# Automorphisms and Coordinates of Polynomial and Free Associative Algebras – 3

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## We fix the notation:

All the algebras are over the field $\mathbb{C}$ (although some of the results hold for fields of characteristic 0 and even over an arbitrary field;
$X_n = \{x_1, \dots, x_n\}$,
$G(X_n)$, $L(X_n)$, $\mathbb{C}\langle X_n\rangle$ – the free group, the free Lie algebra, and the free associative algebra freely generated by $X_n$.

## Abhyankar-Sathaye embedding conjecture

Let $A$ be a commutative ring containing $\mathbb{Q}$ and let $f(X_n) \in A[X_n]$. If the factor algebra $A[X_n]/(f(X_n))$ is isomorphic to $A[X_{n-1}]$, then $f(X_n)$ is a coordinate of $A[X_n]$.

$A = \mathbb{C}$, $n = 2$:

Abhyankar-Moh (1975), also Suzuki(1974):
If $\mathbb{C}[x,y]/(f(x,y)) \cong \mathbb{C}[x]$, then $f(x,y)$ is a coordinate of $\mathbb{C}[x,y]$.

This follows from the famous Abhyankar-Moh theorem: If the polynomials $f(x), g(x) \in \mathbb{C}[x] \setminus \mathbb{C}$ generate the whole algebra $\mathbb{C}[x]$ and $\deg(f) > \deg(g)$, then $\deg(g)$ divides $\deg(f)$. As a consequence one easily obtain a new proof of the Jung-van der Kulk theorem.

### Russel and Sathaye (1979):

Let $A$ be a commutative $K$-algebra such that $A[t] \cong K[x,y,z]$ and let $A$ contains a polynomial of the form

$$a(x,y,z) = g(x) + zh(x,y,z), \quad h(x,y,z) \in K[x,y,z],$$

where $f(z) \in K[z]$ is separable. Then $A \cong A[t]/(t) \cong K[u,v]$. In particular, $t$ is a coordinate of $K[x,y,z]$.

### Counterexample in positive characteristic

If $\mathrm{char}(K) = p > 0$, and

$$f(x, y) = y^{p^2} - x - x^{2p},$$

then $K[x, y]/(f(x, y)) \cong K[x]$.

## Vénéreau polynomials

Vénéreau (2001), Berson (2004) independently introduced the following polynomials, known today as the Vénéreau polynomials:

$$b_m(x,y,z,u) = y + x^m(xz + y(yu + z^2)) \in \mathbb{C}[x,y,z,u], \quad m \geq 1.$$

These polynomials satisfy the condition

$$\mathbb{C}[x,y,z,u]/(b_m(x,y,z,u)) \cong \mathbb{C}[x,y,z].$$

Vénéreau proved that $b_m(x,y,z,u)$ is a coordinate for $m \geq 3$. Drew Lewis (arXiv 2010-2012, Ph. D. Thesis, 2012): $b_2(x,y,z,u)$ is also a coordinate.

## Counterexample for free associative algebras

Shpilrain and J.-T. Yu (2003):
For any $n \geq 3$, the element

$$f(x_1, x_2, x_3) = x_1 - (x_1^2 + x_2 x_3)x_3 \in K\langle X_n\rangle$$

is not a coordinate but

$$K\langle X_n\rangle/(f(x_1, x_2, x_3)) \cong K\langle X_{n-1}\rangle$$

When $K$ is of characteristic 0, this gives a counterexample to the problem of G. M. Bergman whether there exists a principal ideal $(u)$ of $K\langle X_n\rangle$ which is not an automorphic image of the ideal generated by $x_1$ but $K\langle X_n\rangle/(u) \cong K\langle X_{n-1}\rangle$.

Drensky and Yu (2003):
Let $(f)$ be the principal ideal of $K\langle x, y\rangle$ generated by $f = f(x, y)$. The factor algebra $K\langle x, y\rangle/(f)$ is isomorphic to the polynomial algebra $K[z]$ if and only if $f(x, y)$ is a coordinate.

Shpilrain and Yu (2002):
If $f(x,y) \in K[x,y] \subset K[x,y,z_3,\dots,z_n]$ is a coordinate in the polynomial algebra $K[x,y,z_3,\dots,z_n]$, then $f(x,y)$ is a coordinate in $K[x,y]$.

Drensky and Yu (2003):
If $f(x,y)$ is a coordinate in the free associative algebra $K\langle x,y,z_3,\dots,z_n\rangle$, then $f(x,y)$ is a coordinate also in $K\langle x,y\rangle$.

## Zariski cancellation conjecture

If $\mathbb{C}[x_1,\dots,x_{n+m}] \cong A[y_1,\dots,y_m]$, is $A \cong \mathbb{C}[x_1,\dots,x_n]$?
The conjecture was discussed in details in the talk of Neena Gupta.
**Results:** The conjecture is true for $n = 1$ and $n = 2$:
♣ $n = 1$: Abhyankar, Eakin, and Heinzer (1972), Miyanishi (1973).
♣ $n = 2$: Fujita (1979), Miyanishi and Sugie (1980).
♣ Kambiyashi (1980) showed that the proofs can be modified to work for positive characteristic.

**Be wary about cancellation!**

(Danielewski surfaces, 1989)

$$A=\mathbb{C}[x,y,z]/(xy-(1-z^2))\cong\mathbb{C}[x,y,z]/(x^2y-(1-z^2))=B,$$

$A[t]\cong B[t]$ but $A$ and $B$ are not isomorphic.

## Free associative algebras

Drensky and Yu (2008):

**Theorem.** Let $R$ be an algebra over an arbitrary field $K$. If the free product $R * K[z]$ is $K$-isomorphic to $K\langle x, y\rangle$, then $R$ is $K$-isomorphic to $K[x]$.

**Theorem.** (New proof.) Let $R$ be an algebra over a field $K$ of zero characteristic. If $R[z]$ is $K$-isomorphic to $K[x, y]$, then $R$ is isomorphic to $K[x]$.

### Lemma of Bergman, 1969

Let $K$ be an arbitrary field, $f, g \in K\langle X_n\rangle$. Then $f$ and $g$ are algebraically dependent over $K$ if and only if $[f, g] = 0$. The centralizer

$$\mathcal{C}(f) = \{g \in K\langle X_n\rangle \mid [f, g] = 0\}$$

has the form $\mathcal{C}(f) = K[u]$ for some $u \in K\langle X_n\rangle$.

## Proof of the cancelation conjecture for $K\langle x, y\rangle$

Let $R * K[z] \cong K\langle x, y\rangle$. Clearly, $(R * K[z])/(z) \cong R$. The algebra $R * K[z] \cong K\langle x, y\rangle$ is two-generated and the same holds for its homomorphic image $(R * K[z])/(z) \cong R$. Let $R$ be generated by $v, w \in R$. Since $R$ is a subalgebra of $R * K[z] \cong K\langle x, y\rangle$, if $v$ and $w$ are algebraically independent over $K$, then $R$ is isomorphic to the free algebra $K\langle t_1, t_2\rangle$. Hence $R * K[z] \cong K\langle t_1, t_2, z\rangle$, which is impossible. Hence $v$ and $w$ are algebraically dependent and the same holds for $v$ and any element $f \in R$. By the lemma of Bergman, $R \subset K[u]$ for some $u \in R * K[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$ and $u_1$ contains all monomials of $u$ with $z$-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, we obtain $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. It forces $R = K[u_0]$. Hence $R$ is $K$-isomorphic to $K[x]$.

## Partial case of Jacobi (1841)

Let $\mathrm{char}(K) = 0$. The polynomials $f, g \in K[x, y]$ are algebraically dependent if and only if $f_x g_y - g_x f_y = 0$ for the Jacobian of $f$ and $g$.

## Partial case of Shestakov and Umirbaev (2004)

Let $f \in K[x, y] \setminus K$ $(\mathrm{char}(K) = 0)$. Then

$$\mathcal{C}(f) = \{g \in K[x, y] \mid f_x g_y - g_x f_y = 0\} = K[u]$$

for some $u \in K[x, y]$.

## Proof of the cancelation conjecture for $K[x, y]$

As $R[z] \cong K[x, y]$, the transcendental degree of $R$ over $K$ is equal to 1. Hence there exists a $g \in R \setminus K$ such that $f$ and $g$ are algebraically dependent over $K$ for all $f \in R$. Therefore $R \subset K[u]$ for some $u \in R[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$ and $u_1$ contains all monomials of $u$ with $z$-degree at least 1. The elements $f \in R$ have the form $f = h(u) = h(u_0 + u_1)$, where $h$ is a polynomial over $K$ in one variable. Substituting $z = 0$, we obtain $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$ and $R = K[u_0]$. Hence $R$ is $K$-isomorphic to $K[x]$.

## Jacobian conjecture

Let $\mathrm{char}(K) = 0$ and let $\varphi$ and $\psi$ be endomorphisms of $K[X_n]$ such that $\varphi(x_j) = f_j$ and $\psi(x_j) = g_j$. Let

$$J(\varphi) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

be the Jacobian matrix of $\varphi$. Then the chain rule holds:

$$J(\varphi\psi) = J(\varphi)\varphi(J(\psi)),$$

where $\varphi(J(\psi))$ means that $\varphi$ acts on the entries of the matrix $J(\psi)$. If $\varphi$ is an automorphism, then the chain rule for $\varphi\varphi^{-1} = \iota$ gives that $I_n = J(\iota) = J(\varphi)\varphi(J(\varphi^{-1}))$ and the matrix $J(\varphi)$ is invertible, i.e., its determinant is a nonzero constant.

## Jacobian conjecture

(O. H. Keller, 1939): Let $\varphi$ be an endomorphism of $K[X_n]$ such that its matrix $J(\varphi)$ is invertible. Is $\varphi$ an automorphism?

The answer is unknown for all $n \geq 2$. Good references:
♣ the seminal paper by Bass, Connell, and Wright (1982);
♣ the book by van den Essen;
♣ the recent preprint by Belov, Bokut, Rowen, and Yu.

The Jacobian conjecture and some of its equivalent statements were discussed in the lectures of Makar-Limanov and Belov-Kanel.

## Freiheitssatz (Theorem for freedom), Magnus, 1930

One of the most important theorems of the combinatorial group theory:

Let $G = \langle x_1, \ldots, x_n \mid r = 1 \rangle$ be a group defined by a single cyclically reduced relator $r$. If $x_n$ appears in $r$, then the subgroup of $G$ generated by $x_1, \ldots, x_{n-1}$ is a free group, freely generated by $x_1, \ldots, x_{n-1}$. Magnus gave several applications. The most important is: *The word problem for groups with a single defining relation is algorithmically solvable*.

## Freiheitssatz – generalizations:

♣ free Lie algebras: Shirshov, 1962;

♣ free nilpotent and solvable groups: Romanovskij, 1972;

♣ free associative algebras ($\mathrm{char}(K) = 0$): Makar-Limanov, 1985 (not known if $\mathrm{char}(K) > 0$);

♣ free right-symmetric algebras: Kozybaev, Makar-Limanov, Umirbaev, 2008

right-symmetric identity: $(xy)z - x(yz) = (xz)y - x(zy)$;

♣ free Novikov algebras ($\mathrm{char}(K) = 0$): Makar-Limanov and Umirbaev, 2011

Novikov algebras: right-symmetric algebras with identity $x(yz) = y(xz)$;

## Poisson algebras: (important application by Shestakov and Umirbaev for the Nagata conjecture)

An algebra $P$ is a Poisson algebra if it is commutative-associative and has a second multiplication (Poisson brackets) which makes it a Lie algebra and both multiplications are related with the Leibniz rule $[x \cdot y, z] = [z, x] \cdot y + x \cdot [y, z]$.
**Main example.** $L$ is a Lie algebra with basis $\{f_1, f_2, \ldots\}$, $P(L) = S(L)$ is the symmetric algebra of $L$ (the polynomial algebra $K[f_1, f_2, \ldots]$) with multiplication as in $K[f_1, f_2, \ldots]$ and Poisson brackets as in $L$ extended to $P$ by the Leibniz rule.
**Free Poisson algebras:** Take the free Lie algebra $L = L(X_n)$ and apply the above construction.

## Free Poisson algebras:

♣ Makar-Limanov, Turusbekova, and Umirbaev, 2009:
If $\mathrm{char}(K) = 0$, then in the two-generated free Poisson algebra $P(X_2)$:
All locally nilpotent derivations are triangulizable (analogue of the theorem of Rentschler;
The automorphisms are tame (analogue of the theorem of Jung-van der Kulk).

♣ Makar-Limanov and Umirbaev, 2011:
If $\mathrm{char}(K) = 0$, then the Freiheitssatz holds for $P(X_n)$
(the Freiheitssatz does not hold for Poisson algebras in positive characteristic);
**Corollary.** New proof for the tameness of $\mathrm{Aut}(P(X_2))$.

## Relation with the Jacobian conjecture:

Makar-Limanov and Umirbaev, 2011:
**Theorem.** The Jacobian conjecture holds for $K[X_2]$ if and only if the commutator test (of Dicks) holds for the free Poisson algebra $P(X_2)$:
The endomorphism $\varphi$ of $P(X_2)$ is an automorphism if and only if $[\varphi(x_1), \varphi(x_2)] = \alpha[x_1, x_2]$, $\alpha \in K^*$.

## Generalizations of the Jacobian conjecture:

♣ free groups: Joan Birman, 1973;

♣ free associative algebras: ($n = 2$) Dicks and Lewin, 1982, (any $n$) Yagdzhev, 1980, Schofield, 1985;

♣ free Lie algebras: Umirbaev, 1990, Reutenauer, 1992, Shpilrain, 1993;

♣ absolutely free algebras, free commutative algebras: Jagdzhev, 2000.

## Fox derivatives in $K\langle X_n\rangle$

The elements of $K\langle X_n\rangle$ have the form

$$f = f(X_n) = \alpha + \sum_{i=1}^{n} x_i f_i(X_n),\ \alpha \in K, f_i \in K\langle X_n\rangle.$$

The (right) partial Fox derivatives of $f$ are

$$\frac{\partial f}{\partial x_i} = \frac{\partial_r f}{\partial_r x_i} = f_i(X_n), \quad i = 1, \ldots, n,$$

The idea comes from the Fox derivatives for free groups as used by Joan Birman in her proof of the Jacobian conjecture for groups.

## The Jacobian matrix

If $\varphi$ is an endomorphism of $K\langle X_n\rangle$, the (*right*) *Jacobian matrix* of $\varphi$ is the $n \times n$ matrix with entries from $K\langle X_n\rangle$

$$J(\varphi) = J_r(\varphi) = \left(\frac{\partial_r \varphi(x_j)}{\partial_r x_i}\right).$$

## The chain rule

If $\varphi$ and $\psi$ are endomorphisms of $K\langle X_n\rangle$, then

$$J_r(\varphi\psi) = J_r(\varphi)\varphi(J_r(\psi)),\ J_l^t(\varphi\psi) = \varphi(J_r^t(\psi))J_l^t(\varphi),$$

where $^t$ states for the transpose matrix.

If $\varphi \in \mathrm{Aut}(K\langle X_n\rangle)$, then $J_r(\varphi)$ has right inverse in $GL_n(K\langle X_n\rangle)$.

## Jacobian conjecture for free Lie algebras

Embedding the free Lie algebra $L(X_n)$ in $K\langle X_n\rangle$, every endomorphism $\varphi$ of $L(X_n)$ can be extended to an endomorphism of $K\langle X_n\rangle$, and hence we can define its Jacobian matrix.

## Reutenauer-Shpilrain-Umirbaev

The endomorphism $\varphi$ of $L(X_n)$ is an automorphism if and only if its right Jacobian matrix $J_r(\varphi)$ is invertible from the right in $GL_n(K\langle X_n\rangle)$.

We shall present the proof of Umirbaev which is really elementary and uses basic facts on Lie algebras only.

## Enveloping algebras of Lie algebras

If $R$ is an associative algebra, then $R^{(-)}$ is the Lie algebra on the vector space $R$ with multiplication $[u, v] = uv - vu$, $u, v \in R$. If the Lie algebra $L$ is isomorphic to a subalgebra of $R^{(-)}$, we say that $R$ is an *enveloping algebra* of $L$. The associative algebra $U = U(L)$ is the *universal enveloping algebra of the Lie algebra* $L$, if $L$ is a subalgebra of $U^{(-)}$ and $U$ has the following universal property: For any associative algebra $R$ and any homomorphism of Lie algebras $\varphi : L \longrightarrow R^{(-)}$ there exists a unique homomorphism of associative algebras $\psi : U \longrightarrow R$ which extends $\varphi$, i.e., $\psi$ is equal to $\varphi$ on $L$.

## The Poincaré-Birkhoff-Witt (PBW) theorem

Every Lie algebra $L$ possesses a unique (up to an isomorphism) universal enveloping algebra $U(L)$. If $L$ has a basis $\{e_i \mid i \in I\}$, and the set of indices $I$ is ordered, then $U(L)$ has a basis (called the PBW-basis)

$$e_{i_1} \cdots e_{i_p},\ i_1 \leq \cdots \leq i_p,\ i_k \in I,\ p = 0, 1, 2, \ldots .$$

## The Witt theorem

The Lie subalgebra $L(X_n)$ of $K\langle X_n\rangle^{(-)}$ generated by $X_n$ is isomorphic to the free Lie algebra with $X$ as a set of free generators; $U(L(X_n)) = K\langle X_n\rangle$.

### Proposition

Let $H$ be a subalgebra of the Lie algebra $L$, and let $H$ be generated by $u_1, \dots, u_m$. Let the right $U(L)$-module $LU(L)$ be generated by $u_1, \dots, u_m$. Then $H = L$.

## Proof

Let $H \neq L$. We fix a basis $\{h_i \mid i \in I\}$ of $H$ and extend it by a set $\{g_j \mid j \in J\} \subset L \setminus H$ to a basis of $L$, assuming that $h_i < g_j$. Since the right $U(L)$-module $LU(L)$ is generated by $H$, if $g \in G \setminus H$ is a basis element of $G$, then $g = g \cdot 1 \in LU(L)$ and

$$g = \sum_{k=1}^{m} u_k f_k, \quad f_k \in U(L).$$

Using the PBW theorem, we express $f_k$ as:

$$f_k = \sum_{p,q} \beta_{kpq} h_{p_1} \ldots h_{p_s} g_{q_1} \cdots g_{q_t}, \; \beta_{kpq} \in K.$$

We present $g$ in terms of the PBW-basis of $U(L)$. Since $u_k, h_{p_l} \in H$, each product $h_{kp} = u_k h_{p_1} \cdots h_{p_s}$ belongs to $U(H)$ and is a linear combination of products of $h_i$'s,

$$h_{kp} = \sum_{i \in I} \alpha_{ikp} h_i.$$

Hence, each summand of $g$ contains some $h_i$:

$$g = \sum_{k=1}^{m} u_k f_k = \sum_{k,p,q} \beta_{kpq} u_k h_{p_1} \dots h_{p_s} g_{q_1} \cdots g_{q_t}$$

$$= \sum_{i,k,p,q} \beta_{kpq} \alpha_{ikp} h_i g_{q_1} \cdots g_{q_t}.$$

This contradicts with the linear independence of the basis elements of $U(L)$. Hence $H = L$.

## Proof of the Jacobian conjecture for free Lie algebras

Let $\varphi$ be an endomorphism of the free Lie algebra $L(X_n)$ such that the right Jacobian matrix $J(\varphi) = J_r(\varphi)$ is invertible over $K\langle X_n\rangle$ and let $J_r(\varphi)A = I_n$ for some $n \times n$ matrix $A$ with entries from $K\langle X_n\rangle$. Since $f_j = \varphi(x_j)$, $j = 1, \dots, n$, are polynomials without constant terms, we obtain that

$$f_j = x_1 \frac{\partial f_j}{\partial x_1} + \cdots + x_m \frac{\partial f_j}{\partial x_n}, \, j = 1, \dots, n,$$

and, in matrix form,

$$(f_1, \dots, f_n) = (x_1, \dots, x_n) J(\varphi).$$

Since $J_r(\varphi)A = I_n$, we obtain

$$(f_1, \ldots, f_n)A = (x_1, \ldots, x_n)J(\varphi)A = (x_1, \ldots, x_n).$$

Hence $x_1, \ldots, x_n$ belong to the right ideal generated by $f_1, \ldots, f_n$ in the universal enveloping algebra $U(L(X_)) = K\langle X_n\rangle$. Obviously $f_1, \ldots, f_n$ belong to the right ideal generated by $x_1, \ldots, x_n$, i.e. these two right ideals of $U(L(X_n))$ coincide. By the proposition, the Lie subalgebra of $L(X_n)$ generated by $f_1, \ldots, f_n$ coincides with the whole $L(X_n)$ and the endomorphism $\varphi$ is an epimorphism. This implies that $\varphi$ is an automorphism.

## Counterexample for associative algebras

(Reutenauer)
Let $\varphi \in \mathrm{End}K\langle x, y\rangle$ be defined by

$$\varphi(x) = x + yxy, \quad \varphi(y) = y.$$

Then the right and left Jacobian matrices are

$$J(\varphi)_r = \begin{pmatrix} 1 & 0 \\ xy & 1 \end{pmatrix}, \quad J(\varphi)_l = \begin{pmatrix} 1 & 0 \\ yx & 1 \end{pmatrix}.$$

Both matrices are invertible. The commutator test of Dicks gives

$$[\varphi(x), \varphi(y)] = [yxy, y] = yxy^2 - y^2xy \neq \alpha[x, y], \quad \alpha \in K^*.$$

Hence $\varphi$ is not an automorphism.

## The correct partial derivatives

Let $\mathcal{M}(R)$ be the algebra of multiplications of the algebra $R$. This is the algebra of linear operators on $R$ generated by the left multiplications $\lambda(r)$ and right multiplications $\rho(r)$, $r \in R$, defined by

$$\lambda(r) : s \to rs, \quad \rho(r) : s \to sr, \quad s \in R.$$

For $K\langle X_n\rangle$, $n > 1$,

$$\mathcal{M}(K\langle X_n\rangle) \cong K\langle X_n\rangle \otimes_K K\langle X_n\rangle^{\mathrm{op}},$$

where $R^{\mathrm{op}}$ is the opposite algebra of $R$. Here we identify $\lambda(r)$ with $r \otimes 1$ and $\rho(r)$ with $1 \otimes r$.

We define the partial derivatives on the variables of $K\langle X_n\rangle$ by

$$\partial_i(x_j) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

and on the monomials of $K\langle X_n\rangle$ by

$$\partial_i(x_{j_1} \cdots x_{j_m}) = \sum_{k=1}^{m} \lambda(x_{j_1} \cdots x_{j_{k-1}})\mu(x_{j_{k+1}} \cdots x_{j_m}).$$

Then we construct the Jacobian matrix of the endomorphisms of $K\langle X_n\rangle$ using these partial derivatives.

## The Jacobian conjecture for free associative algebras

Dick and Lewin ($n = 2$), Schofield (any $n$):
An endomorphism of $K\langle X_n\rangle$, $n > 1$, is an automorphism if and only if its Jacobian matrix $J(\varphi)$ has right invertible in

$$GL_n(\mathcal{M}(K\langle X_n\rangle)) \cong GL_n(K\langle X_n\rangle \otimes_K K\langle X_n\rangle^{\mathrm{op}}).$$

## Example of Reutenauer

$\varphi \in \mathrm{End} K\langle x, y\rangle$ is defined by

$$\varphi(x) = x + yxy, \quad \varphi(y) = y.$$

Then its Jacobian matrix is

$$J(\varphi) = \begin{pmatrix} 1 + \lambda(y)\rho(y) & 0 \\ \rho(xy) + \lambda(yx) & 1 \end{pmatrix} = \begin{pmatrix} 1 + y \otimes y & 0 \\ 1 \otimes (yx) + (yx) \otimes 1 & 1 \end{pmatrix}$$

which is not invertible over $K\langle X_n\rangle \otimes_K K\langle X_n\rangle^{\mathrm{op}}$.