Introduction to Khovanov homology

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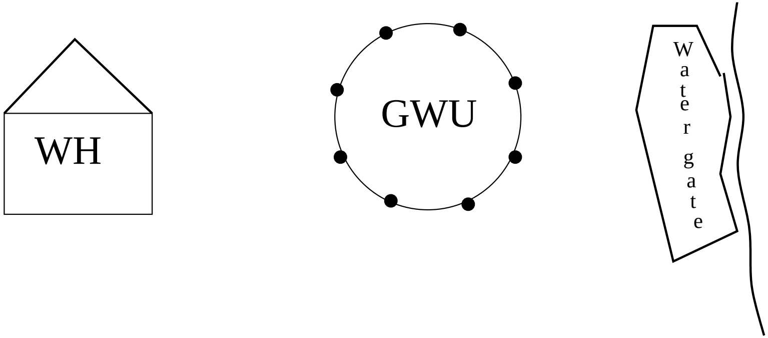
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Abstract

Introduction to Khovanov homology: from enhanced Kauffman states to applications of the long exact sequence of homology.

Lecture 1 (Aug. 26).
Introduction to Khovanov homology: from enhanced Kauffman states to applications of the long exact sequence of homology.
Study materials: J.H.Przytycki, Khovanov Homology: categorification of the Kauffman bracket relation, arXiv:math/0512630 [math.GT]

M. Asaeda, J. H. Przytycki, A. S. Sikora, Categorification of the Kauffman bracket skein module of I-bundles over surfaces, *Algebraic & Geometric Topology (AGT)*, 4, 2004, 1177-1210;
e-print: `http://front.math.ucdavis.edu/math.QA/0403527`.

S.Mukherjee, J. H. Przytycki, M.Silvero, X.A.Wang, S.Y.Yang, Search for torsion in Khovanov homology, *Experimental Mathematics*, Volume 27, Issue 4, 2018, pp. 488–497;
e-print: `arXiv:1701.04924 [math.GT]`.

J. H. Przytycki, R.Sazdanovic, Torsion in Khovanov homology of semi-adequate links (with R.Sazdanovic), *Fundamenta Mathematicae*, 225, May, 2014, 277-303;
e-print: `arXiv:1210.5254 [math.QA]`.

J. H. Przytycki, M. Silvero, Homotopy type of circle graphs complexes motivated by extreme Khovanov homology, *Journal of Algebraic Combinatorics*, 48(1), August 2018, 119-156; e-print: `arXiv:1608.03002 [math.GT]`.

Historical Introduction I

The theory of invariants of knots and links was revolutionized with the discovery of the Jones polynomial in May of 1984. One of its most spectacular applications is the proof of the first Tait's conjecture, via the Kauffman bracket polynomial. Soon after its announcement, this development motivated several generalizations such as the HOMFLYPT polynomial. At the end of the XX century, Mikhail Khovanov (PhD 1997) announced a novel construction of a new and very powerful link invariant: a homology theory categorifying the Jones polynomial, containing more information, and having a richer algebraic structure.

Historical Introduction II

Khovanov homology (KH) offers a nontrivial generalization of the Jones polynomial (and the Kauffman bracket polynomial) of links in $\mathbb{R}^3$. A more powerful invariant than the Jones polynomial, this special type of categorification has been extensively developed over the last 20 years. In particular, KH detects the unknot which at the moment of writing is still unknown for the Jones polynomial. The gist of KH is that it is a bigraded chain complex associated to a link, in such a way that the homology of the complex is a link invariant. Furthermore, the graded Euler characteristic of the chain complex is the Jones polynomial, which explains the phrase associated with KH that it categorifies the Jones polynomial. With the idea of achieving an elementary exposition of KH, we present the following construction after Oleg Viro (it was his lecture in Gdansk in the summer of 2002 which directed my attention to Khovanov homology).

The Kauffman bracket polynomial

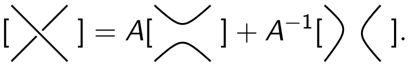
We start from the description of the Khovanov homology for framed links, after Viro.

Definition

(Kauffman 1985) The unreduced Kauffman polynomial is defined by initial conditions

$$[U_n] = (-A^2 - A^{-2})^n,$$

where U_n is the crossingless diagram of a trivial link of n components,
and the skein relation:



Just to recall: the classical Kauffman bracket $< D >$ satisfies $(-A^2 - A^{-2}) < D >= [D]$.

Kauffman state

A Kauffman state is a function from a set of crossings to the two element set $\{A, B\}$, that is $s : cr(D) \to \{A, B\}$, see Figure A. We denote by D_s the diagram (system of circles) obtained from D by smoothing all crossings of D according to s; $|D_s|$ denotes the number of circles in D_s.

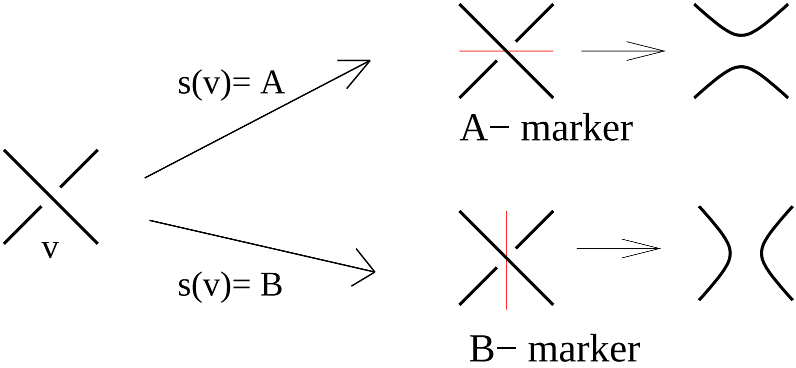


Figure A, Interpretation of Kauffman states

Kauffman state sum

Proposition

(Kauffman) The unreduced Kauffman bracket polynomial can be written as the state sum (over all Kauffman states):

$$[D] = \sum_{s \in KS} A^{|s^{-1}(A)| - |s^{-1}(B)|} (-A^2 - A^{-2})^{|D_s|},$$

where KS is the set of all Kauffman states of the diagram D.

Notice that the Kauffman bracket associates to every trivial circle of D_s the polynomial $-(A^2 + A^{-2})$. In order to have state sum with monomial entries Viro considers two type of circles: positive with A^2 associated to it, and negative with A^{-2} associated to it. These lead to Enhanced Kauffman States (EKS).

Enhanced Kauffman State

Definition

(i) An enhanced Kauffman state, S, is a Kauffman state s together with a function $h : D_s \to \{+1, -1\}$.

(ii) The enhanced Kauffman state formula for the unreduced Kauffman bracket is the Kauffman state formula written using the set of enhanced Kauffman states EKS:

$$[D] = \sum_{S \in EKS} (-1)^{|D_s|} A^{\sigma(s)+2\tau(S)},$$

where the signature of s is $\sigma(s) = |s^{-1}(A)| - |s^{-1}(B)|$ and $\tau(S) = |h^{-1}(+1)| - |h^{-1}(-1)|$ that is the number of positive circles minus the number of negative circles in D_s with enhanced Kauffman state function h of S.

With the above preparation we can define the Khovanov chain complex and Khovanov homology of a diagram.

Definition

Consider bidegree on the Enhanced Kauffman States as follows:

$$\mathcal{S}_{a,b} = \{S \in EKS \mid \sigma(s) = a, \;\; \sigma(s) + 2\tau(S) = b\}$$

(a) The chain groups are free abelian groups with basis $\mathcal{S}_{a,b}$, that is $C_{a,b}(D) = \mathbb{Z}\mathcal{S}_{a,b}$.

(b) Boundary maps are $\partial_{a,b} : C_{a,b}(D) \to C_{a-2,b}(D)$ given by the formula:

$$\partial_{a,b}(S) = \sum_{S' \in \mathcal{S}_{a-2,b}} (-1)^{t(S,S')}[S; S']S'$$

Conditions for signs of circles

where $[S, S']$ is 1 or 0 and it is 1 if and only if the following two conditions hold:

(i) S and S' differ at exactly one crossings, say v, at which $s(v) = A$ and $s'(v) = B$. In particular $\sigma(s') = \sigma(s) - 2$.

(ii) $\tau(S') = \tau(S) + 1$ and common circles to D_s and $D_{s'}$ have the same sign. The possible signs of circles around the crossing v is shown in Figure B.

The sign $(-1)^{t(S,S')}$

To define the sign $(-1)^{t(S,S')}$ we need to order crossings of D. Then $t(S,S')$ is equal to the number of crossings with label A smaller than the crossing v in the chosen ordering.
(c) The Khovanov homology is defined in the standard way as: $H_{a,b}(D) = ker(\partial_{a,b})/im(\partial_{a+2,b})$.

Incident states

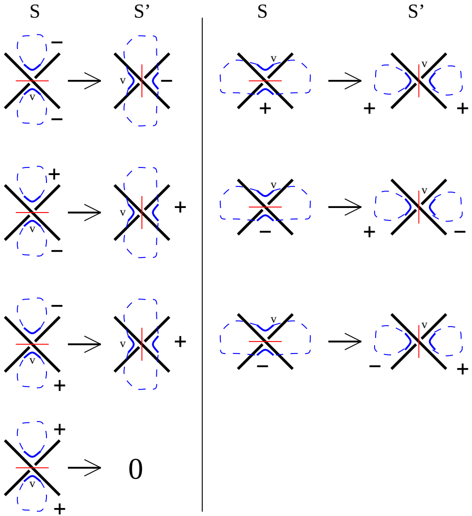


Figure B, List of neighboring states with $[S, S'] = 1$

Fusion as multiplication

The fusion of two components into one component can be thought as a multiplication m. Similarly, the splitting of one circle can be understood as comultiplication Δ. In our settings, from Figure B we have that the multiplication m is given by the following table:

Multiplication and comultiplication

m	-	$+$
-	-	$+$
$+$	$+$	0

Table: Table of the multiplication *m*.

and the comultiplication Δ is given by $\Delta(+) = (+,+)$ and $\Delta(-) = (+,-) + (-,+)$.

Rotate the signs in the table

To make the table more convincing, we rotate the signs $+$ and $-$ to obtain x and 1, respectively. Then, the multiplication m is given by Table 2 below. The comultiplication now has the form $\Delta(1) = (x, 1) + (1, x)$, $\Delta(x) = (x, x)$, which describes the Frobenius algebra $\frac{\mathbb{Z}[x]}{(x^2)}$ used by Khovanov.

m	1	x
1	1	x
x	x	0

Table: Table of the multiplication m.

The most important property of Khovanov homology for knot theorist, is its invariance under second and third Reidemeister moves.

Theorem

(Khovanov) Let D be a link diagram. With the notation given above, the homology groups

$$H_{a,b}(D) = \frac{ker(\partial_{a,b})}{im(\partial_{a+2,b})}$$

are invariant under Reidemeister moves of second and third type. Therefore they are invariants of unoriented framed links. Moreover, the effect of the first Reidemeister move (positive or negative) R_1*, is the shift in the homology,* $H_{a,b}(R_{1+}(D)) = H_{a+1,b+3}(D)$ *and* $H_{a,b}(R_{1-}(D)) = H_{a-1,b-3}(D)$*. These groups categorify the unreduced Kauffman bracket polynomial and are called the framed Khovanov homology groups.*

Khovanov homology of the Trefoil knot

b | **a**	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}_2$			
-9	$\mathbb{Z}$			

Table: Khovanov homology, $H_{a,b}(\bar{3}_1)$, table for the right-handed trefoil knot (Viro notation).

Khovanov (co)homology of the positive trefoil knot

The classical Khovanov (co)homology $\mathcal{H}^{i,j}(\vec{D})$, and the framed version of KH $H_{a,b}(D)$, are related by the following equalities:

$$\mathcal{H}^{i,j}(\vec{D}) = H_{w-2i,3w-2j}(D) = H_{a,b}(D) = \mathcal{H}^{\frac{w-a}{2},\frac{3w-b}{2}}(\vec{D}),$$

Where $w(\vec{D}) = \sum_{v \in cr(\vec{D})} sgn(v)$ is the write of the oriented diagram $\vec{v}$.

j | **i**	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}_2$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

Table: Khovanov cohomology table for the right-handed trefoil knot (Khovanov notation).

Khovanov homology for the Hopf link

b | **a**	-2	0	2
6			$\mathbb{Z}$
2			$\mathbb{Z}$
-2	$\mathbb{Z}$		
-6	$\mathbb{Z}$		

Khovanov homology for the Hopf link

b | **a**	-2	0	2
6			$\mathbb{Z}$
2			$\mathbb{Z}$
-2	$\mathbb{Z}$		
-6	$\mathbb{Z}$		

j | **i**	0	1	2
6			$\mathbb{Z}$
4			$\mathbb{Z}$
2	$\mathbb{Z}$		
0	$\mathbb{Z}$		

Table: Khovanov homology tables for the Hopf link.

Long exact sequence of Khovanov homology

We have the following short exact sequence of chain complexes of diagrams:

$$\ldots \to H_{a+1,b+1}(D_B) \to H_{a,b}(D) \to H_{a-1,b-1}(D_A) \stackrel{\partial^{con}}{\to} H_{a-1,b+1}(D_B) \to \ldots$$

Corollary from the long exact sequence of homology

Corollary

(1) *If* $H_{a+1,b+1}(D_B) = 0$ *then* $H_{a,b}(D) \to H_{a-1,b-1}$ *is a monomorphism.*

(2) *If* $H_{a-1,b+1}(D_B) = 0$ *then* $H_{a,b}(D) \to H_{a-1,b-1}$ *is an epimorphism.*

Khovanov homology of $T(2, n)$ via long exact sequence of homology

Theorem

Let $T(2, n)$ be a torus knot of type $(2, n)$, n odd then $H_{a,b}(T(2, n))$ is described by:

$$H_{n,n+4}(T(2,n)) = H_{n,n}(T(2,n)) = Z,$$

$$H_{n-2s,*}(T(2,n)) = \begin{cases} Z & \text{when } s \text{ is even, } 2 \leq s \leq n \\ Z \oplus Z_2 & \text{when } s \text{ is odd, } 3 \leq s \leq n. \end{cases}$$

More precisely:
$H_{n-2s,n-4s+4}(T(2,n)) = Z$ *for even* s, *and* $2 \leq s \leq n$, *and*
$H_{n-2s,n-4s+4})(T(2,n)) = Z_2$ *for* s *odd and* $3 \leq s \leq n$,
$H_{n-2s,n-4s}(T(2,n)) = Z$ *for* s *odd and* $3 \leq s \leq n$,
In all other cases $H_{a,b}(T(2,n)) = 0$.

Similar theorem holds for torus links of two components.

Khovanov homology of the torus knot $T(2,11)$

b | **a**	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11
15												$\mathbb{Z}$
11												$\mathbb{Z}$
7										$\mathbb{Z}$		
3									$\mathbb{Z}_2$			
-1								$\mathbb{Z}$	$\mathbb{Z}$			
-5							$\mathbb{Z}_2$					
-9						$\mathbb{Z}$	$\mathbb{Z}$					
-13					$\mathbb{Z}_2$							
-17				$\mathbb{Z}$	$\mathbb{Z}$							
-21			$\mathbb{Z}_2$									
-25		$\mathbb{Z}$	$\mathbb{Z}$									
-29	$\mathbb{Z}_2$											
-33	$\mathbb{Z}$											

Khovanov homology of the torus link $T(2, 12)$

b | **a**	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12
16													$\mathbb{Z}$
12													$\mathbb{Z}$
8											$\mathbb{Z}$		
4										$\mathbb{Z}_2$			
0									$\mathbb{Z}$	$\mathbb{Z}$			
-4								$\mathbb{Z}_2$					
-8							$\mathbb{Z}$	$\mathbb{Z}$					
-12						$\mathbb{Z}_2$							
-16					$\mathbb{Z}$	$\mathbb{Z}$							
-20				$\mathbb{Z}_2$									
-24			$\mathbb{Z}$	$\mathbb{Z}$									
-28		$\mathbb{Z}_2$											
-32	$\mathbb{Z}$	$\mathbb{Z}$											
-36	$\mathbb{Z}$												

2-torsion in Khovanov homology.

Torsion in Khovanov homology is an area of active research. One of the main goals is to interpret the torsion information in terms of topological properties of the link. In 2003, Shumakovitch stated the following Conjecture, which if proved, would yield a way of detecting the unknot. Marta Asaeda and I partially solved Shumakovitch's conjecture by proving it for some adequate diagrams.

Conjecture

(Shumakovitch) The Khovanov homology of every link except the unknot, the Hopf link, their disjoint unions, and their connected sums, has torsion of order 2.

Torsion in Khovanov homology different from $\mathbb{Z}_2$

Alexander Shumakovitch verified that for prime knots up to 14 crossings only $\mathbb{Z}_2-$torsion appears, and up to 16 crossings the only other torsion that appears is $\mathbb{Z}_4-$torsion. In fact, 38 knots with 15 crossings (one of them being the $T(4,5)$ torus knot) and 129 knots with 16 crossings have $\mathbb{Z}_4-$torsion. In other words, very few knots up to 16 crossings contain $\mathbb{Z}_4-$torsion in its KH and there is no knot with $\mathbb{Z}_3-$torsion or with larger order torsion.

New torsion in Khovanov homology

Computers and good algorithms allowed to detect odd torsion in Khovanov homology: $\mathbb{Z}_3-$ and $\mathbb{Z}_5-$torsion appear in the KH of the torus knot $T(5,6)$, and $\mathbb{Z}_7-$torsion in the KH of the torus knot $T(7,8)$. Following this algorithm, in 2014 Lukas Lewark computed the KH of the torus knot $T(8,9)$ and found $\mathbb{Z}_8-$torsion. Using the long exact sequence of Khovanov homology, in 2016 we (Mathathoners) constructed infinity families of knots and links containing $\mathbb{Z}_3-$, $\mathbb{Z}_5-$, and $\mathbb{Z}_7-$torsion in their Khovanov homology.

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THANK YOU!

Thank you very much...Next talk tomorrow.

"From Fox 3-coloring to Yang-Baxter homology".