From Fox 3-coloring to Yang-Baxter homology

Jozef H. Przytycki

Department of Mathematics
The George Washington University

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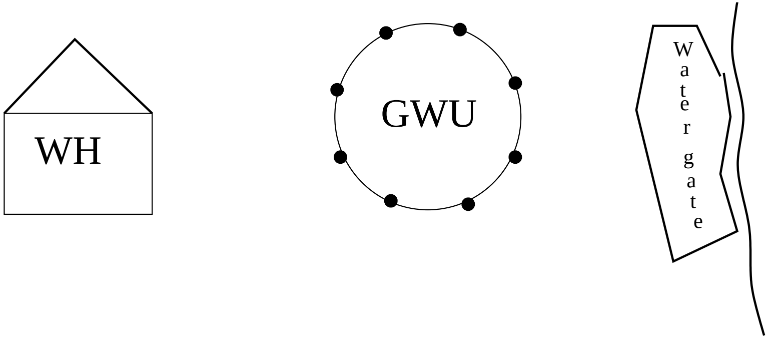
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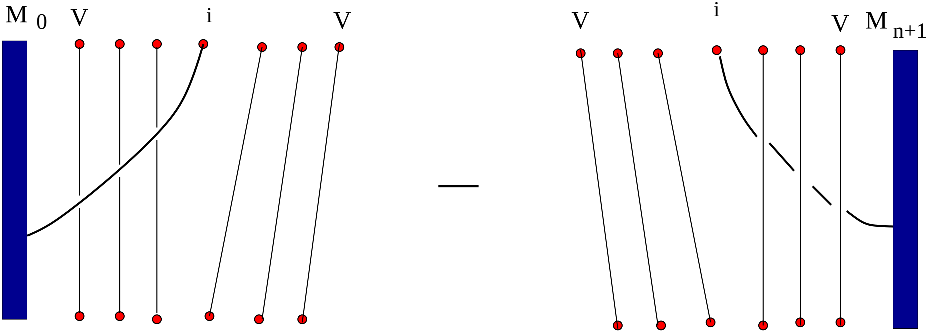
Welcome!



Diagramatic interpretation of a face map d_i^{Y-B}

Curtain to Yang-Baxter homology

Abstract

We start from naive invariants of arc colorings and survey distributive magmas and their homology with relation to knot theory. We outline potential relations to Khovanov homology and categorification, via Yang-Baxter operators. We use here the fact that Yang-Baxter equation can be thought of as a generalization of self-distributivity. We show how to define and visualize Yang-Baxter homology.

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$$\partial_3 : V \otimes V \otimes V \to V \otimes V$$

$$\partial_3 = d_3^r + d_2^\ell + d_1^r - (d_1^\ell + d_2^r + d_3^l)$$

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Despite the fact that I made this observation in May of 2015, the 2-(co)cycle invariants from Yang-Baxter operators which are not set theoretic, are still waiting to be studied.

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The work of Victoria Lebed, Takefumi Nosaka, Maciej Niebrzydowski, Krzysztof Putyra, Xiao Wang, Seung Yeop Yang, and many others should be acknowledged here. Specially my students who wrote a joint project with Nosaka and myself: Rhea Palak Bakshi, Dionne Ibarra, Sujoy Mukherjee, Takefumi Nosaka, J.H.Przytycki, Schur Multipliers and Second Quandle Homology, *Journal of Algebra*, Volume 552, June, 2020, 52-67; e-print: `arXiv:1812.04704 [math.GT]`

Study materials:

J.H.Przytycki, 3-coloring and other elementary invariants of knots, Banach Center Publications, Vol. 42, *Knot Theory*, Warsaw, 1998, 275-295;
e-print: `http://arxiv.org/abs/math.GT/0608172`

J.H.Przytycki, Knots and distributive homology: from arc colorings to Yang-Baxter homology, Chapter in: *New Ideas in Low Dimensional Topology*, World Scientific, Vol. 56, March-April 2015, 413-488; e-print: `arXiv:1409.7044 [math.GT]`.

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J.H.Przytycki, Knots and distributive homology: from arc colorings to Yang-Baxter homology, Chapter in: *New Ideas in Low Dimensional Topology*, World Scientific, Vol. 56, March-April 2015, 413-488; e-print: `arXiv:1409.7044 [math.GT]`.
Youtube Lectures:
Progress in Yang-Baxter homology:
https://www.youtube.com/watch?v=ZJwUVy8laO0
Lecture VIII: https://youtu.be/dB1rk7Us-Ag
Lecture VII: https://youtu.be/gNOANJIZMjg
Lecture VI: https://youtu.be/OjpOqoHjcls
(Lectures I-V are very elemetary undergraduate lectures)

Study materials: II

J.H.Przytycki, Distributivity versus associativity in the homology theory of algebraic structures, *Demonstratio Math.*, 44(4), 2011, 823-869; e-print: `arXiv:1109.4850 [math.GT]`

J.H.Przytycki, W.Rosicki, Cocycle invariants of codimension 2 embeddings of manifolds, *Banach Center Publications*, 103, December 2014, 251-289; e-print: `arXiv:1310.3030 [math.GT]`.

J.H.Przytycki, K. K. Putyra, The degenerate distributive complex is degenerate, *European Journal of Mathematics*, Volume 2, Issue 4, December 2016, 993-1012; e-print: `arXiv:1411.5905 [math.GT]`

J.H.Przytycki, Xiao Wang, Equivalence of two definitions of set-theoretic Yang-Baxter homology, *Journal of Knot Theory and Its Ramifications*, 27(7), June 2018, 1841013 (15 pages); e-print: `arXiv:1611.01178 [math.GT]`.

J.H.Przytycki, Xiao Wang, The second Yang-Baxter homology for the Homflypt polynomial, Preprint, April 2020, e-print: `arXiv:2004.07413 [math.GT]`.

Historical Note: pioneers of distributivity

Historical notes on authors of 1929 paper: C. Burstin and W. Mayer, Distributive Gruppen von endlicher Ordnung, *J. Reine Angew. Math.*, vol. 160, 1929, pp. 111-130; Translation "Finite Distributive Groups" by Ansgar Wenzel is available at e-print: `arXiv:1403.6326 [math.GR]`

Walter Mayer (1887–1948) is well known for Mayer-Vietoris sequence and for being assistant to A.Einstein at Institute for Advanced Study, Princeton. In Princeton he considered n chain complexes with $\partial^n = 0$.

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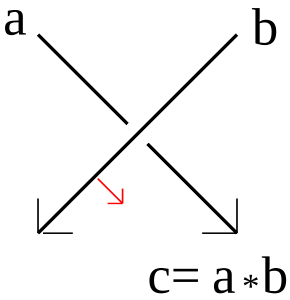
Celestyn Burstin (1888-1938) was born in Tarnopol, where he obtained "Matura" in 1907, he moved to Vienna where in 1911 he completed university. In 1929 he moved to Minsk where he was a member of the Belarusian National Academy of Sciences, and a Director of the Institute of Mathematics of the Academy. In December 1937, he was arrested on suspicion of activity as a spy for Poland and Austria. He died in October 1938, when interrogated in a prison in Minsk ("Minskaja Tjurma"); he was rehabilitated March 2, 1956.

Magma coloring

Magma $(X, *)$, with a binary operation $* : X \times X \to X$.



Convention for a magma coloring of a crossing

Magma coloring: $f\colon \text{Arcs}(D) \to X$ so that $c = a * b$.
$col_X(D)$ denotes the number of colorings of the diagram D by $(X, *)$

$col_X(D)$ as a link invariant

If we want our number $col_X(D)$ to be a link invariant, we check Reidemeister moves and obtain, after Joyce and Matveev, the algebraic structure satisfying conditions (1),(2),(3) below, which Joyce in his 1979 PhD thesis called *quandle*.

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(1) $a * a = a$, for any $a \in X$ (idempotence condition).

(2) The operation $\bar{*}$ is inverse to $*$, that is for any pair $a, b \in X$ we have $(a * b)\bar{*}b = a = (a\bar{*}b) * b$ (invertibility condition. Equivalently we define $*_b : X \to X$ by $*_b(a) = a * b$, and invertibility condition means that $*_b$ is invertible; we denote $*_b^{-1}$ by $\bar{*}_b$.

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(3) $(a * b) * c = (a * c) * (b * c)$ (distributivity), for any $a, b, c \in X$. Figure 1.2 illustrate the relation of (3) to the third Reidemeister move, and in fact can be takes as a "proof without words" that $col_X(D)$ is preserved by the positive third Reidemeister move if and only if $*$ is right self-distributive.

Wrack-rack, kei

If only conditions (2) and (3) hold, then $(X; *, \bar{*})$ is called a rack (or wrack like in "wrack and ruin"); the name coined by J.H.Conway in 1959 (in correspondence with G.Wraith).
If $* = \bar{*}$ in the condition (2), that is $(a * b) * b = a$ then the quandle is called an involutive quandle or kei 圭 (the last term coined in 1942 by M. Takasaki).

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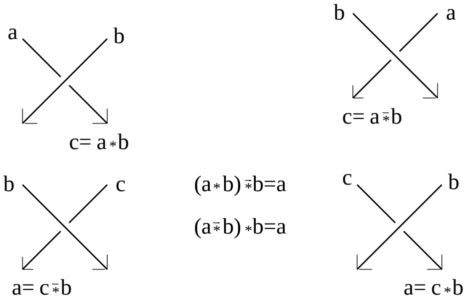
Historical note: Mituhisa Takasaki worked at Harbin Technical University, likely as an assistant to Kôshichi Toyoda. Both perished when Red Army entered Harbin in August 1945.

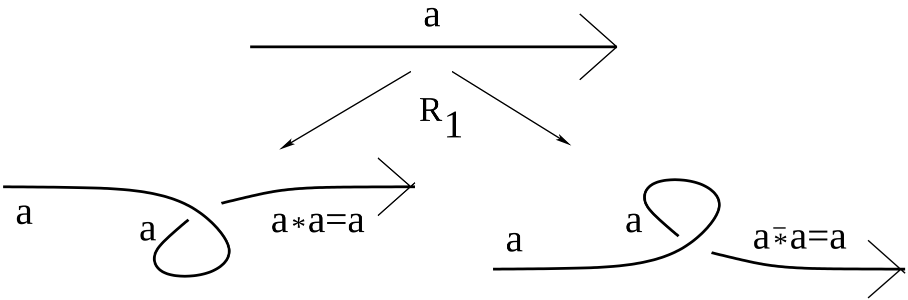
Second Reidemeister move and magma coloring



Second Reidemeister move and magma coloring

$col_X(D)$ is preserved by the second Reidemeister move if $(X, *)$ is invertible.

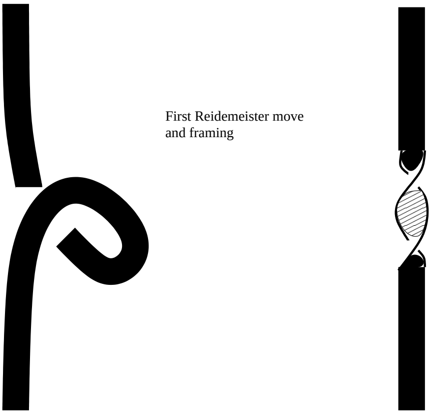
First Reidemeister move and idempotency



First Reidemeister move leads to idempotent condition

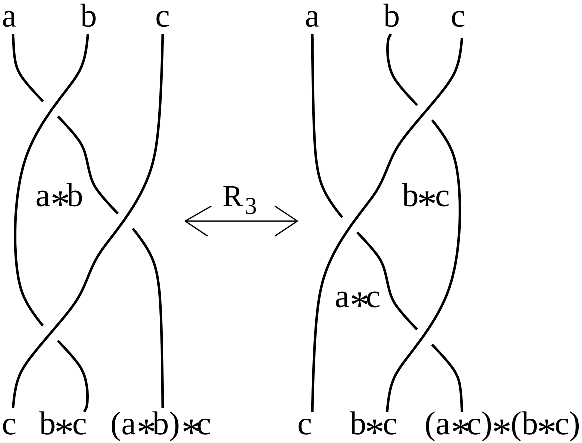
$$a * a = a$$

First Reidemeister move as framing change



First Reidemeister move as framing change

Third Reidemeister move and distributivity



Third Reidemeister move leads to distributivity

$$(a * b) * c = (a * c) * (b * c)$$

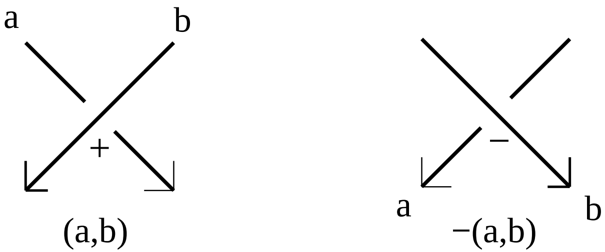
2-(co)cycle invariants of links

Let X be a finite set and $*$ and $\bar{*}$ two binary operations. We define here, after Carter-Kamada-Saito 2-(co)cycle invariants of links:

(1) A 2-chain, $\Psi(D, \phi)$ associated to the diagram D and coloring of its arcs by $\phi : arcs(D) \to X$ is an element of ZX^2 defined as a sum over all crossings of D of the pair $\pm(a, b)$ according to conditions in figure below, that is $\Psi(D, \phi) = \sum_v sgn(v)(a, b)$, where the sum is taken over all crossings of D.

(2) A 2-cochain with coefficients in an abelian group A is a function $\alpha : X^2 \to A$ or equivalently an element of $Hom(ZX^2, A)$. A 2-cochain associated to the diagram D and coloring of its arcs by $\phi : arcs(D) \to X$ is an element of $Hom(ZX^2, A)$, defined by $\Psi(D, \phi, \alpha) = \Sigma_v sgn(v)\alpha(a, b)$.

Origin of rack homology



Figure; Convention for the 2-chain

One would like to have our 2-chain a 2-cycle in some homology theory and to have Reidemeister moves to change the 2-cycle by a boundary. This motivated initially authors of [CJKS] and led to the discovery that what they need is essentially rack homology introduced by Fenn, Rourke and Sanderson.

The notion of homology groups was introduced in 1895 by Jules Henri Poincaré (1854-1912) in his famous *Analisis Situs* paper to study properties of manifolds, see Lecture 2 for a historical remark on the relation between algebraic topology and knot theory.
We now recall the definition of a chain complex and its homology.

Definition

Consider the following descendant sequence of abelian groups or modules $\mathcal{C}_n$ connected by the homomorphisms $\partial_n : \mathcal{C}_n \longrightarrow \mathcal{C}_{n-1}$, called boundary operators:

$$\cdots \xrightarrow{\partial_{n+3}} \mathcal{C}_{n+2} \xrightarrow{\partial_{n+2}} \mathcal{C}_{n+1} \xrightarrow{\partial_{n+1}} \mathcal{C}_n \xrightarrow{\partial_n} \mathcal{C}_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

The system $(\mathcal{C}_n, \partial_n)$ satisfying $\partial_{n-1} \circ \partial_n = 0$, is called a **chain complex**. The previous condition is equivalent to requiring that $im(\partial_{n+1}) \subset ker(\partial_n)$, where $im(\partial_{n+1})$ denotes the image of the boundary map and $ker(\partial_n)$ denotes its kernel.

Chain complexes and homology II

We define the n^{th} **homology group** as the quotient

$$\mathcal{H}_n := \frac{ker(\partial_n)}{im(\partial_{n+1})}.$$

The elements of $\mathcal{H}_n$ are called **homology classes**. Elements of $ker(\partial_n)$ are called n-cycles and denoted by $Z_n = ker(\partial_n)$; elements of $im(\partial_{n+1})$ are called n-boundaries and denoted by $B_n = im(\partial_{n+1})$. In particular, if the equality $im(\partial_{n+1}) = ker(\partial_n)$ holds for all n, then the chain complex $(\mathcal{C}_n, \partial_n)$ is said to be **exact**.

Homology of algebraic structures

The homology theory of associative structures, such as groups and rings, has been extensively studied beginning with the work of Heinz Hopf (1894-1971), Witold Hurewicz (1904-1956), Samuel Eilenberg (1913-1998), and Gerhard P. Hochschild (1915-2010). On the other hand, the homology of non-associative distributive structures, such as quandles, was, in a way, undervalued until recently. Distributive structures have been studied for a long time, we remark that in 1880 Charles S. Peirce (1839–1914) emphasized the importance of right self-distributivity in algebraic structures.

Homology of algebraic structures II

However, the first homology theory related to a self-distributive structure was constructed in the early 1990s by Roger Fenn, Colin Rourke, and Brian Sanderson. They introduced the homology theory of racks motivated by higher dimensional knot theory in [FRS]. In 1998, J. Scott Carter, Seiichi Kamada, and Masahico Saito refined the ideas to define the homology of quandles. Then in 2004, Carter, Saito, and Mohamed Elhamdadi generalized this homology by defining a (co)homology theory for set theoretic Yang-Baxter operators, and they also constructed cocycle invariants. The homology theory of general Yang-Baxter operators was independently developed by Victoria Lebed and myself (2012). This homology theory is equivalent to that defined by Carter, Kamada, and Saito when restricted to set-theoretic Yang-Baxter operators.

To define Yang-Baxter homology it is convenient to have the terminology of **presimplicial**[1] and **precubic sets** and **modules**. These concepts take into account the fact that, in most homology theories, the boundary map $\partial_n : \mathcal{C}_n \longrightarrow \mathcal{C}_{n-1}$ can be decomposed as an alternating sum of other maps.

[1]The concept was introduced in 1950 by Samuel Eilenberg (1913-1998) and Joseph A. Zilber (1923-2009) under the name *semi-simplicial complex*.

Presimplicial sets and modules

Let X_n, $n \geq 0$ be a sequence of sets and $d_i = d_{i,n} : X_n \to X_{n-1}$, $0 \leq i \leq n$ maps (called face maps) such that:

$$(1)\ d_i d_j = d_{j-1} d_i \text{ for any } i < j.$$

Then the system $(X_n, d_i$ satisfying the above equality is called a presimplicial set.

similarly if C_n, $n \geq 0$ is a sequence of$k$-modules (e.g. $C_n = kX_n$) and $d_i = d_{i,n} : C_n \to C_{n-1}$, $0 \leq i \leq n$ are homomorphisms satisfying

$$(1)\ d_i d_j = d_{j-1} d_i \text{ for any } i < j,$$

then (C_n, d_i) satisfying the above equality is called a presimplicial module The important basic observation is that if (C_n, d_i) is a presimplicial module then (C_n, ∂_n), for $\partial_n = \Sigma_{i=0}^{n} (-1)^i d_i$, is a chain complex.

Precubic set and module

Definition

Let X_n with $n \geq 0$ be a sequence of sets and let $d_i^\varepsilon := d_{i,n}^\varepsilon : X_n \longrightarrow X_{n-1}$ be maps called **face maps** or **face operators**, where $\varepsilon \in \{0,1\}$, $0 \leq i \leq n$. If the condition

$$d_i^\varepsilon d_j^\delta = d_{j-1}^\delta d_i^\varepsilon \quad \textit{for} \quad i < j$$

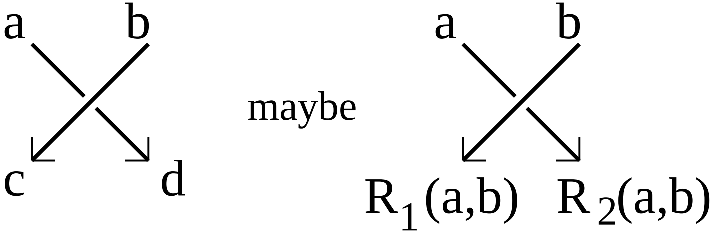
is satisfied, then the system (X_n, d_i^ε) is called a **precubic set**. If $\mathcal{C}_n = \daleth X_n$, then the system $(\mathcal{C}_n, \partial_n)$ is a chain complex for

$$\partial_n = \sum_{i=1}^{n} (-1)^i (d_i^0 - d_i^1).$$

The notion of **Precubic module** is defined analogously to the presimplicial module.

More general colorings, overcrossing can change color

Back to diagram colorings. We do not have to keep the same color at overpass.

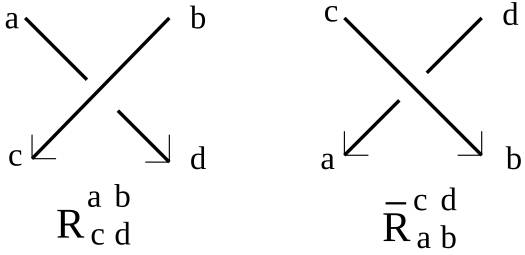


Try more general colorings

We define $R : X \times X \to X \times X$ by $R(a, b) = (R_1(a, b), R_2(a, b)$, leading to set theoretic Yang-Baxter equation via the third Reidemeister move.

From distributive homology to Yang-Baxter homology

We can extend the basic construction from the introduction, still using very naive point of view, as follows: Fix a finite set X and color semi-arcs of D (parts of D from a crossing to a crossing) by elements of X allowing different weights from some ring k for every crossing (following statistical mechanics terminology we call these weights Boltzmann weights). We allow also differentiating between a negative and a positive crossing; see Figure below.



Boltzmann weights $R^{a,b}_{c,d}$ and $\bar{R}^{c,d}_{a,b}$ for positive and negative crossings

We can now generalize the number of colorings to state sum (basic notion of statistical physics) by multiplying Boltzmann weight over all crossings and adding over all colorings:

$$col_{(X;BW)}(X) = \sum_{\phi \in col_X(D)} \prod_{p \in \{crossings\}} \hat{R}^{a,b}_{c,d}(p)$$

where $\hat{R}^{a,b}_{c,d}$ is $R^{a,b}_{c,d}$ or $\bar{R}^{a,b}_{c,d}$ depending on whether p is a positive or negative crossing (see [Jones 1986]). Our state sum is an invariant of a diagram but to get a link invariant we should test it on Reidemeister moves.

To get analogue of a shelf invariant we start from the third Reidemeister move with all positive crossings. Here we notice that, in analogy to distributivity, where passing through a positive crossing was coded by a map $R : X \times X \to X \times X$ with $R(a, b) = (b, a * b)$. Thus in the general case passing through a positive crossing is coded by a linear map $R : kX \otimes kX \to kX \otimes kX$ and in basis X the map R is given by the $|X|^2 \times |X|^2$ matrix with entries $(R^{a,b}_{c,d})$. The third Reidemeister move leads to the equality of the following maps $V \otimes V \otimes V \to V \otimes V \otimes V$ where $V = kX$:

$$(R \otimes Id)(Id \otimes R)(R \otimes Id) = (Id \otimes R)(R \otimes Id)Id \otimes R),$$

as illustrated in Figure X below.

This is called the Yang-Baxter equation. Older names include: the star-triangle relation, the triangle equation, and the factorization equation, [Jimb]. and R is called a pre-Yang-Baxter operator. If R is additionally invertible it is called a Yang-Baxter operator. If entries of R^{-1} are equal to $\bar{R}^{a,b}_{c,d}$ then the state sum is invariant under “parallel” (directly oriented) second Reidemeister move, see Figure Y.

We should stress that to find link invariants it suffices to use directly oriented second and third Reidemeister moves in addition to both first Reidemeister moves, as we can restrict ourselves to braids and use the Markov theorem. This point of view was used in [Turaev].

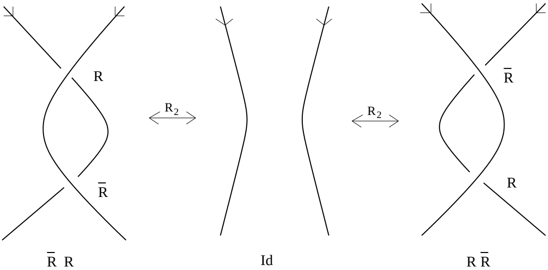


Figure Y; Invertibility of R and
the parallel second Reidemeister move

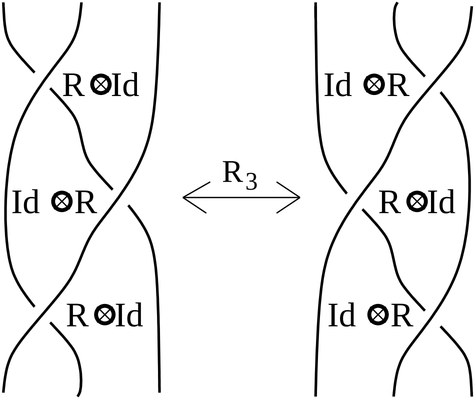


Figure X; Yang-Baxter equation
from the positive third Reidemeister move

The presimplicial set corresponding to (two term) Yang-Baxter homology has the following visualization. In the case of a set-theoretic Yang-Baxter equation we recover the homology of J. S. Carter, M. Elhamdadi, M. Saito, Homology Theory for the Set-Theoretic Yang-Baxter Equation and Knot Invariants from Generalizations of Quandles, *Fund. Math.* 184, 2004, 31–54.

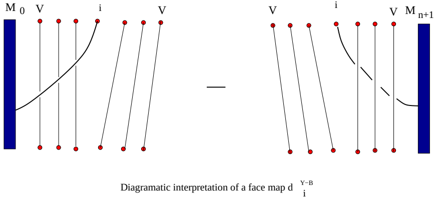
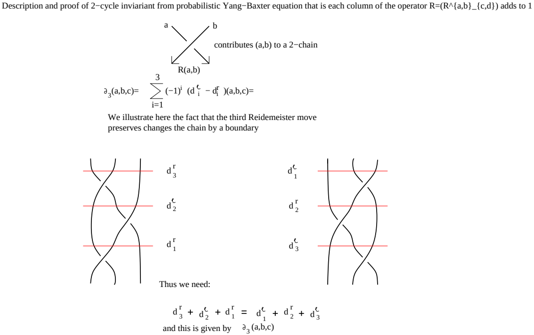


Figure 12.3; Graphical interpretation of the face map d_i
(This visualization was discovered independently by Victoria Lebed in her PhD thesis (December 2012).)

Decomposition of the third Reidemeister move into cubic face maps

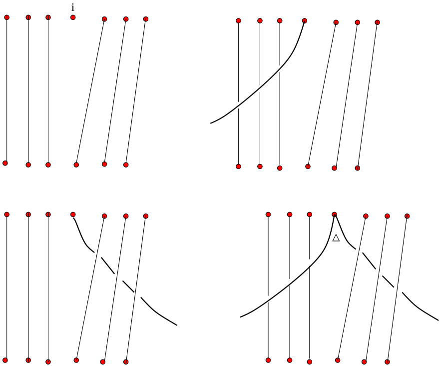
The main idea is illustrated by the following picture:



Reidemeister third move and face maps d_i^{ε}

The idea leads to (co)cycle invariants of links, at least for stochastic (or more generally column unital) Yang-Baxter matrices. Example is developed in paper with Xiao Wang (and in Xiao PhD thesis).

Figure for "goodbye":
the figure below illustrate various graphical interpretation of the generating morphism d_i of the presimplicial category. They are related to homology of a set-theoretic Yang-Baxter equation of Carter-Kamada-Saito and Fenn, and to homology of Yang-Baxter equation of Eisermann. We should also acknowledge stimulating observations by Ivan Dynnikov (May 2012).



Various interpretation of the graphical face map d_i

Jones examples of Yang-Baxter operators

$V = kX^2$ and $R : V \otimes V \to V \otimes V$ is given by (Jones, Turaev 1986):

$$\begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}$$

or using stochastic (or more generally column unital) matrix by balancing every column to have the sum equal to 1 (Xiao Wang):

Column unital Yang-Baxter matrix for the Jones polynomial

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-y^2 & 1 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Conjecture (Xiao Wang)
Let $k = Z[y]$
For a column unital Yang-Baxter operator giving the Jones polynomial

$$H_n(R) = k^2 \bigoplus (k/(1-y^2)^{a_n} \bigoplus (k/(1-y^4))^{s_n - 2},$$

where $s_n = \sum_{i=1}^{n+1} f_i$ is the partial sum of Fibonacci sequence, where $f_1 = f_2 = 1$ and a_n is given by
$2^n = 2 + a_{n-1} + s_{n-2}a_n + s_{n-1}$ with $a_1 = 0$. We verified conjecture for $n \leq 10$ (in 2016). Possibly more now.

Yang-Baxter operator for HOMFLYPT polynomial

Let k be a commutative ring and let $V = kX$ be a free k-module over the basis $X = \{v_1, v_2, \ldots, v_m\}$ with the ordering $v_a \leq v_b$ if and only if $a \leq b$. Recall that a k-linear map $R : V \otimes V \longrightarrow V \otimes V$ is a Yang-Baxter operator if it satisfies the Yang-Baxter equation and it is invertible. Jones discovered that the Yang-Baxter operator on level m given by the formula below leads to the Homflypt polynomial:

$$R_{c\ d}^{a\ b} = \begin{cases} q, & \text{if } a = b = c = d; \\ 1, & \text{if } d = a \neq b = c; \\ q - q^{-1}, & \text{if } c = a < b = d; \\ 0, & \text{otherwise.} \end{cases}$$

Xiao Wang and I adjusted the matrix above to be a column unital matrix and showed that for each m, $R_{(m)}$ is also a Yang-Baxter operator.

The second Yang-Baxter homology of the Homflypt polynomial of links

We consider the homology of a column unital Yang-Baxter operator. In particular, we prove that the second homology is given by the following theorem:

Theorem

(P- Xiao Wang)
Let R_m be a unital Yang-Baxter operator giving Homflypt polynomial on level m, then for $k = Z[y]$:

$$H_2(R_m) = k^{1+\binom{m}{2}} \oplus \Big(k/(1-y^2)\Big)^{\binom{m}{2}} \oplus \Big(k/(1-y^4)\Big)^{m-1}.$$

If $R_{(m)}$ is the column unital Yang-Baxter operator giving level m Homflypt polynomial then

Conjecture

(X) *(Xiao Wang)*

$$H_n(R_{(m)}) = k^{u_{m,n}} \bigoplus (k/(1-y^2))^{v_{m,n}} \bigoplus (k/(1-y^4))^{w_{m,n}},$$

(J) *(JHP)*
$H_n(R_{(m)})$ can have arbitrary large torsion, $k/(1-y^2)^N$, for m, n large enough.

THANK YOU!

Thank you very much...

and possibly watch

Knots and graphs part 8

https://youtu.be/dB1rk7Us-Ag

if you do not have enough.