, x^{11})$-volume, of the form

$$V \sim -\frac{1}{2\kappa^2}\int_M d^5x\sqrt{-g}e^{-2\phi}\left(R-2\lambda\right) - \frac{1}{\kappa^2}\int_{\partial M} d^4x\sqrt{-g_B}e^{-2\phi}K,$$

where g_B is the metric at the boundary and K is the extrinsic curvature of the boundary.

- Using the above two metrics we first need to calculate separately the actions $V(BH)$ and $V(TH)$. Now for the regularization of the action at the boundary for both the solution, we integrate up to a UV cut-off $r = r_\Lambda$ and introduce boundary cosmological counter terms (in the large-N limit) proportional either to $\int_{r_\Lambda}\sqrt{-g_B^{BH}}$ (gravity dual with a black hole) or $\int_{r_\Lambda}\sqrt{-g_B^{TH}}$ (thermal background) but will take the limit of $r_\Lambda \to \infty$ at the end.

- Assuming $\mathcal{R}_{D5/\overline{D5}} = \sqrt{3}a$, $|\mu_{\rm Ouyang}|^{\frac{2}{3}} = \delta r_0$ and comparing the two UV finite energy densities and assuming (inspired by K.Dasgupta et al [2012]) $a(r_h) = \left(\alpha + (\beta + \gamma\log r_h)\frac{g_sM^2}{N}\right) r_h$, we get in the large N limit:

$$r_0 = r_h\left(\left|\frac{9\alpha^4-1}{2(\delta^{\frac{8}{3}}-1)}\right|\right)^{1/4} + \mathcal{O}\left(\frac{1}{\log N}\right).$$

- For $\alpha = 0.58, \delta = 1.008$ (for obtaining RHIC-compatible $\frac{\eta}{s}$'s variation with temperature and a lattice compatible T_c), $r_0 = \frac{r_h}{2^{\frac{1}{4}}}$, like for conformal backgrounds. Also, up to LO in N, $T_h = \left(\frac{r_h}{2\pi^{3/2}\sqrt{g_s N}}\right)$ so that the transition temperature is obtained as:

$$T_c = \frac{2^{1/4} r_0}{2\pi^{3/2}(g_s N)^{1/2}}$$

$D7$ brane embedding: Inclusion of $U(1)$ gauge field

- $D7$ probe branes are included in our set up via Ouyang embedding P.Ouyang [2003], with $\mu(\neq 0) \in \mathbb{R}(/\mathbb{C})$ being the embedding parameter is defined by the following equation:

$$r^{\frac{3}{2}} e^{\frac{i}{2}(\psi-\phi_1-\phi_2)} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} = \mu.$$

  The above is satisfied for $\psi = \phi_1 + \phi_2 + \pi$ and $r^{\frac{3}{2}} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} = |\mu|$.

- The DBI action for the $D7$ brane is given as:

$$S_{\rm DBI} \sim \int d^8\xi e^{-\phi} \sqrt{\rm det\left(i^* \left(g + B\right) + F\right)}.$$

- In the 'MQGP' limit, after the angular integration one obtains at large r:

$$S_{D7} \sim \int_{r=r_h}^{\infty} dr \left[\sqrt{|\mu|} r^{\frac{9}{4}} \sqrt{1 - F_{rt}^2} + \mathcal{O}\left(r^{\frac{3}{2}}, \left[\frac{1}{\sqrt{g_s N}}, \frac{g_s M^2}{N}\right]\right)\right].$$

- In the MQGP limit, one obtains from the above:

$$A_t = r\ {}_2F_1\left(\frac{2}{9}, \frac{1}{2}, \frac{11}{9}, -\frac{r^{\frac{9}{2}}\left(\frac{1}{g_s} - \frac{N_f \log \mu}{2\pi}\right)^2}{C^2}\right),$$

whose large-r-limit (we assume this limit to also be applicable in the IR apart from the IR-UV interpolating region and the UV) is given by: $\gamma_1 - \frac{\gamma_2}{r^{\frac{5}{4}}} + \frac{\gamma_3}{r^{\frac{23}{4}}} + \ldots$ with,

$$\gamma_1 = \left(\frac{2^{4/9}\Gamma\left(\frac{5}{18}\right)\Gamma\left(\frac{11}{9}\right)(Cg_s)^{4/9}}{\pi^{1/18}(g_s N_f \log(\mu) - 2\pi)^{4/9}}\right)$$

$$\gamma_2 = \left(\frac{36\pi\Gamma\left(\frac{11}{9}\right)Cg_s}{5\Gamma\left(\frac{2}{9}\right)(g_s N_f \log(\mu) - 2\pi)}\right)$$

$$\gamma_3 = \left(\frac{72\pi^3\Gamma\left(\frac{11}{9}\right)(Cg_s)^3}{23\Gamma\left(\frac{2}{9}\right)(g_s N_f \log(\mu) - 2\pi)^3}\right).$$

- We Choose a quantity γ such that:

$$\int_{r_h}^{r_\Lambda} \sqrt{g}\,(A_t-\gamma)^2 \sim \int_{r_h}^{r_\Lambda} r^3\,(A_t-\gamma)^2 < \infty,$$

for some UV cut-off scale r_Λ, then the following equation,

$$\frac{8}{11}\gamma_2 r_\Lambda^{11/4}(\gamma-\gamma_1)+\frac{1}{4}r_\Lambda^4(\gamma-\gamma_1)^2+\frac{2}{3}\gamma_2^2 r_\Lambda^{3/2}=0,$$

can be solved for γ as,

$$\gamma=\frac{\gamma_3}{r_\Lambda^{23/4}}+\frac{1}{33}\gamma_2\left(-\frac{33}{r_\Lambda^{5/4}}+\frac{2\left(24+5i\sqrt{6}\right)}{r_\Lambda^{5/4}}\right).$$

- Impose a Dirichlet boundary condition at a cut-off r_0 : $A_t(r_0)-\gamma=0$ and writing $C\equiv m_\rho^{\frac{9}{4}}$ on dimensional grounds, where m_ρ provides the mass scale of the lightest vector boson, one obtains:

$$m_\rho=\left(\frac{23}{10}\right)^{2/9}\frac{r_0\,(2\pi-g_sN_f\log(|\mu|))^{4/9}}{(g_s\pi)^{4/9}}.$$

If $m_\rho=760$ MeV the cut-off r_0 in units of MeV, from the above is given by:

$$r_0=760\times\left(\frac{10}{23}\right)^{2/9}\frac{(g_s\pi)^{4/9}}{(2\pi-g_sN_f\log(|\mu|))^{4/9}}.$$

- Replacing r_0 by m_ρ in the expression for T_c one get

$$N_f \sim \frac{1}{23\log(|\mu|)}\left(\frac{46\pi}{g_s} \pm \frac{144167}{\pi^{19/8} g_s^{9/8} N^{9/8} T_c^{9/4}}\right);$$

where, for $g_s \stackrel{\sim}{<} 1$, the embedding parameter $\mu \sim 13.6\imath$ and the number of light flavors $N_f = 3$, one obtains $T_c = 175$ MeV consistent with lattice data in the MQGP limit from the type IIB background.

- Now, dimensionally, $[\mu] = [r^{\frac{3}{2}}]$ and using the AdS/CFT dictionary, hence mass dimensions of 3/2. Curiously, if one were to write $m_q : \sqrt{|\mu|} = m_q^{\frac{3}{4}}$, one would obtain, *in units of MeV*, $m_q \approx 5.6$ - *exactly the mass scale of the first generation light quarks!*

σ, χ, Einstein Relation from Gauge Correlators

- The DBI action for the $D7$-brane is given as:

$$\mathcal{I}_{D7} = T_7 N_f \int d^8\xi e^{-\phi(r)} \sqrt{det[i^*(g+B)+F]}.$$

- We consider $U(1)$ gauge field fluctuation about a non-zero temporal component of gauge field background:

$$\hat{A}_\mu(u = \frac{r_h}{r}, \vec{x}) = \delta^0_\mu A_t(u) + \tilde{A}_\mu(\vec{x}, u),$$

with

$$\tilde{A}_\mu(x, u) = \int \frac{d^4q}{(2\pi)^4} e^{-i\omega t + iqx} \tilde{A}_\mu(q, u)$$

- The EOM for components of $\tilde{A}_\mu$ are:

$$\partial_\nu \left[\sqrt{|detG|} \times \left(G^{\mu\nu} G^{\sigma\gamma} - G^{\mu\sigma} G^{\nu\gamma} - G^{[\nu\sigma]} G^{\gamma\nu} \right) \partial_{[\gamma} \tilde{A}_{\mu]} \right] = 0,$$

where $G \equiv i^*(g + B) + F$.

- In terms of the gauge invariant field components $E_x = w\tilde{A}_x + q\tilde{A}_t$, $E_\alpha = w\tilde{A}_\alpha$, the on-shell DBI action can be written as Erdmenger et al [2007]:

$$\mathcal{I}^{(2)}_{D7} = T_7 \int \frac{dwd^3q}{(2\pi)^4} \left[e^{-\phi(u)} r_h^{1/4} u^{7/4} \left(\frac{E_x \partial_u E_x}{\left(q^2 - \frac{w^2}{g_1}\right)} - \frac{1}{w^2} E_\alpha \partial_u E_\alpha \right) \right] \Bigg|_{u=1}^{u=0} .$$

- Using the prescription of D.T.Son, A. Starinets [2002], we first calculate the retarded greens function for E_x from the above action and hence the DC conductivity as:

$$\sigma \sim (g_s N)^{\frac{1}{8}}\, T^{-\frac{3}{4}} .$$

Interestingly, this mimicks a one-dimensional interacting system - Luttinger liquid - for appropriately tuned Luttinger parameter T. Giamarchi [1991].

- Following J. Mas et al [2008], the charge susceptibility χ, is thermodynamically defined as response of the charge density to the change in chemical potential, is given by :

$$\chi = \frac{\partial n_q}{\partial \mu_C}\Big|_T,$$

where $n_q = \frac{\delta S_{DBI}}{\delta F_{rt}}$, and the chemical potential μ_C is defined as $\mu_C = \int_{r_h}^{r_B} F_{rt} dr$.

- One gets the following charge susceptibility KS, A. Misra [2015]:

$$\chi = \left[\int_{r_h}^{\infty} dr \frac{dF_{rt}}{dn_q}\right]^{-1} \sim \sqrt{|\mu|}\,(g_s N)^{\frac{5}{8}}\, T^{5/4}.$$

- Given that one is in the regime of linear response theory, one expects the Einstein's relation: $\frac{\sigma}{\chi} = D \sim \frac{1}{T}$ M. Dhuria, A Misra[2014], to hold. However, a naive application yields $\frac{\sigma}{\chi} \sim \frac{1}{\sqrt{|\mu| g_s N}} \frac{1}{T^2}$.
- One expects the Ouyang embedding parameter could be related to the deformation parameter if there were supersymmetry. In the MQGP limit, there is approximate supersymmetry; the resolution parameter possesses an r_h-dependence F. Chen et al [2012]. If one assumes that $|\mu| \sim \frac{1}{r_h^2}$ (in $\alpha' = 1$-units), then the Einstein's relation is preserved.

Part 3: Transport Coefficients [Diffusion Constant, Speed of Sound, $\frac{\eta}{s}$]

- We consider a linear order perturbation of the black $M3$ brane metric as $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$, where we assume the perturbation to depend on x_1 and t only with the fourier decomposed form given as:

$$h_{\mu\nu}(\vec{x}, t, u) = \int \frac{d^4 q}{(2\pi)^4} e^{-iwt+iqx_1} h_{\mu\nu}(q, w, u).$$

- Depending on the spin of the different perturbations under the $SO(2)$ rotation in $x_2 - x_3$ plane these metric perturbations can be categorized as:
  - (i) vector modes: $h_{x_1 x_2}, h_{tx_2} \neq 0$ or $h_{x_1 x_3}, h_{tx_3} \neq 0$, with all other $h_{\mu\nu} = 0$.
  - (ii) Scalar modes: $h_{x_1 x_1} = h_{x_2 x_2} = h_{x_3 x_3} = h_{tt} \neq 0$, $h_{x_1 t} \neq 0$, with all other $h_{\mu\nu} = 0$.
  - (iii) Tensor modes: $h_{x_2 x_3} \neq 0$, with all other $h_{\mu\nu} = 0$.
- With these perturbations we need to solve the Einstein's equation which is given up to first order in metric perturbation as ,

$$\mathcal{R}^{(1)}_{\mu\nu} = \frac{2}{d-2} \Lambda h_{\mu\nu},$$

where d is the dimension of the non-compact space-time.

- An easier way of solving the Einstein's equation is to consider the gauge invariant combination of the perturbations. In this way, we would have a single equation to solve instead of several coupled EOMs.

- These combinations are given as P. K. Kovtun, A. O. Starinets [2005],

  (i) Vector type: $Z_v = qH_{tx_2} + wH_{x_1x_2}$

  (ii) Scalar type: $Z_s =$
  $$-q^2(1-u^4)H_{tt}+2wqH_{x_1t}+w^2H_{x_1x_1}+q^2(1-u^4)\left(1+\frac{g_{x_1x_1}(-4u^3)}{g'_{x_1x_1}(1-u^4)}-\frac{w^2}{q^2(1-u^4)}\right)H_{x_2x_2}.$$

  (iii) Tensor type: $Z_t = H_{x_2x_3}$,

  where $H_{tt} = -g^{tt}h_{tt}$, $H_{x_1x_1} = g^{x_1x_1}h_{x_1x_1}$, $H_{x_2x_2} = g^{x_1x_1}h_{x_2x_2}$, $H_{x_1t} = g^{x_1x_1}h_{x_1t}$, $H_{x_1x_2} = g^{x_1x_1}h_{x_1x_2}$.

Vector Mode Perturbations and Diffusion Constant up to NLO in N in the MQGP LimitK. Sil, A. Misra [2016]

- The equations of motion for the vector perturbation modes up next-to-leading order in N, can be reduced to the following single equation of motion in terms of a gauge-invariant variable $Z_v(u)$ P. Kovtun, A. Starinets [2005]:

$$Z_v''(u) - m_V(u)Z_v'(u) - l_V(u)Z_v(u) = 0.$$

  The horizon $u = 1$ is a regular singular point.

- Using the Frobenius method:

$$Z_v(u) = (1-u)^{-\frac{i\omega_3}{4}+\frac{3ig_s{}^2M^2N_f\omega_3\log^2(N)}{256\pi^2N}}\left(1+\sum_{n=1}^{\infty}a_n(u-1)^n\right).$$

- The quasi-normal modes are obtained via the Dirichlet boundary condition $Z_v(u=0)=0$ P. K. Kovtun and A. O. Starinets [2005].
- Truncating the infinite series at $\mathcal{O}\left((u-1)^2\right)$, in the hydrodynamical limit retaining terms only up to $\mathcal{O}(\omega_3^m q_3^n)$: $m+n=4$ reduces to: $a\omega_3^4+b\omega_3^3+c\omega_3^2+f\omega_3+g=0$. One of the four roots of $Z_v(u=0)=0$ is:

$$\omega_3=-8.18i+\frac{0.14ig_s^2M^2N_f(\log N)^2}{N}+\left(-0.005i-\frac{0.002ig_s^2M^2N_f(\log N)^2}{N}\right)q_3^2+\mathcal{O}(q_3^3).$$

- Using the Frobenius method and going up to $\mathcal{O}((u-1)^3)$, the Dirichlet condition $Z_v(u=0)=0$ reduces to $a'\omega_3^4+b'\omega_3^3+c'\omega_3^2+f'\omega_3+g'=0$. One of the four roots of the quartic in ω_3 is:

$$\omega_3=\left(-0.73i+\frac{0.003ig_s^2M^2N_f(\log N)^2}{N}\right)q_3^2+\mathcal{O}(q_3^3).$$

The leading order coefficient of q_3^2 is not terribly far off the correct value $-\frac{i}{4}$ already at the third order in the infinite series.

- At $\mathcal{O}\left((u-1)^4\right)$, one finds in the hydrodynamical limit the Dirichlet boundary condition $Z(u=0)=0$ reduces to $a''\omega_3^4+b''\omega_3^3+c''\omega_3^2+f''\omega_3+g''=0$. Incredibly, one of the roots of the quartic equation in ω_3 is:

$$\omega_3=\left(-\frac{i}{4}+\frac{3ig_s^2M^2N_f\log N\left(5+2\log N\right)}{512\pi^2N}\right)q_3^2+\mathcal{O}\left(q_3^3\right).$$

- Hence, the leading order (in N) yields a diffusion constant $D=\frac{1}{4\pi T}$, exactly the conformal result! Including the non-conformal corrections which appear at NLO in N, one obtains:

$$D=\frac{1}{\pi T}\left(\frac{1}{4}+\frac{3g_s^2M^2N_f\log N\left(5+2\log N\right)}{512\pi^2N}\right).$$

NLO correction to the Speed of Sound KS, A. Misra [2016]

- Here we need to consider the non zero scalar modes of metric perturbations and solve the solve the Einstein's equation for the scalar type gauge invariant variable Z_s:

  $Z_s''(u) - m_s(u)Z_s'(u) - l_s(u)Z_s(u) = 0.$

- The EOM can be solved by considering an ansatz $Z_s(u) = (1-u)^r F(u)$ where $F(u)$ is regular near the horizon $u = 1$. Substituting this ansatz into the equation, one determines the value of r and is given by $\pm\frac{i\omega_3}{4}$. We choose the negative sign here as it represents an incoming wave. The evaluation of the function $F(u)$ can be done perturbatively using hydrodynamic approximation, given as: $\omega_3 \ll 1$, $q_3 \ll 1$. For analytic solution the momentum has to be light-like, means ω_3 and q_3 would be of the same order. Hence we can rescale ω_3 and q_3 by a same parameter λ as:$\omega_3 \to \lambda\omega_3$, $q_3 \to \lambda q_3$ and expand to first order in λ, where the limit $\lambda \ll 1$ ensure that we are working in the hydrodynamic regime. We choose the following series expansion of $F(u)$ for small frequency and momentum as:

$$F(u) = F_0(u) + \omega_3 F_1(u) + \mathcal{O}(\omega_3^2, q_3^2, \omega_3 q_3).$$

Obtaining the Conformal Value

- Ensuring regularity of $F_0(u)$ near the horizon $u = 1$ and imposing a boundary condition $F_1(u = 0) = 0$:

$$Z_s(u) = c_1(1-u^4)^{-i\omega_3/4}\left(\frac{q_3^2\left(u^4+1\right)-3\omega_3^2}{17q_3^2-3\omega_3^2} - \frac{iq_3^2\omega_3\left(1-u^4\right)}{17q_3^2-3\omega_3^2}\right).$$

Assuming the prescription of C. Nunez, A. O.Starinets [2003]; P. K. Kovtun, A. O. Starinets [2005] that the quasinormal frequencies are obtained by imposing Dirichlet boundary condition $Z_s(u = 0) = 0$, to be valid, we get ,

$$\omega_3 = \pm\frac{q_3}{\sqrt{3}} - \frac{iq_3^2}{6} + \mathcal{O}(q_3^3).$$

or equivalently the following dispersion relation:

$$\omega = \pm\frac{q}{\sqrt{3}} - \frac{iq^2}{6\pi T}.$$

- Comparing with the dispersion relation corresponding to the sound wave mode,

$$\omega = \pm q v_s - i\Gamma_s q^2$$

where v_s is the speed of sound and Γ_s is the attenuation constant, we get their exact conformal values.

Dropping $\log r \log N$ As Compared to $(\log N)^2$

- The horizon $u = 1$ is a regular singular point of the $Z(u)$ EOM which can be rewritten as:

$$(u-1)^2 Z_s''(u) + (u-1)P(u-1)Z_s'(u) + Q(u-1)Z_s(u) = 0,$$

in which $P(u-1) = \sum_{n=0}^{\infty} p_n(u-1)^n$ and $Q(u-1) = \sum_{m=0}^{\infty} q_n(u-1)^n$. The Frobenius method then dictates that the solution is given by:

$$Z_s(u) = (1-u)^{\frac{3{g_s}^2 M^2 Nf \log(N)\left(8{q_3}^2{\omega_3}^2 \log(N) + \left({\omega_3}^2+4\right)\left(10{q_3}^2 - 27{\omega_3}^2\right)\right)}{2048\pi^2 N q_3^2 \omega_3 (-1)^{3/2}} - \frac{i\omega_3}{4}} \left(1 + \sum_{m=1} a_m (u-1)^m\right).$$

Imposing Dirichlet boundary condition $Z_s(u=0) = 0$ and going up to second order in powers of $(u-1)$ and considering in the hydrodynamical limit $\omega_3^n q_3^m : m+n=2$ one obtains:

$$\omega_3 = -\frac{2q_3}{\sqrt{3}} - \frac{9i{q_3}^2}{32},$$

which yields the same result for the speed of sound as to be discussed momentarily with the inclusion of the non-conformal NLO terms in the EOM.

- Going up to the fourth order in powers of $(u-1)$, substituting the expressions for $a_{1,2,3,4}$ into $Z(u)$ and implementing the Dirichlet boundary condition: $Z_s(u=0)=0$, in the hydrodynamical limit, going up to $\mathcal{O}(\omega_3^4)$ one sees that one can write the Dirichlet boundary condition as a quartic: $a\omega_3^4+b\omega_3^3+c\omega_3^2+f\omega_3+g=0$. One of the four roots yields:

$$\omega_3 \approx 0.46q_3 - 0.31iq_3^2,$$

- The coefficient of q_3 is not too different from the conformal value of $\frac{1}{\sqrt{3}} \approx 0.58$ We expect the leading order term in the coefficient of q_3 to converge to $\frac{1}{\sqrt{3}}$. Also, the coefficient of q^2 term turns out to be $\frac{0.31}{\pi}$ which is not terribly far from the conformal result of $\frac{0.17}{\pi}$. We are certain that the inclusion of higher order terms will ensure that we get a perfect match with the conformal result; given the exclusion of the non-conformal log r-terms, we do not expect to obtain the $\frac{1}{N}$-corrections.

Inclusion of Non-Conformal $\log r$ Terms - First Pass

- $$Z_s''(u) = m_S(u)Z_s'(u) + l_S(u)Z_s(u).$$

- The horizon $u = 1$ due to inclusion of the non-conformal corrections to the metric, becomes an irregular singular point. One then tries the ansatz: $Z_s(u) = e^{S(u)}$ near $u = 1$. Assuming that $(S')^2 >> S''(u)$ near $u = 1$ the differential equation can be written as $Z_s''(u) = m_S(u)Z_s' + l_S(u)Z_s(u)$ can be approximated by C. M. Bender, S. O. Orzag, *Asymptotic Methods and Perturbation Theory*, Springer [1999]:

$$(S')^2 - m_S(u)S'(u) - l_S(u) \approx 0.$$

- By staying close to $\omega_3 = \frac{2}{\sqrt{3}} q_3$ and near $u = 1$ (this will effect Dirichlet boundary condition on $F(u)$ in the conformal calculation), one can estimate $S(u)$ as:

$$Z_s(u) \sim (1-u)^{-\frac{1}{2} + \frac{15 g_s^2 M^2 N_f \omega_3^2 \log\left(\frac{1}{N}\right)}{256 \pi^2 N \left(2 q_3^2 - 3\omega_3^2\right)}} \mathcal{F}(u),$$

- Obtain $\omega = \omega(q)$ from $Z_s(u=0) = 0$. As the solution to $\mathcal{F}(u)$'s EOM involves $\frac{1}{\Gamma(\Omega(q_3,\omega_3;g_s,N,M,N_f))}$, this can be effected by requiring

$$\Omega(q_3, \omega_3; g_s, N, M, N_f) = -n \in \mathbb{Z}^-$$

or

$$\omega = q_3 \left(\frac{\sqrt{14n+17}}{\sqrt{21n+\frac{51}{2}}} + \frac{5(2n+5){g_s}^2 M^2 N_f \log N}{128\pi^2 \sqrt{14n+17}\sqrt{84n+102N}} \right),$$

implying the following estimate of the speed of sound:

$$v_s \approx \frac{\sqrt{14n+17}}{\sqrt{21n+\frac{51}{2}}} + \frac{5(2n+5){g_s}^2 M^2 N_f \log N}{128\pi^2 \sqrt{14n+17}\sqrt{84n+102N}}.$$

Given that one is working with an approximate solution, one expects to obtain an expression for v_s from an $M3$-brane uplift, to be of the form $v_s \approx \frac{\mathcal{O}(1)}{\sqrt{3}} + \mathcal{O}\left(\frac{g_s M^2}{N}\right)$, and the above is exactly of this form for $n = 0, 1$; for a p-brane solution, the conformal result for the speed of sound is: $v_s = \frac{1}{\sqrt{p}}$ C.Herzog [2003].

Non-Conformal corrections to v_s and Γ - Second Pass

- We have reconsidered this issue with A.Czajka, K.Dasgupta, C.Gale, S.Jeon, A.Misra, M.Richard, KS [to appear soon], such that one obtains the following dispersion relation:

$$\omega_3 = \left(\frac{1}{\sqrt{3}} + \alpha \frac{g_s M^2}{N}\right) q_3 + \left(-\frac{i}{6} + \beta \frac{g_s M^2}{N}\right) q_3^2,$$

NLO correction to η, $\frac{\eta}{s}$ and Comparison with lattice simulation of QGP KS, A. Misra [2016]

- In this case, we need to consider the nonzero tensor mode of metric perturbation and solve the Einstein's equation for Z_t. Realizing the horizon is a regular singular point, one makes the following double perturbative ansatz in ω_3 and q_3:

$$Z_t(u) = (1-u)^{-i\omega_3\left(\frac{1}{4} - \frac{3{g_s}^2 M^2 N_f \log(N)\log r_h}{128\pi^2 N}\right)} \left(z_{00}(u) + \omega_3 z_{01}(u) + q_3 z_{10}(u) + \mathcal{O}(q_3^m \omega_3^n)\right).$$

- Using Kubo formula the Shear Viscosity can be obtained from the solution Z_t as:

$$\eta = \Upsilon \frac{N^{\frac{2}{5}}}{g_s^2 \alpha_N \alpha_{\theta_1}^2} \lim_{\omega_3 \to 0}\left(\frac{1}{\omega_3 T}\lim_{u\to 0}\left[\frac{r_h^4}{u^3}\mathrm{Im}\left(\frac{Z_t'(u)}{Z_t(u)}\right)\right]\right).$$

where Υ is an overall multiplicative constant.

- Using the result of M.Dhuria, A.Misra [2013] for the entropy density s defined in terms of the UV finite Gibbons-Hawking-York (GHY) surface term $S_{\rm GHY}$ the $D = 11$ supergravity action as

$$s = -T\frac{\partial S^{\rm UV-finite}_{\rm GHY}}{\partial T} - S^{\rm UV-finite}_{\rm GHY},$$

yields:

$$\frac{\eta}{s} = \mathcal{O}(1)\times$$

$$\left[\frac{\frac{1}{4\pi} - 0.00051\log(r_h)}{1 - 0.064\gamma + 0.004\gamma^2 + \sum_{n=1}^{4} a_n(\beta,\gamma)\log^n r_h + \frac{\sum_{n=0}^{4} b_n(\beta,\gamma)\log^n r_h}{\sum_{n=0}^{2} c_n(\beta,\gamma)\log^n r_h}}\right],$$

where a_n, b_n, c_n are known functions of β and γ.
As per lattice QGP data, one must have

$$\left.\frac{\eta}{s}\right|_{T=T_c} \sim 0.1, \qquad \left.\frac{d\left(\frac{\eta}{s}\right)}{d\tilde{t}}\right|_{\tilde{t}>0} > 0$$

Numerically, one sees that setting $(\beta,\gamma) = (4,4)$ and the $\mathcal{O}(1)$ constant equal to 5.8, fits the bill.

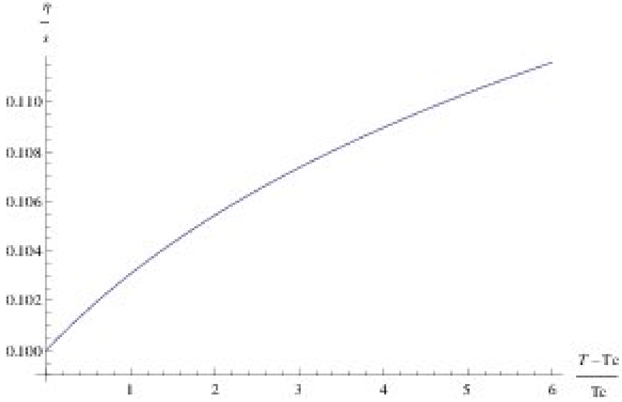


Figure : $\frac{\eta}{s}\,(N_f = 3, M = 3, g_s = 0.9, N = 100)$ vs. $\frac{T-T_c}{T_c}$ for $T \geq T_c$ assuming $\frac{\eta}{s}\Big|_{T=T_c} = 0.1$

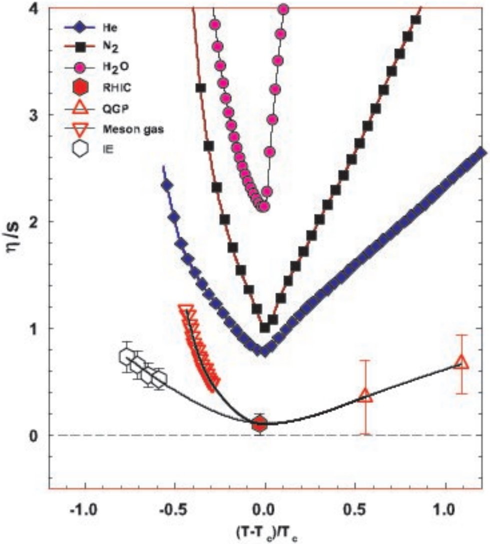


Figure : $\frac{\eta}{s}$ vs. $\frac{T-T_c}{T_c}$ reproduced from R. Lacey et al [2006].

We draw a third graph in which the previous plots are given on the same graph.

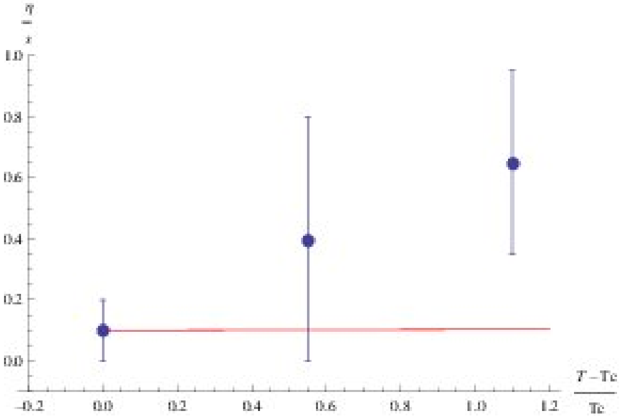


Figure : Combined plots: the graph in red is from our calculations and the set of three points with error bars are from R. Lacey et al [2006].

- The combined plots make the comparison of our results with those of lattice simulation of QGP, very clear. We conclude the following:
  - $\frac{\eta}{s}(T = T_c) = 0.1$, and $\left.\frac{d(\frac{\eta}{s})}{dT}\right|_{T>T_c} > 0$.
  - The numerical values, unlike lattice simulation of QGP, remain close to the value at $T = T_c$. In other words, unlike lattice simulation of QGP, $\frac{\eta}{s}$ is found to be a much more slowly varying function of $\tilde{t} = \frac{T-T_c}{T_c}$. Also, $\frac{d^2(\frac{\eta}{s})}{d\tilde{t}^2} < 0$ in our calculations and $\frac{d^2(\frac{\eta}{s})}{d\tilde{t}^2} > 0$ in the lattice simulation of QGP. The error bars appearing in the third figure, for $\frac{T-T_c}{T_c} \in [0, 1.1]$ - the range covered in R. Lacey et al [2006] - permit our deviations from the lattice simulation of QGP at least for $\frac{T-T_c}{T_c} \in [0, 0.6]$.

Electircal/Thermal Conductivity and Wiedemann-Franz Law

KS, A.Misra[2016]

- Consider the fluctuation of $U(1)$ gauge field on the $D7$ brane along with the metric fluctuations of the form:

$$A_\mu = A^0_\mu + \mathcal{A}_\mu \qquad g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

where we write the fluctuations as the following fourier decomposed form

$$\mathcal{A}_\mu = \phi(u) e^{-iwt+iqx} \delta^y_\mu$$
$$h_{\mu\nu} = H^\mu_\nu(u) e^{-iwt+iqx}.$$

- Now considering the Einstein-Hilbert action together with the $D7$ brane DBI action:

$$S = a_{EH} \int dud^4x \sqrt{-g_{(5)}}(R-2\lambda) + T_{D_7} \int d^8\xi \sqrt{-\det(g+B+\hat{F})},$$

substituting the metric and gauge fluctuations into the action, performing the angular integration in the $D7$ DBI action, we solve the equations of motion for the gauge field and metric fluctuations near $u=0$ in the $q=0$-limit.

- Using this solution and the prescription of D. T. Son, A. Starinets [2002], we computed the temperature dependance of electrical/thermal conductivity and the Wiedemann-Franz law as:

$$\sigma = (0.39)\frac{\sqrt{|\mu|}\,(g_s N)^{\frac{3}{4}}}{2^{\frac{1}{4}}T^{\frac{1}{4}}\pi^{\frac{3}{8}}},$$

$$\kappa_T = \frac{9\times 0.39}{200\times 2^{3/4}}\frac{\sqrt{|\mu|}g_s^{\frac{3}{2}}C^2}{\sqrt{N}\pi^{\frac{17}{8}}T^{\frac{15}{4}}\left(2\pi - g_s N_f\log|\mu|\right)^2};$$

$$\text{Wiedemann}-\text{Franz law}:\ \frac{\kappa_T}{\sigma T} = \frac{9}{200\sqrt{2}}\frac{g_s^{\frac{3}{4}}C^2}{N^{\frac{5}{4}}\pi^{\frac{7}{2}}\left(2\pi - g_s N_f\log|\mu|\right)^2 T^{\frac{9}{2}}}.$$

- Assuming the Ouyang embedding parameter to depend on the temperature via the horizon radius as $|\mu_{\rm Ouyang}| \sim r_h^{\alpha}$,

  (a) for $\alpha \leq 0$, the temperature dependence of σ, κ_T and the consequent deviation from the Wiedemann-Franz law, upon comparison with Table 2 of A. Garg et al [2009], all point to the remarkable similarity with $D = 1 + 1$ Luttinger liquid with '$\frac{1}{3}$-doping'.

  (b) For $\alpha > 0$ and in particular with $\alpha = \frac{5}{2}$ one is able to reproduce the expected linear large-T variation of DC electrical conductivity for most strongly coupled gauge theories with five-dimensional gravity duals with a black hole S.Jain [2010].

- In all calculations $M \to M(r), N_f \to N_f(r)$; given that there should be no net $D5$-brane and $D7$-brane charges in the UV [assumed to be defined $\forall r > \mathcal{R}_{D5/\overline{D5}}$ (the $D5 - \overline{D5}$ separation)] to effect UV conformality, assuming $\kappa_{M,N_f} \gg 1$ (motivated by a similar profile for $D5$-branes in M. Mia et al [2010]):

$$M(r) = \frac{M}{1 + e^{\kappa_M (r - \mathcal{R}_{D5/\overline{D5}})}},$$
$$N_f(r) = \frac{N_f}{1 + e^{\kappa_{N_f} (r - \mathcal{R}_{D5/\overline{D5}})}}.$$

Part 4

Glueball Spectroscopy

Introduction

- Glueballs are color-neutral bound states of gluons (gg, ggg, etc.). They are represented by quantum numbers J^{PC}, where J, P and C correspond respectively to total angular momentum, parity and charge conjugation respectively.
- The 11D metric obtained as the uplift of the delocalized SYZ type IIA metric, up to LO in N, can be interpreted as a black $M5$-brane wrapping a two-cycle, i.e. a black $M3$-brane M.Dhuria, A. Misra [2014]. Taking this as the starting point, compactifying again along the M-theory circle, we land up at the type IIA metric and then compactifying again along the periodic temporal circle (with the radius given by the reciprocal of the temperature), one obtains QCD_3 corresponding to the three non-compact directions of the black $M3$-brane world volume. The Type IIB background of K.DSasgupta et al [2009], in principle, involves $M_4\times$ RWDC($\equiv$ Resolved Warped Deformed Conifold); asymptotically the same becomes $AdS_5 \times T^{1,1}$. To determine the gauge theory fields that would couple to appropriate supergravity fields a la gauge-gravity duality, ideally one should work the same out for the $M_4\times$ RWDC background (which would also involve solving for the Laplace equation for the internal RWDC). We do not attempt to do the same here.

- Motivated by S.Gubser [1998], we calculated:
  - type IIB dilaton fluctuations, which we refer to as 0^{++} glueball
  - type IIB complexified two-form fluctuations that couple to $d^{abc}Tr(F^a_{\mu\rho}F^{b\ \rho}_{\lambda}F^{c\ \lambda}{}_{\nu})$, which we refer to as 0^{--} glueball
  - type IIA one-form fluctuations that couple to $Tr(F \wedge F)$, which we refer to as 0^{-+} glueball
  - M-theory metric's scalar fluctuations which we refer to as another (lighter) 0^{++} glueball
  - M-theory metric's vector fluctuations which we refer to as 1^{++} glueball, and
  - M-theory metric's tensor fluctuations which we refer to as 2^{++} glueball.
- For each of the above mentioned glueballs, the spectrum are obtained for
  - the thermal background with an IR cut-off r_0,
  - the black hole background with horizon r_h.
- We have used two different methods for the solution separately for the above two backgrounds, namely
  - WKB method
  - Diritchlet/Neumann boundary condition at r_0/r_h.

- General form of the supergravity field EOM with the field perturbation $H(r) = \tilde{H}(r)e^{ikx}$, $k^2 = -m^2$ is given as:

$$\tilde{H}'' + f_1(r)\tilde{H}' + m^2 f_2(r)\tilde{H} = 0.$$

- Following J.A.Minahan [1999] we consider the redefinition of the variables as:
  - Background with a black hole: $r \to \sqrt{y}$, $r_h \to \sqrt{y_h}$ and then $y \to y_h\,(1 + e^z)$
  - Background with an IR cut-off: $r \to \sqrt{y}$, $r_0 \to \sqrt{y_0}$ and then $y \to y_0\,(1 + e^z)$.

  The modified equation is given as:

$$\partial_z \left(f_3(z)\partial_z \tilde{H} \right) + m^2 f_4(z)\tilde{H} = 0.$$

- Making the transformation of $\tilde{H}$ as $\psi(z) = \sqrt{f_3(z)}\tilde{H}(z)$, the above equation reduces to the following schrodinger like form,

$$\left(\frac{d^2}{dz^2} + V(z) \right) \psi(z) = 0,$$

where $V(z)$ is the potential term.

- The WKB quantization condition is given as:

$$\int_{z_1}^{z_2} \sqrt{V(z)}dz = \left(n + \frac{1}{2}\right)\pi$$

,

where z_1 and z_2 are the turning points obtained by solving for the roots, the equation $V(z) = 0$, for $V(z) > 0$.

0^{PC} Glueballs spectra from type IIB/IIA supergravity background

- 0^{++}: The dilaton equation that has to be solved is given as:

$$\partial_\mu \left(e^{-2\Phi} \sqrt{g} g^{\mu\nu} \partial_\nu \phi \right) = 0.$$

- 0^{--}: The equation for the two form $A_{MN} = B_{MN} + iC_{MN}$ is given as:

$$\partial_\mu \left(\sqrt{-g} g^{MP} g^{NR} g^{\mu\nu} \partial_\nu \delta A_{MN} \right) = 0.$$

The above equation is solved for the perturbation $A_{MN} = A^{(0)}_{MN} + \delta A_{MN}$ with $\delta A^{MN} = \delta^M_{x_2} \delta^N_{x_3} \delta A_{x_2 x_3}$.

- 0^{-+}: The EOM involving type IIA one form A_μ is given as:

$$\partial_\nu \left(\sqrt{g^{\rm IIA}} g^{\mu\sigma}_{\rm IIA} g^{\nu\rho}_{\rm IIA} \left(\partial_{[\sigma} A_{\rho]} \right) \right) = 0,$$

where the indices $\mu, \nu, ... = \{0, 1, 2, ..., 9\}$. The above equation is solved for the perturbation $A_\mu = A^0_\mu + \mathcal{A}_\mu$ with $\mathcal{A}_\mu = \delta^{\theta_2}_\mu a_{\theta_2}(r) e^{ik\cdot x}$.

Glueball spectrum from M-theory background

- The glueball spectrum for spin $0^{++}, 1^{++}$ and 2^{++} is calculated from the M-theory perspective.
  The 11 dimensional M-theory action is given as:

$$S_M = \frac{1}{2k_{11}^2}\int_M d^{11}x\sqrt{G^{\mathcal{M}}}\left(R - \frac{1}{2}G_4 \wedge *_{11} G_4\right),$$

  where $G_4 = dC_3 + A_1 \wedge dB_2 + dx_{10} \wedge dB_2$, and $C^M_{\mu\nu 10} = B^{IIA}_{\mu\nu}, C^M_{\mu\nu\rho} = C^{IIA}_{\mu\nu\rho}$.

- Using the results of M. Dhuria, A. Misra [2013] for H_3, A_1 and the inverse of 11-dimensional metric we calculate at leading order in N, the following flux-generated cosmological constant term:

$$\left(G_{MNOP}G^{MNOP}\right)\Big|_{\theta_1\sim N^{-\frac{1}{5}},\theta_2\sim N^{-\frac{3}{10}}} = \frac{\mathcal{A}(r)}{N^{7/10}}.$$

- After performing the integration on the six compact coordinated the M-theory action reduces to 5-dimension:

$$S_M \sim \int_M d^5x\,\sqrt{G^5}\,(R + Cosmological\ constant\ term)\,,$$

- The background metric $g^{(0)}_{\mu\nu}$ is linearly perturbed as $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} = \epsilon_{\mu\nu}(r)e^{ikx_1}$.

- 0^{++}: The non zero perturbations are given as Mathur et al [2000]

$$h_{tt} = g_{tt} e^{iqx_1} q_1(r),$$
$$h_{x_1 r} = h_{r x_1} = iq\, g_{x_1 x_1} e^{iqx_1} q_3(r),$$
$$h_{rr} = g_{rr} e^{iqx_1} q_2(r),$$

- 2^{++}: The non zero perturbations are given as Mathur et al [2000]

$$h_{x_2 x_3} = h_{x_3 x_2} = g_{x_1 x_1} H(r) e^{ikx_1}$$
$$h_{x_2 x_2} = h_{x_3 x_3} = g_{x_1 x_1} H(r) e^{ikx_1},$$

- 1^{++}: The only non zero perturbation is given as Mathur et al [2000]

$$h_{ti} = h_{it} = g_{x_1 x_1} G(r) e^{ikx_1}, i = x_2, x_3.$$

2^{++} Glueball spectrum from type IIB background

- The 10-dimensional type IIB supergravity action is given by,

$$S = \frac{1}{2k_{10}^2}\left\{\int d^{10}x\, e^{-2\phi}\sqrt{-G}\left(R - \frac{1}{2}H_3^2\right) - \frac{1}{2}\int d^{10}x\,\sqrt{-G}\left(F_1^2 + \widetilde{F}_3^2 + \frac{1}{2}\widetilde{F}_5^2\right)\right\},$$

- The EOM is given as:

$$R_{\mu\nu} = \left(\frac{5}{4}\right) e^{2\phi}\widetilde{F}_{\mu p_2 p_3 p_4 p_5}\widetilde{F}_{\nu}^{p_2 p_3 p_4 p_5} - \left(\frac{g_{\mu\nu}}{8}\right) e^{2\phi}\widetilde{F}_5^2 + \left(\frac{3}{2}\right) H_{\mu\alpha_2\alpha_3}H_{\nu}^{\alpha_2\alpha_3}$$
$$- \left(\frac{g_{\mu\nu}}{8}\right) H_3^2 + \left(\frac{3}{2}\right) e^{2\phi}\widetilde{F}_{\mu\alpha_2\alpha_3}\widetilde{F}_{\nu}^{\alpha_2\alpha_3} - \left(\frac{g_{\mu\nu}}{8}\right) e^{2\phi}\widetilde{F}_3^2 + \left(\frac{1}{2}\right) e^{2\phi}F_\mu F_\nu.$$

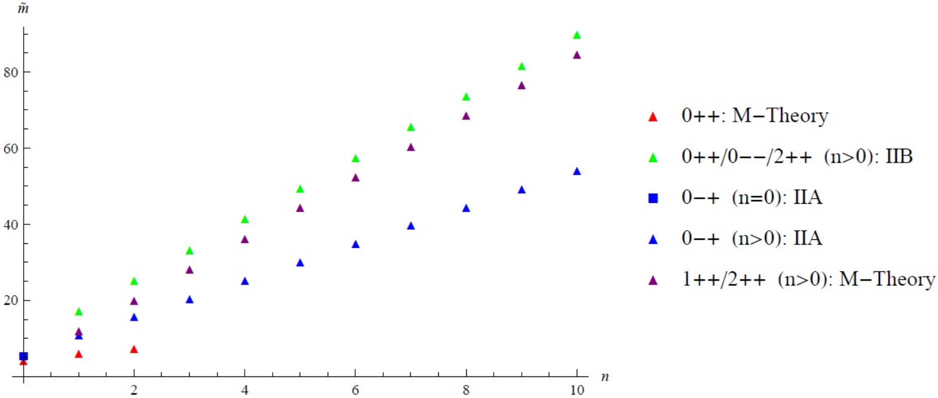
The above EOM has to be solved with the perturbation of the metric as $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, where the only non zero component of the perturbation is $h_{x_2 x_3}$.

Glueball	$\tilde{m}$ using WKB $r_h \neq 0$	$\tilde{m}$ using WKB $r_h = 0$
0^{++} (Fluctuations: $h_{00,rx_1,rr}$ in M-theory metric)	(M theory) $\frac{\sqrt{35+70n}}{\sqrt{\mathcal{P}\mathcal{L}}(13.15+26.31n)}$ ▲	(M theory) No turning points
0^{++} (Dilaton Fluctuations)	(Type IIB) $9.18 + 8.08n$ ▲	(Type IIB) $\frac{(4.64+6.28n)}{(\delta^2-1)}\left[1 - 0.01\frac{g_s M^2}{N}(g_s N_f)\log N \log r_0\right]$
0^{-+} (1-form fluctuation a_{θ_2})	(Type IIA) $11.12\left(n+\frac{1}{2}\right), n = 0$ ■ $(6.22 + 4.80n), n \in \mathbb{Z}^+$ ▲	(Type IIA) $\frac{3.72+4.36n}{(\delta^2-1)}, n = 0$ $4.8\left(n+\frac{1}{2}\right), n \in \mathbb{Z}^+$
0^{--} 2-form fluctuation A_{23}	(Type IIB) $= m_n^{0^{++}}(\text{dilaton}, T)$ ▲	(Type IIB) $\frac{6.28n+4.71}{(\delta^2-1)}\left(1 + \frac{0.01 g_s{}^2 \log N M^2 N_f \log(r_0)}{N}\right), n = 0$ $(7.87 + 6.93n), n \in \mathbb{Z}^+$
1^{++} (Fluctuations: $h_{it} = h_{ti}, i = x_{2,3}$ in M-theory metric)	(M theory) $8.08\left(n+\frac{1}{2}\right)$ ▲	(M theory) $m_n^{1^{++}}(n = 0, r_h = 0) = m_n^{0^{--}}(n = 0, r_h = 0)$ $(3.46 + 6.93n), n \in \mathbb{Z}^+$

Table : Glueball Spectra - I: $m = \tilde{m}\frac{r_h}{L^2}$ from Type IIB, IIA and M Theory using WKB quantization condition for $r_h \neq 0$, and $m = \tilde{m}\frac{r_0}{L^2}$ for $r_h = 0$ (equalities in the $r_h = 0$ column, are valid up to NLO in N); the colored triangles/square correspond to the colored triangles/square that appear in the combined plot of $r_h \neq 0$ supergravity calculations of glueballs

Glueball	$\tilde{m}$ using WKB $r_h \neq 0$	$\tilde{m}$ using WKB $r_h = 0$
2^{++} (Fluctuations: $h_{x_2x_3} = h_{x_3x_2}$, $h_{x_2x_2} = h_{x_3x_3}$ in M-theory metric)	(M theory) $8.08\left(n+\frac{1}{2}\right) = m_n^{1^{++}}(T)$ ▲	(M theory) $= m_n^{1^{++}}(r_h = 0)$
2^{++} (Fluctuation $h_{x_2x_3} = h_{x_3x_2}$ in type IIB metric)	(Type IIB) $9.18 + 8.08n = m_n^{0^{++}}(IIB, T)$ ▲	(Type IIB) $= m_n^{1^{++}}(r_h = 0)$

Table : Glueball Spectra - II from Type IIB, IIA and M Theory using WKB quantization condition for $r_h \neq 0$ and $r_h = 0$



$\tilde{m}$
80
60
40
20
2
4
6
8
10
n
0++: M−Theory
0++/0−−/2++ (n>0): IIB
0−+ (n=0): IIA
0−+ (n>0): IIA
1++/2++ (n>0): M−Theory

Salient features

- Interestingly, via a WKB quantization condition using coordinate/field redefinitions of J.A.Minahan [1998], the lightest 0^{++} glueball spectrum for $r_h \neq 0$ coming from scalar metric fluctuations in M theory compares rather well with the $N \to \infty$ lattice results of M.J.Teper [1998]. Also, similar to R.C.Brower[1999], the 0^{++} coming from the scalar fluctuations of the M theory metric is lighter than the 0^{++} coming from type IIB dilaton fluctuations.

State	$N \to \infty$ from M.J.Teper[1998] in units of $\sqrt{\sigma}$	M-theory scalar (metric perturbations in units of $\frac{r_h}{L^2}$)	Type IIB $\delta\phi$ of C.Csaki et al [1998] in units of reciprocal of temporal circle's diameter
0^{++}	4.065 ± 0.055	4.267	4.07 (normalized to match lattice)
0^{++*}	6.18 ± 0.13	6.251	7.02
0^{++**}	7.99 ± 0.22	7.555	9.92
0^{++**}	-	8.588	12.80
0^{++***}	-	9.464	15.67

Table : Comparison of M.J.Teper[1998]'s $N \to \infty$ lattice results for 0^{++} glueball with our supergravity results obtained using WKB quantization condition and redefinitions of J.A.Minahan [1998] for M theory scalar metric fluctuations

- $m_{n>0}^{2^{++}} > m_{n>0}^{0^{++}}$(scalar metric perturbations), similar to R.C.Brower[1999].
- The higher excited states of the type IIA 0^{-+} glueball, for both $r_h \neq 0$ and $r_h = 0$, are isospectral. This is desirable because large-n corresponds to the UV and that takes one away from the BH geometry, i.e., towards $r_h = 0$.
- As per a more recent lattice calculation Y.Chen et al [2006], the 0^{++}-glueball has a mass $4.16 \pm 0.11 \pm 0.04$ (in units of the reciprocal of the 'hadronic scale parameter' of R.Sommer [1993]), which compares rather well with $m_{n=0}^{0^{++}} = 4.267$ (in units of $\frac{r_h}{L^2}$) coming from scalar fluctuations of the M theory metric. Similarly, the 0^{-+}-glueball in Y.Chen et al [2006] has a mass $6.25 \pm 0.06 \pm 0.06$ which matches rather nicely with $m_{n=0}^{0^{-+}}(\delta = 1.26) = 6.25$ (in units of $\frac{r_0}{L^2}$) coming from type IIA one-form fluctuation.

- The ground state and the $n \gg 1$ excited states of 1^{++} and 0^{--} glueballs are isospectral.
- The higher excited $r_h \neq 0$ 2^{++} glueball states corresponding to metric fluctuations of the M-theory metric and the ones corresponding to fluctuations of the type IIB metric, are isospectral. The $r_h = 0$ 2^{++} glueball states corresponding to metric fluctuations of the M-theory/type IIB string theory, are isospectral.
- $m_n^{2^{++}}(\mathrm{NLO}, r_h = 0) = m_n^{1^{++}}(\mathrm{NLO}, r_h = 0) \stackrel{n \gg 1}{\longrightarrow} m_n^{0^{--}}(\mathrm{NLO}, r_h = 0)$, where the ‘NLO’ implies equality with the inclusion of NLO-in-N corrections.

- From a comparison with $N \to \infty$ lattice results, it appears that WKB quantization-based spectra are closer to $N \to \infty$ lattice results than the computations involving imposing Neumann/Dirichlet boundary conditions at the horizon/IR cut-off. In particular, it is pleasantly surprising that the WKB quantization method applied to the $0^{++}, 0^{-+}$ glueball spectra, is able to provide a good agreement (in fact for the lightest 0^{++} glueball spectrum, better than the classic computations of C.Csaki et al [1998]) with lattice results even for the ground and the lower excited states.

Conclusion

- Part 2
  1. Obtained a Lattice-Compatible Deconfinement temperature T_c both for constant dilaton profile
  2. Temperature dependance of DC conductivity $\sigma \sim T^{-\frac{3}{4}}$, charge susceptibility $\chi \sim \sqrt{\mu} T^{\frac{5}{4}}$ and Einstein's relation $\frac{\sigma}{\chi} = D$(Diffusion Constant) requires $\mu \sim \frac{1}{r_h^2}$.

- Part 3
  1. The speed of sound $v_s = \frac{\mathcal{O}(1)}{\sqrt{3}} + \frac{g_s M^2 N_f \log N}{N}$.
  2. The diffusion constant $D = \frac{1}{\pi T}\left(\frac{1}{4} + \frac{3 g_s^2 M^2 N_f \log N (5+2\log N)}{512\pi^2 N}\right)$.
  3. $\frac{\eta}{s}$-vs-T behavior within RHIC error bars for $T_c \leq T \leq 1.6 T_c$.
  4.

$$\sigma = (0.39)\frac{\sqrt{|\mu|}\,(g_s N)^{\frac{3}{4}}}{2^{\frac{1}{4}} T^{\frac{1}{4}} \pi^{\frac{3}{8}}},$$

$$\kappa_T = \frac{9 \times 0.39}{200 \times 2^{3/4}} \frac{\sqrt{|\mu|} g_s^{\frac{3}{2}} C^2}{\sqrt{N} \pi^{\frac{17}{8}} T^{\frac{15}{4}} \left(2\pi - g_s N_f \log|\mu_{\rm Ouyang}|\right)^2};$$

$$\text{Wiedemann} - \text{Franz law}: \frac{\kappa_T}{\sigma T} = \frac{9}{200\sqrt{2}} \frac{g_s^{\frac{3}{4}} C^2}{N^{\frac{5}{4}} \pi^{\frac{7}{2}} \left(2\pi - g_s N_f \log|\mu|\right)^2 T^{\frac{9}{2}}};$$

- Part 4
  1. Lattice compatible glueball spectrum.

Thanks.

IR Flavor-Color Enhancement of Length Scale for $\mathcal{O}(1)$ M

- The ten-dimensional warp factor h, disregarding the angular part, can be written in terms of the five-form flux $N_{\rm eff}$ as:

$$h = \frac{4\pi g_s}{r^4}\left[N_{\rm eff}(r) + \frac{9g_s M_{\rm eff}^2 g_s N_f^{\rm eff}}{2\,(2\pi)^2}\log r\right],$$

where K. Dasgupta et al [2010]

$$N_{\rm eff}(r) = N\left[1 + \frac{3g_s M_{\rm eff}^2}{2\pi N}\left(\log r + \frac{3g_s N_f^{\rm eff}}{2\pi}(\log r)^2\right)\right],$$

$$M_{\rm eff}(r) = M + \frac{3g_s N_f M}{2\pi}\log r + \sum_{m\geq 1}\sum_{n\geq 1} N_f^m M^n f_{mn}(r) \equiv M + M' + \tilde{M} \equiv M + \tilde{\tilde{M}},$$

$$N_f^{\rm eff}(r) = N_f + \sum_{m\geq 1}\sum_{n\geq 0} N_f^m M^n g_{mn}(r).$$

The terms in the double summation arise, e.g., from the terms higher order in $g_s N_f$ in $\tilde{F}_3$ and the NLO terms in h_i's, both of which though in principle calculable from the solutions to the IIB supergravity equations of motion, are very cumbersome to work out.

- Seiberg duality is then effected via $r \to re^{-\frac{2\pi}{3g_s(M+M')}}$ P. Ouyang [2003], under which $N_{\rm eff} \to N_{\rm eff} - M + \frac{M^2}{(M+M')^2}N_f$. For $r = \Lambda : \log\Lambda \ll \frac{2\pi}{3g_sN_f}$, $\frac{M^2}{(M+M')^2}N_f = N_f\left\{1 - \frac{3g_sN_f}{\pi}\log\Lambda + \mathcal{O}\left[\left(\frac{3g_sN_f}{2\pi}\log\Lambda\right)^2\right]\right\}$. Hence, up to $\mathcal{O}\left(g_sN_f^2\log\Lambda\right)$, $N_{\rm eff} \to N_{\rm eff} - (M - N_f)$. Continuing this process until, as written earlier, one cascades almost (as one has to consider higher order terms in $\frac{3g_sN_f}{\pi}\log\Lambda$ in the MQGP limit that involves $g_s \stackrel{\sim}{<} 1$ and $N_f \sim \mathcal{O}(1)$ and $\Lambda : \log\Lambda < \frac{2\pi}{3g_sN_f}$) the entire $N_{\rm eff}$ away, i.e., $N_{\rm eff}(\Lambda) \approx 0$, one ends up with:

$$h(\Lambda) \sim \frac{4\pi g_s}{r^4}M^2N_f^3\left(\frac{3g_s}{2\pi}\sum_{m\geq 0}\sum_{n\geq 0}N_f^mM^nf_{mn}(\Lambda)\right)^2\sum_{l\geq 0}\sum_{p\geq 0}N_f^lM^pg_{lp}(\Lambda).$$

- Hence, the length scale of the OKS-BH metric in the IR will be given by:

$$
\begin{aligned}
L_{\rm OKS-BH} &\sim \sqrt{M} N_f^{\frac{3}{4}} \sqrt{\left(\sum_{m\geq 0}\sum_{n\geq 0} N_f^m M^n f_{mn}(\Lambda)\right)} \left(\sum_{l\geq 0}\sum_{p\geq 0} N_f^l M^p g_{lp}(\Lambda)\right)^{\frac{1}{4}} g_s^{\frac{1}{4}} \sqrt{\alpha'} \\
&\equiv N_f^{\frac{3}{4}} \sqrt{\left(\sum_{m\geq 0}\sum_{n\geq 0} N_f^m M^n f_{mn}(\Lambda)\right)} \left(\sum_{l\geq 0}\sum_{p\geq 0} N_f^l M^p g_{lp}(\Lambda)\right)^{\frac{1}{4}} L_{\rm KS}\Bigg|_{\Lambda:\log\Lambda < \frac{2\pi}{3 g_s N_f}},
\end{aligned}
$$

which implies that **in the IR, relative to KS, there is a color-flavor enhancement of the length scale in the OKS-BH metric**. Hence, in the IR, even for $N_c^{\rm IR} = M = 3$ and $N_f = 6$ upon inclusion of of $n, m > 1$ terms in $M_{\rm eff}$ and $N_f^{\rm eff}$, $L_{\rm OKS-BH} \gg L_{\rm KS}(\sim L_{\rm Planck})$ in the MQGP limit involving $g_s \stackrel{\sim}{<} 1$.