## SUM RULES FROM HOLOGRAPHY

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## INTRODUCTION AND MOTIVATION

- Sum rules constrain spectral densities in quantum field theories.

$$
\int \frac{d \omega}{\omega^{n}} \rho(\omega)=\text { constant }
$$

$\rho(\omega)=\operatorname{Im} G(\omega)$

- In QCD there are sum rules for the bulk spectral density that depend on one point functions like energy and pressure which can be determined from lattice calculations.
Thus the sum rule constraints 2-point correlators which is hard to do in lattice.
- The Ferrell-Glover sum rule in BCS theory determines the coherence length of the cooper pair in terms of an integral over frequency dependent conductivities.
- Sum rules result form unitarity and causality of the QFT. Studying them from holographic duals helps to understand how these are encoded in the dual geometry.
- Sum rules can constrain QCD/CMT applications of holography.
- Sum rules relations which involve the entire frequency domain, they go beyond hydrodynamics.


## THE STRUCTURE OF SUM RULES

- Let $G(\omega, q=0)$ be the retarded correlator at temperature $T$ of the:
stress tensor $T_{x y}$
R-currents $J_{x}^{i}$ of the following theories:
$\mathcal{N}=4$ Yang-Mills
M2-brane theory
M5-brane theory.
- Let $\rho_{T}^{i}(\omega)=\operatorname{Im} G^{i}(\omega)$ be the corresponding spectral density. $i$ labels either the R-current or the stress tensor .

$$
\int_{-\infty}^{\infty} \frac{d \omega}{\pi \omega}\left(\rho_{T}^{i}(\omega)-\rho_{T=0}^{i}(\omega)\right)=G^{i}(\omega=0)+C_{i i}^{k}\left\langle\mathcal{O}_{k}\right\rangle_{T}
$$

$C_{i i}^{k}$ are the structure constants of certain chiral primaries $\mathcal{O}_{k}$ which appear in the OPE of the stress tensor/R-currents.
-The first term in the sum rule is determined by hydrodynamics.
The second terms is determined by the short distance properties of the theory.

- The sum rule connects information of the 1-point , 2-point, and the 3-point functions of the theory.
- The sum rule provides an alternative approach to determine 3 -point functions. As a consistency check they can be compared to that obtained from the corresponding Witten diagrams.


## SUM RULE GENERALITIES

- Sum rules are determined by the analytic properties of the Green's function in the $\omega$-plane.
- Consider $G(\omega)$ with

1. $G(\omega)$ is holomorphic in the upper half plane, including the real axis.
2. $\lim _{|\omega| \rightarrow \infty} G(\omega)=0$ if $\operatorname{Im}(\omega) \geq 0$

- By Cauchy's theorem

$$
\begin{align*}
G(\hat{\omega}+i \epsilon) & =\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{G(z) d z}{z-\hat{\omega}-i \epsilon}  \tag{1}\\
0 & =\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{G(z) d z}{z-\hat{\omega}+i \epsilon} . \tag{2}
\end{align*}
$$

$\hat{\omega}, \epsilon \in R$ and $\epsilon>0$. The contour is chosen as a large semi-circle in the upper half plane.

- Taking (1) $+(2)^{*}$

$$
G(\hat{\omega})=\lim _{\epsilon \rightarrow 0^{+}} \int_{-\infty}^{\infty} \frac{d z}{\pi} \frac{\rho(z)}{z-\hat{\omega}-i \epsilon},
$$

where $\rho(z)=\operatorname{Im} G(z)$.

- Setting $\hat{\omega}=0$ we obtain

$$
G(0)=\lim _{\epsilon \rightarrow 0^{+}} \int_{-\infty}^{\infty} \frac{d \omega}{\pi} \frac{\rho(\omega)}{\omega-i \epsilon}
$$

- Deriving sum rules essentially reduces to proving the analytic properties of the Green's function.
- In AdS/CFT the Green's function of interest is determined by solving certain differential equations.
- Thus determining the analytic behaviour of the Green's function can be cast into a problem of establishing the analytic behaviour of solutions of the differential equations which determine the relevant Green's function.
- We will demonstrate this in detail for the simplest situation.


## THE SHEAR SUM RULE

- This sum rule was originally derived and verified numerically by Romatschke, Son (2009).
The proof we will discuss is a variation of the one developed by Gulotta, Herzog, Kaminski (2010).
- Consider the retarded Green's function

$$
\tilde{G}(t, \vec{x})=i \theta(t)\left\langle\left[T_{x y}(t, \vec{x}), T_{x y}(0,0)\right]\right\rangle_{T},
$$

in $\mathcal{N}=4$ Yang-Mills.
$T_{x y}$ is the $x y$ component of the stress tensor.
Its Fourier transform

$$
G(\omega, 0)=\int d^{4} x e^{i(\omega t)} \tilde{G}_{R}(t, x)
$$

- To evaluate the shear correlator we consider the non-extremal D3-brane geometry

$$
\begin{aligned}
d s^{2} & =\frac{r^{2}}{L^{2}}\left(-f d t^{2}+d x^{2}+d y^{2}+d z^{2}\right)+\frac{L^{2}}{r^{2} f} d r^{2} \\
f & =1-\frac{r_{+}^{4}}{r^{4}}
\end{aligned}
$$

- The stress tensor $T_{x y}$ is dual to the fluctuation

$$
\delta g_{x y}=\phi(r) e^{-i \omega t} \frac{r^{2}}{L^{2}}
$$

This fluctuation obeys the equation of motion of a minimally coupled massless scalar

$$
\partial_{r}^{2} \phi+\left(\frac{F^{\prime}}{F}+\frac{3}{r}\right) \partial_{r} \phi+\frac{\omega^{2}}{F^{2}} \phi=0
$$

where

$$
F=\frac{r^{2}}{L^{2}} f
$$

- The Green's function is obtained by imposing ingoing boundary conditions at the horizon $r_{+}$and obtaining the behaviour of $\phi$ at the boundary $r \rightarrow \infty$.

$$
\begin{aligned}
G(\omega, T) & =\hat{G}(\omega, T)-P(T)+G_{\text {counter }}(\omega) \\
\hat{G}(\omega) & =-\frac{N^{2}}{8 \pi^{2} L^{6}} \lim _{r \rightarrow \infty} \frac{F r^{3} \phi^{\prime}}{\phi}
\end{aligned}
$$

$G_{\text {counter }}(\omega)$ is the contribution from the counter terms required to cancel the $r^{2}$ and $\log (r)$ divergences in $\hat{G}(\omega, T)$.
$P(T)$ is the pressure, independent of $\omega$.
$G_{\text {counter }}(\omega)$ is independent of temperature $T$.

- Essentially the behaviour of the Green's function is determined by

$$
g(\omega)=\lim _{r \rightarrow \infty} \frac{F r^{3} \phi^{\prime}}{\phi}
$$

- The behaviour of $g(\omega)$ in the $\omega$-plane can be obtained by studying the solution to the differential equation.
- Near the horizon the solutions are wave like and is given by

$$
\phi(r) \sim\left(r-r_{+}\right)^{ \pm \frac{i \omega}{F_{h}}}, \quad r \rightarrow r_{+}
$$

where

$$
F_{h}=4 \frac{r_{+}}{L^{2}}
$$

- At the boundary the two independent solutions are given by

$$
\begin{aligned}
& \phi(r) \rightarrow \frac{L^{4} \omega^{2}}{r^{2}} J_{2}\left(\frac{L^{2} \omega}{r}\right) \sim r^{-4}, \quad r \rightarrow \infty \\
& \phi(r) \rightarrow \frac{L^{4} \omega^{2}}{r^{2}} K_{2}\left(i \frac{L^{2} \omega}{r}\right) \sim \text { constant, } r \rightarrow \infty
\end{aligned}
$$

- The differential equation can be obtained as the equations of motion of the following action

$$
S_{\phi}=\int_{r_{h}}^{\infty} d r F r^{3}\left(\left|\phi^{\prime}(r)\right|^{2}-\frac{\omega^{2}}{F^{2}}|\phi(r)|^{2}\right) .
$$

## No poles in $g(\omega)$ for $\operatorname{Im} \omega>0$

- Poles/divergences in $g(\omega)$ correspond to quasi-normal modes of the differential equation. They obey the following boundary conditions.

$$
\begin{aligned}
& \phi(r) \sim\left(r-r_{+}\right)^{-i \frac{\omega}{F_{h}}}, \quad r \rightarrow r_{+} \\
& \phi(r) \sim r^{-4}, \quad r \rightarrow \infty
\end{aligned}
$$

- Quasi-normal modes with $\operatorname{Im} \omega>0$ do not exist. Intuitive reason: from the time dependence $\exp (-i \omega t)$, we see such modes are instabilities if they exists.
- A proof from the differential equation:

Let $\phi(r)$ be a quasi-normal mode with complex frequency $\omega$.
The coefficients of the differential equation is real.
$\phi(r)^{*}$ is also a quasi-normal mode with frequency $\omega^{*}$.

- Consider the identity

$$
0=S_{\phi}-S_{\phi}
$$

Use equation of motion of $\phi^{*}$ in first $S_{\phi}$. equation of motion of $\phi$ in the second $S_{\phi}$.

$$
0=\left.F r^{3}\left(\phi^{* \prime} \phi-\phi^{*} \phi^{\prime}\right)\right|_{r_{h}} ^{\infty}+\left(\omega^{* 2}-\omega^{2}\right) \int_{r_{h}}^{\infty} d r \frac{r^{3}}{F}|\phi|^{2}
$$

- From the boundary conditions and using $\operatorname{Im} \omega>0$ can show the boundary terms vanish. Since the integrand in the second term is positive definite we have

$$
\omega^{2}=\omega^{* 2}
$$

- So if at all a quasi-normal mode exists in for $\operatorname{Im} \omega>0$ it is restricted to the positive imaginary axis.

$$
\omega^{2}<0
$$

- Evaluating the action for such a mode one obtains

$$
S_{\phi}=\left.F r^{3} \phi^{*}(r) \phi^{\prime}(r)\right|_{r_{h}} ^{\infty}=0
$$

- But $S_{\phi}$ is positive definitive for $\omega^{2}<0$.

Thus no such quasi-normal mode exists.

## No poles for $\omega$ real and $\omega \neq 0$

- For real $\omega, \phi, \phi^{*}$ are linearly independent and the following is the Wronskian.

$$
W=-i \frac{2 \omega r_{+}^{3}}{r^{3} F}
$$

- The solution for the Wronskian consistent with ingoing boundary conditions at the horizon

$$
W=-i \frac{2 \omega r_{+}^{3}}{r^{3} F}
$$

- $r^{3} \mathrm{FW}$ is a non-zero constant. Evaluating it at $r \rightarrow \infty$ for the quasi-normal mode indicates that it must vanish.
- Contradiction: No quasi-normal mode and hence poles or divergences of $g(\omega)$ does not exist in this domain.


## No poles for $\omega=0$

- $g(\omega)$ admits a power series expansion around the $\omega=0$.
- Define

$$
\tilde{g}(r)=\frac{\phi^{\prime}(r)}{\omega \phi(r)} .
$$

It satisfies

$$
\tilde{g}^{\prime}(r)+\omega \tilde{g}^{2}(r)+\left(\frac{F^{\prime}}{F}+\frac{3}{r}\right) \tilde{g}(r)+\frac{\omega}{F^{2}}=0 .
$$

This admits a solution in terms a power series in $\omega$.

- The leading term in the solution consistent with the ingoing boundary conditions

$$
g=-i \omega r_{+}^{3}+O\left(\omega^{2}\right)
$$

## Absence of branch cuts for $\operatorname{Im} \omega \geq 0$

- A theorem in Ordinary differential equations, Arnold: The solutions of a differential equation is smooth with respect to parameter $\omega$ provided the differential equation and the boundary conditions are smooth with respect to $\omega$.
- Applying this for our case we conclude that $\phi$ and $\phi^{\prime}$ is smooth with respect to $\omega$ at $r \rightarrow \infty$.
- Thus $g(\omega)$ is smooth with respect to $\omega$.

The only locations of possible singularities in the $n$-th order derivative of $g(\omega)$ with respect to $\omega$ are if $\phi$ vanishes at the boundary. This can occur at possible quasi-normal modes.
We have shown that they do not occur at $\operatorname{Im} \omega \geq 0$.

## $\omega \rightarrow \infty$ behaviour

- Define

$$
y=\lambda \frac{r_{+}}{r}, \quad i \lambda=\frac{L^{2}}{r_{+}} \omega .
$$

Note the boundary is now at $y=0$.
The differential equation:

$$
\begin{gathered}
\phi^{\prime \prime}(y)-\frac{1}{y f(y)}\left(3+\frac{y^{4}}{\lambda^{4}}\right) \phi^{\prime}(y)-\frac{1}{f(y)^{2}} \phi(y)=0, \\
f=1-\frac{y^{4}}{\lambda^{4}}
\end{gathered}
$$

- Note that for $\lambda \rightarrow \infty$, the equation is that of a minimally coupled scalar in $A d S_{5}$.
The solutions are given in terms of Bessel functions.
- A systematic expansion of the solutions around $\lambda \rightarrow \infty$ consistent with ingoing boundary condition at the horizon can be performed.
- The results in the following Green's function.

$$
\begin{gathered}
\lim _{\lambda \rightarrow \infty} g(\lambda)=-\frac{r_{+}^{4}}{L^{2}}\left(\lim _{y \rightarrow 0} \frac{1}{y^{3}} \lambda^{4} g_{0}^{(1)}-\frac{6}{5}+O\left(\frac{1}{\lambda^{4}}\right)\right) . \\
g_{0}^{(1)}=-\frac{K_{1}(y)}{K_{2}(y)}
\end{gathered}
$$

- The leading term contains the $1 / y^{2}, \log (y)$ divergence.

That will be regulated by $G_{\text {counter }}$.
It is independent of temperature and goes as $\omega^{4}$.

- The finite term implies that the Green's function does not satisfy the fall off property required for the derivation of the sum rule.
The presence of the $-P(T)$, pressure in the Green's function also does the same.
- Thus define

$$
\delta G_{R}(\omega)=G_{R}(\omega, T)-G_{R}(\omega, 0)+\frac{N^{2}}{8 \pi^{2} L^{6}} \frac{r_{+}^{4}}{L^{2}} \frac{6}{5}+P
$$

This satisfies the required fall off property as $\omega \rightarrow \infty$.

$$
\begin{aligned}
\operatorname{Im} \delta G_{R}(\omega) & =\operatorname{Im} G_{R}(\omega, T)-\operatorname{Im} G_{R}((\omega, 0) \\
& =\rho(\omega, T)-\rho(\omega, 0)
\end{aligned}
$$

- Applying Cauchy's theorem we obtain shear sum rule

$$
\frac{2}{5} \epsilon=\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \omega}{\omega}\left(\rho(\omega)-\rho_{T=0}(\omega)\right)
$$

## Sum rule from OPE

- The constant term resulted from the high frequency analysis of the Green's function.
It should be possible to understand it from OPE arguments.
- Consider the OPE of the stress tensor.

$$
T_{\mu \nu}(x) T_{\rho \sigma}(0) \sim C_{T} \frac{I_{\mu \nu, \rho \sigma}}{x^{8}}+\hat{A}_{\mu \nu \rho \sigma \alpha \beta}(x) T_{\alpha \beta}(0)+B_{\mu \nu \rho \sigma}^{a}(x) \mathcal{O}_{a}(0)
$$

- Taking the Fourier transform, and by a simple scaling analysis.
Constant terms at high frequency in the Green's function result from the presence of the $T_{\mu \nu}$ and operators of dimension $\Delta=4$ in the OPE.
- For the uncharged D3-brane, in the dual background there are no other operators turned on.
The constant term must result from the expectation value of the stress tensor.
- It can be shown that

$$
\frac{2}{5} \epsilon
$$

in the sum rule results form this term in the OPE.

## R-CHARGE SUM RULES

- Consider the R-charge correlator in $\mathcal{N}=4$ Yang-Mills.

$$
G^{i}(t, \vec{x})=i \theta(t)\left\langle\left[J_{x}^{i}(t, \vec{x}), J_{x}^{i}(0,0)\right]\right\rangle,
$$

$J_{x}^{j}$ is the $x$ component of the $i$-th R-symmetry current and $i \in\{1,2,3\}$.

- Study the Green's function at finite temperature with the 3-chemical potentials turned on and obtain sum rules.
- The gravity dual is the R-charged black hole in $A d S_{5}$. The system is parametrized by:
The radius of the horizon $r_{+}$; temperature.
The three R-charges $k_{i}$, chemical potentials.
- The solution also has 2 scalars, $\vartheta_{1,2}$ turned on.

The scalars correspond to chiral primary operators of conformal dimension $\Delta=2$. They are linear combinations of

$$
\operatorname{Tr}(Z \bar{Z}), \quad \operatorname{Tr}(Y \bar{Y}), \quad \operatorname{Tr}(Z \bar{Z})
$$

which excludes the Konishi scalar

$$
\operatorname{Tr}(X \bar{X})+\operatorname{Tr}(Y \bar{Y})+\operatorname{Tr}(Z \bar{Z})
$$

Call this as $\mathcal{O}_{1,2}$.

- The expectation values of these scalars from the gravity solution is

$$
\begin{aligned}
\left\langle\mathcal{O}_{1}\right\rangle & =\frac{N^{2}}{8 \pi^{2}} \frac{r_{+}^{2}}{L^{4}} \frac{2}{\sqrt{6}}\left(k_{1}+k_{2}-2 k_{3}\right), \\
\left\langle\mathcal{O}_{2}\right\rangle & =\frac{N^{2}}{8 \pi^{2}} \frac{r_{+}^{2}}{L^{4}} \frac{2}{\sqrt{2}}\left(k_{1}-k_{2}\right) .
\end{aligned}
$$

- To evaluate the Green's function from gravity consider.

$$
\begin{array}{r}
A_{x}^{i}=A_{x}^{i(0)}+a_{x}^{i}(r, t), \\
g_{x t}=g_{x t}^{(0)}+h_{x t}^{i(r, t),}, \\
g_{l t}=g_{l t}^{(0)}, \\
g_{l m}=g_{l m}^{(0)},
\end{array} \vartheta_{1,2}=\vartheta_{1,2}^{(0)}, ~ \$
$$

Here $I \in\{y, z, t, r\}$ and the superscript ${ }^{(0)}$ refers to the background values and

$$
a_{x}^{i}(r, t)=\phi^{i}(r) e^{i \omega t}
$$

- One can obtain the coupled equations of motion for the gauge field fluctuations.

$$
\begin{aligned}
& \phi^{i \prime \prime}+\left(\ln \left(\frac{F H_{i}^{2}}{\mathcal{H}}\right)^{\prime}+\frac{1}{r}\right) \phi^{i \prime}+\frac{\omega^{2} \mathcal{H}}{F^{2}} \phi^{i} \\
& -\left(1+k_{i}\right) \frac{m_{i}}{H_{i}^{2}} \sum_{j=1}^{3}\left(\frac{4 r_{+}^{6}}{r^{6} L^{2} F}\left(1+k_{j}\right) m_{j} \phi^{j}\right)=0, \\
& m_{i}=\sqrt{k_{i}} \prod_{j=1}^{3}\left(1+k_{j}\right)^{1 / 2}
\end{aligned}
$$

- The equations of motion can be obtained by a variation of $\phi^{i *}$ on the following action.

$$
S_{\phi}=\int_{r_{h}}^{\infty} d r \frac{F r H_{i}^{2}}{\mathcal{H}}\left(\frac{d \phi^{i *}}{d r} \delta_{i j} \frac{d \phi^{j}}{d r}\right)+\phi^{* i}\left(M_{i j}-\frac{\omega^{2} H_{i}^{2} r}{F} \delta_{i j}\right) \phi^{j}
$$

$$
M_{i j}=\frac{4 r_{+}^{6}}{L^{2} r^{5} \mathcal{H}}\left(1+k_{i}\right) m_{i}\left(1+k_{j}\right) m_{j}
$$

- The retarded Green's function is obtained by imposing ingoing boundary conditions

$$
\phi^{i}(r) \sim\left(r-r_{+}\right)^{-i \alpha \omega}
$$

and evaluating

$$
\begin{aligned}
G_{T}^{i}(\omega) & =\hat{G}^{i}(\omega, T)+G_{\text {counter }}(\omega, T), \\
\hat{G}^{i}(\omega, T) & =-\left.\frac{N^{2}}{8 \pi^{2} L^{3}} \lim _{r \rightarrow \infty} \frac{r F \phi^{i}}{L \phi^{i}}\right|_{\phi_{\infty}^{j}=0, j \neq i}
\end{aligned}
$$

$G_{\text {counter }}(\omega, T)$ is the counter term needed to cancel the $\log (r)$ divergences.

- By a similar analysis one can show that the conditions required to derive the sum rule holds for

$$
\begin{gathered}
\delta G^{1}(\omega)=G^{1}(\omega, T)-G^{1}(\omega, 0)-\frac{1}{e^{2}} \frac{2 r_{+}^{2}}{3 L^{3}}\left(-2 k_{1}+k_{2}+k_{3}\right) \\
\frac{1}{e^{2}}=\frac{N^{2}}{8 \pi^{2} L^{3}}
\end{gathered}
$$

A similar definition holds for the other two Green's functions.

- The sum rules are

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{d \omega}{\pi \omega}\left(\rho^{1}(\omega)-\rho_{T=0}^{1}(\omega)\right)= & \lim _{\omega \rightarrow 0} \omega \operatorname{Im} \sigma^{1}(\omega) \\
& -\frac{1}{e^{2}} \frac{2 r_{+}^{2}}{3 L^{3}}\left(-2 k_{1}+k_{2}+k_{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{d \omega}{\pi \omega}\left(\rho^{2}(\omega)-\rho_{T=0}^{2}(\omega)\right)= & \lim _{\omega \rightarrow 0} \omega \operatorname{Im} \sigma^{2}(\omega) \\
& -\frac{1}{e^{2}} \frac{2 r_{+}^{2}}{3 L^{3}}\left(k_{1}-2 k_{2}+k_{3}\right)
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} \frac{d \omega}{\pi \omega}\left(\rho^{3}(\omega)-\rho_{T=0}^{3}(\omega)\right)=\lim _{\omega \rightarrow 0} \omega \operatorname{Im} \sigma^{3}(\omega)
$$

$$
-\frac{1}{e^{2}} \frac{2 r_{+}^{2}}{3 L^{3}}\left(k_{1}+k_{2}-2 k_{3}\right)
$$

-The high frequency contribution in the sum rule can be obtained from an OPE analysis.
The OPE of two R-currents is

$$
J_{\mu}^{i}(x) J_{\nu}^{j}(0) \sim \frac{\mathcal{C} \delta_{i j} I_{\mu \nu}(x)}{x^{6}}+\mathcal{A}_{\mu \nu} C_{i j}^{\hat{k}} \mathcal{O}_{\hat{k}}(0)+\mathcal{B}_{\mu \nu ; k}^{i j \rho} J_{\rho}^{k}(0)+\cdots,
$$

- Taking the Fourier transform, and by a scaling analyis: The finite terms at high frequency of the Green's function arises due to the presence of operators of $\Delta=2$.

$$
\lim _{\omega \rightarrow \infty} G_{T}^{i}(\omega)-G_{T=0}^{i}(\omega)=4 \pi^{2} C_{i i} \hat{k}^{\prime}\left\langle\mathcal{O}_{\hat{k}}(0)\right\rangle_{T} .
$$

Note that this analysis is entirely based on conformal invariance and does not involve the gravity dual.

- Comparing this to the high frequency terms in the sum rule and using the expectation values of the scalars $\mathcal{O}_{1,2}$ we obtain the following structure constants.

$$
\begin{aligned}
& C_{11}^{\hat{1}}=-\frac{1}{(2 \pi)^{2} L^{2} \sqrt{6}}, C_{11}^{\hat{2}}=-\frac{1}{(2 \pi)^{2} L^{2} \sqrt{2}}, \\
& C_{22}^{\hat{1}}=-\frac{1}{(2 \pi)^{2} L^{2} \sqrt{6}}, C_{22}^{\hat{2}}=\frac{1}{(2 \pi)^{2} L^{2} \sqrt{2}}, \\
& C_{33}^{\hat{1}}=\frac{2}{(2 \pi)^{2} L^{2} \sqrt{6}}, \quad C_{33}^{\hat{2}}=0 .
\end{aligned}
$$

- These structure constants can also be directly evaluated using Witten diagrams from the following cubic coupling in the supergravity Lagrangian

$$
\frac{1}{4 e^{2}} \vec{a}^{i} \cdot \int d^{5} x \sqrt{g} g^{\mu \rho} g^{\nu \sigma} \vec{\vartheta} F_{\mu \nu}^{i} F_{\nu \rho}^{i},
$$

where we have organized the two scalars $\vartheta_{1}, \vartheta_{2}$ into a two dimensional vector and the two dimensional vectors $\vec{a}^{i}$ with $i=1,2,3$ are given by

$$
\vec{a}^{1}=2\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right), \quad \vec{a}^{2}=2\left(\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{2}}\right), \quad \vec{a}^{3}=2\left(-\frac{2}{\sqrt{6}}, 0\right) .
$$

- Proceeding with the evaluation of the Witten diagrams and extracting the structure constants we indeed do recover the ones obtained from the high frequency contribution to the sum rule precisely.


## SUM RULES IN OTHER SYSTEMS

## M2-branes

- The sum rules are similar in form.

The high frequency contribution arises from expectation values of three chiral primaries of $\Delta=1$.
Here there are 12 structure constants that match precisely with that obtained from the Witten diagrams.

## M5-branes

- The high frequency contribution arises from expectation values of 2 chiral primaries of $\Delta=4$. There are 4 structure constants which agree with that obtained from the Witten diagrams.


## Shear sum rules at finite chemical potential

- We have also obtained the modifications in the shear sum rule for the case of D3, M2 and M5-branes at finite chemical potential.
These modifications can also be explained due to the expectation values of the corresponding scalars in these background.


## CONCLUSIONS

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$$

- Sum rules carry important information of the theory.

They can be derived holographically.
The analytic properties of the Green's function involved can be extracted from the properties of the corresponding differential equations.

- It will be interesting to derive sum rules for spectral densities sensitive to the $U(1)^{3}$ anomaly present in $\mathcal{N}=4$ Yang-Mills. Anomalies are present both at long distances and short distances.
It will be interesting to see how they occur in the sum rules.
- It will be useful to examine putative holographic duals of QCD and obtain sum rules.
These sum rules will a-priori be different from that of QCD since the UV properties of these theories are different form QCD. This might provide tight constraints on the validity of these models.

