Automorphisms and Coordinates of Polynomial and Free Associative Algebras – 1

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The Cremona group

$$Cr(\mathbb{P}^n) = Cr(\mathbb{P}^n(\mathbb{C})) = \{f : \mathbb{P}^n \to \mathbb{P}^n \mid f \text{ birational automorphism}\}.$$

This means that $f \in Cr(\mathbb{P}^n)$ is an invertible map $\mathbb{P}^n \to \mathbb{P}^n$ such that

$$f[x_1 : \ldots : x_n : x_{n+1}] = [f_1 : \ldots : f_n : f_{n+1}],\ [x_1 : \ldots : x_n : x_{n+1}] \in \mathbb{P}^n,$$

where the f_i are homogeneous polynomials in the variables x_i, of the same degree d, and without common factor of positive degree. This degree d is the degree of f. The group is introduced by Cremona (1863, 1865).

Equivalent definition

(Up to antiisomorphism:)

$$Cr(\mathbb{P}^n) = \mathrm{Aut}(\mathbb{C}(x_1, \dots, x_n)),$$

the group of automorphisms of the $\mathbb{C}$-algebra $\mathbb{C}(x_1, \dots, x_n)$ of the rational functions in n variables.

Easy, $n = 1$:

$$\mathrm{Aut}(\mathbb{C}(x)) = \left\{ \varphi : x \to \frac{ax+b}{cx+d} \mid ad - bc \neq 0 \right\} = PGL_2(\mathbb{C}).$$

Here $PGL_{n+1}(\mathbb{C})$ is the group of linear projective transformations.

$n = 2$

Obvious automorphisms: $\varphi \in PGL_3(\mathbb{C})$,

$$\chi : (x_1, x_2) \to (x_1 + f(x_2), x_2) \mid f(x_2) \in \mathbb{C}(x_2)\}$$

(Héron transformations),

$$\gamma : (x_1, x_2) \to (ax_1, f(x_1)x_2), \quad a \in \mathbb{C}^*, \quad f(x_1) \in \mathbb{C}(x_1) \setminus \{0\}$$

(partial case of de Jonquiéres automorphisms),

$$\sigma : (x_1, x_2) \to \left(\frac{1}{x_1}, \frac{1}{x_2}\right)$$

(the standard quadratic involution).

Theorem (Max Noether (1870), Castelnuovo (1901))

The group $\mathrm{Aut}(\mathbb{C}(x_1, x_2))$ is generated by $PGL_3(\mathbb{C})$ and the standard quadratic involution σ. It is also generated by the second de Jonquiére group $\mathrm{Jonq}_{2,1}(\mathbb{C})$ and the involution $\tau : (x_1, x_2) = (x_2, x_1)$.

The de Jonquiére group $\mathrm{Jonq}_{n,r}(\mathbb{C})$ is the subgroup of the automorphisms of $\mathbb{C}(x_1, \ldots, x_n)$ mapping the subfield $\mathbb{C}(x_1, \ldots, x_r)$ into itself for some $r < n$.

The picture for Cr_n, $n \geq 2$, is not very clear even for $n = 2$.

Theorem

(Cantat, Lamy, arXiv, 2010,
journal version: Cantat, Lamy, de Cornulier, Acta Math. 2013)

The group $\mathrm{Aut}(\mathbb{C}(x_1, x_2))$ is not simple as an abstract group. It contains an uncountable family of distinct normal subgroups.

References

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Polynomial automorphisms of the affine space

These are isomorphisms of $\mathbb{A}^n$ of the form

$$f : \mathbb{A}^n \to \mathbb{A}^n, \quad f = (f_1, \ldots, f_n), f_i \in \mathbb{C}[x_1, \ldots, x_n], i = 1, \ldots, n.$$

An equivalent definition is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{C}[x_1, \ldots, x_n])$ of the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n complex variables.

Typical problems

Let K be any field of arbitrary characteristic, $X_n = \{x_1, \dots, x_n\}$.
(1) Describe $\text{Aut}(K[X_n])$. Find generators and defining relations.
(2) How to construct automorphisms?
(3) If φ is an endomorphism of $K[X_n]$, is it an automorphism? If "yes", how to find its inverse?
(4) Solve similar problems for "parts" of the automorphisms. (We know only $\varphi(x_1)$.)
(5) Find noncommutative analogues of the results for $\text{Aut}(K[X_n])$.

Obvious automorphisms

(1) Affine automorphisms

$$\alpha : x_i \to \sum_{k=1}^{n} \alpha_{ik} x_k + \beta_i, \quad i = 1, \dots, n, \alpha_{ik}, \beta_i \in K,$$

the matrix (α_{ik}) is invertible;
(2) Triangular (de Jonquiére) automorphisms

$$\tau : x_i \to \alpha_i x_i + f_i(x_{i+1}, \dots, x_n), \quad \alpha_i \in K^*, f_i \in K[x_{i+1}, \dots, x_n];$$

More complicated automorphisms:

(3) Exponential automorphisms.
Let $\text{char}(K) = 0$, let δ be a locally nilpotent derivation of $K[X_n]$ and let $w \in K[X_n]^{\delta}$. Then $\Delta = w\delta$ is also a locally nilpotent derivation and

$$\exp(\Delta) = \sum_{k \geq 0} \frac{\Delta^k}{k!} = 1 + \frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots$$

is an automorphism of $K[X_n]$.

Tame and wild automorphisms

The automorphisms in the group generated by the affine and the triangular automorphisms are called *tame*. The other automorphisms (if any) are *wild*.

Problem

Are all automorphisms of $K[X_n]$ tame?

$n = 1$ – trivial

$$\mathrm{Aut}(K[x]) = \mathrm{Aff}_1 = \{\alpha : x \to ax + b \mid a \in K^*, b \in B\}$$

$n = 2$

Theorem

Jung (1942): $K = \mathbb{C}$,
van der Kulk (1953): K any filed of arbitrary characteristic:

All automorphisms of $K[X_2]$ are tame.

One of the many proofs (in characteristic 0) was given in the talk of Makar-Limanov.

Mathematics is one science

If you have a problem, look at what the other people have done. Maybe your problem was solved in another language, or at least you may find ideas to work for your problem.

Characterization of $K[X_n]$:

If R is any commutative algebra, then any map

$$\varphi : X_n \to R$$

can be extended to a unique homomorphism

$$\varphi : K[X_n] \to R.$$

Noncommutative analogues of $K[X_n]$:

In Group Theory – the free group $G_n = G(X_n)$;

In Theory of Associative Algebras – the free associative algebra $K\langle X_n\rangle$ (the algebra of polynomials in n noncommuting variables);

In Theory of Lie Algebras – the free Lie algebra $L(X_n)$

Lie algebras

The vector space L is a Lie algebra if it has a binary mapping $[L, L] \to L$ such that

$$[a, a] = 0, \quad a \in L \text{ (anticommutativity)}$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \quad a, b, c \in L \text{ (the Jacobi identity)}.$$

Main example

If R is an associative algebra, then it has a structure of Lie algebra with respect to the new operation $[u, v] = uv - vu$, $u, v \in R$.

The free Lie algebra $L(X_n)$ is isomorphic to the Lie subalgebra of $K\langle X_n\rangle$ generated by X_n.

Group Theory

Theorem. (Nielsen, 1924) The automorphism group $\mathrm{Aut}(G(X_n))$ of the free group $G(X_n)$ is generated by the elementary automorphisms

(i) $\sigma(x_i) = x_{\sigma(i)}$, $\sigma \in S_n$;

(ii) $\theta_1(x_1) = x_1^{-1}$, $\theta_1(x_i) = x_i$, $i \neq 1$;

(iii) $\theta_2(x_1) = x_1x_2$, $\theta_2(x_i) = x_i$, $i \neq 1$.

In other words:

The automorphisms of the free group are tame.

Algebras over a field – elementary automorphisms

(1) Linear automorphisms

$$\alpha : x_i \to \sum_{k=1}^{n} \alpha_{ik} x_k, \quad i = 1, \dots, n, \alpha_{ik} \in K,$$

the matrix (α_{ik}) is invertible;
For unitary algebras one considers affine automorphisms (instead of the linear ones).
(2) Triangular automorphisms

$$\tau : x_i \to \alpha_i x_i + f_i(x_{i+1}, \dots, x_n), \quad \alpha_i \in K^*,$$

f_i does not depend on $x_1, \dots, x_i$.
These automorphisms generate the group of tame automorphisms.

Lie algebras

Theorem. (Cohn, 1964)
The automorphisms of the free Lie algebra $L(X_n)$ are tame for any n.

Groups and Lie algebras

Free groups and free Lie algebras share a remarkable property – their subobjects are free. There is an algorithm which, given a finite system of elements $f_1, \dots, f_m$ in $G(X_n)$ or $L(X_n)$, produces a free generating system of $\langle f_1, \dots, f_m \rangle$.

Free Lie algebras

The elements $f_1, \dots, f_m \in L(X_n)$ are algebraically dependent if there exists a nonzero Lie polynomial $h(y_1, \dots, y_m) \in L(Y_m)$ such that $h(f_1, \dots, f_m) = 0$ in $L(X_n)$.
The following holds: If the homogeneous polynomials $f_1, \dots, f_m \in L(X_n)$ are algebraically dependent, then one of them can be expressed as a polynomial of the others.

Proof of the tameness of the automorphisms of free Lie algebras

Let $\varphi \in \mathrm{Aut}(L(X_n))$, $\varphi(x_i) = f_i$, $i = 1, \dots, n$. If $\deg(f_i) = 1$ for all f_i, then φ is a linear automorphisms, and hence is tame. Let $\deg(f_i) > 1$ for some f_i. Since φ is an automorphism, $x_1 = g(f_1, \dots, f_n)$ for some Lie polynomial $g(y_1, \dots, y_n)$. Hence the highest homogeneous components $\overline{f_i}$ of f_i, $i = 1, \dots, n$, are algebraically dependent. Then one of them can be expressed by the others. Let

$$\overline{f_1} = h(\overline{f_2}, \dots, \overline{f_n}).$$

Consider the triangular automorphism ψ defined by

$$\psi(x_1) = x_1 - h(x_2, \dots, x_n), \psi(x_i) = x_i, \quad i = 2, \dots, n.$$

Compute $\varphi\psi$:

$$\varphi\psi(x_1) = \varphi(x_1 - h(x_2, \dots, x_n)) = f_1 - h(f_2, \dots, f_n),$$
$$\varphi\psi(x_i) = \varphi(x_i), \quad i = 2, \dots, n.$$

Obviously
$$\overline{f_1} = h(\overline{f_2}, \dots, \overline{f_n}) = \overline{h(f_2, \dots, f_n)}.$$

Hence $\deg(\varphi\psi(x_1)) < \deg(f_1) = \deg(\varphi(x_1))$ and by induction

$$\varphi\psi = \tau_1 \cdots \tau_k$$

for some elementary automorphisms $\tau_1, \dots, \tau_k$. Then

$$\varphi = \tau_1 \cdots \tau_k \psi^{-1}.$$

The main difficulty in the proof of the Jung-van der Kulk theorem:

If $\varphi \in \mathrm{Aut} K[x, y]$, $f = \varphi(x)$, $g = \varphi(y)$, then in all of the proofs one tries to show that one of the degrees $p = \deg(f)$ and $q = \deg(g)$ divides the other. If $p = kq > 1$, then the homogeneous components of highest degree (with respect to some grading) satisfy $\overline{f} = \alpha \overline{g}^k$, $\alpha \in K^*$. Then one defines the triangular automorphism ψ by

$$\psi(x) = x - \alpha y^k, \quad \psi(y) = y$$

and obtains that

$$\deg(\varphi\psi(x)) < \deg(\varphi(x)), \quad \varphi\psi(y) = \varphi(y).$$

The proof is completed by obvious induction.

Structure of $\mathrm{Aut}(K[x,y])$:

Some of the proofs give also the structure of $\mathrm{Aut}(K[x,y])$ as an amalgamated free product:

$$\mathrm{Aut}(K[x,y]) \cong \mathrm{Aff}(K[x,y]) *_C \mathrm{Triang}(K[x,y]),$$

$$C = \mathrm{Aff}(K[x,y]) \cap \mathrm{Triang}(K[x,y]).$$

Locally nilpotent derivations and $\mathrm{Aut}(K[x,y])$:

Theorem (Rentschler, 1968) Let δ be a locally nilpotent derivation of $K[x,y]$ ($\mathrm{char}(K)=0$). Then there exists a tame automorphism θ of $K[x,y]$ such that

$$\theta^{-1}\delta\theta = h(y)\frac{\partial}{\partial x}, \quad h(y)\in K[y].$$

This means that, up to a change of the coordinates by a tame automorphism, the only locally nilpotent derivations are $h(y)\dfrac{\partial}{\partial x}$. This gives also a new proof of the tameness of the automorphisms of $K[x,y]$.

The free algebra $K\langle x, y\rangle$

Theorem. (Makar-Limanov, 1970, Czerniakiewicz, 1972)
All automorphisms of the free associative algebra $K\langle x, y\rangle$ are tame. The groups $\mathrm{Aut}(K[x, y])$ and $\mathrm{Aut}(K\langle x, y\rangle)$ are "canonically" isomorphic: If $\pi : K\langle x, y\rangle \to K[x, y]$ is the natural homomorphism, then the mapping $\overline{\pi} : \mathrm{Aut}(K\langle x, y\rangle) \to \mathrm{Aut}(K[x, y])$, defined by $\overline{\pi}(\varphi) = \pi\varphi$, $\varphi \in \mathrm{Aut}(K\langle x, y\rangle)$, i.e., if $g = \pi(f) \in K[x, y]$ for some $f \in K\langle x, y\rangle$, then

$$\overline{\pi}(\varphi) : g = \pi(f) \to \pi(\varphi(f)),$$

defines a group isomorphism $\mathrm{Aut}(K\langle x, y\rangle) \cong \mathrm{Aut}(K[x, y])$.

Commutator test for the automorphisms of $K\langle x, y\rangle$

(Dicks, 1982)
Let φ be an endomorphism of the free associative algebra $K\langle x, y\rangle$. Then φ is an automorphism if and only if

$$[\varphi(x), \varphi(y)] = \alpha[x, y], \quad \alpha \in K^*.$$

Locally nilpotent derivations of $K\langle x, y\rangle$

Exercise. (Use the isomorphism $\mathrm{Aut}(K\langle x, y\rangle) \cong \mathrm{Aut}(K[x, y])$ and the theorem of Rentschler!)
Up to a change of the coordinates, if $\mathrm{char}(K) = 0$, then the only locally nilpotent derivations of $K\langle x, y\rangle$ are $h(y)\dfrac{\partial}{\partial x}$.

In $K[x,y]$, if $h_1(y), h_2(y) \neq 0$, then

$$\ker\left(h_1(y)\frac{\partial}{\partial x}\right) = \ker\left(h_2(y)\frac{\partial}{\partial x}\right).$$

Theorem.

(Drensky, Makar-Limanov, unpublished)
If δ_1 and δ_2 are locally nilpotent derivations of $K\langle x,y\rangle$ ($\mathrm{char}(K)=0$), and $\ker(\delta_1)=\ker(\delta_2)$, then $\delta_1=\alpha\delta_2$ for some nonzero constant $\alpha\in K$.

$n > 2$: Candidates for wild automorphisms

Conjecture. (Nagata, 1972)
If $\text{char}(K) = 0$, then the automorphism $\nu \in \text{Aut}(K[x, y, z])$ defined by

$$\nu(x) = x - 2(y^2 + xz)y - (y^2 + xz)^2 z, \nu(y) = y + (y^2 + xz)z, \nu(z) = z$$

is wild.

Theorem (Nagata)

The Nagata automorphism is wild as an automorphism of the algebra of polynomials $(K[z])[x, y]$ in two variables x and y with coefficients from $K[z]$.

Idea of the proof of Nagata

One can see that if $\varphi \in \mathrm{Aut}((K[z])[x, y])$ is tame, and

$$\varphi(x) = f(x, y, z), \varphi(y) = g(x, y, z), \varphi(z) = z,$$

then the homogeneous components of highest degree with respect to x, y (i.e., $\deg(x) = \deg(y) = 1$, $\deg(z) = 0$) satisfy

$$\overline{f(x, y, z)} = a(z)\overline{g^k(x, y, z)}, \quad a(z) \in K[z]$$

(if $\deg_{x,y}(f) > \deg_{x,y}(g)$). For the Nagata automorphism we have

$$\overline{\nu(x)} = -y^4 z, \overline{\nu(y)} = y^2 z,$$

hence ν is wild as an automorphism of $(K[z])[x, y]$.

How to construct the Nagata automorphism:

The linear operator δ acting on the K-algebra R is a *derivation* of R, if it satisfies the *Leibniz rule*

$$\delta(uv) = \delta(u)v + u\delta(v), \quad u, v \in R.$$

The elements of the kernel of δ are called *constants* and form a subalgebra R^δ of R.
The derivation δ is *locally nilpotent*, if for any $r \in R$ there exists an $n > 0$ such that $\delta^n(r) = 0$.

How to construct the Nagata automorphism – 2:

If δ is a locally nilpotent derivation of the algebra R, then the linear operator

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

is well defined on R and is an automorphism of R as a K-algebra.

$R = K[X_n]$:

If δ is a locally nilpotent derivation of the algebra $K[X_n]$ and $0 \neq w \in K[X_n]^{\delta}$, then the linear operator $\Delta = w\delta$ is also a locally nilpotent derivation and $K[X_n]^{\Delta} = K[X_n]^{\delta}$.

How to construct the Nagata automorphism – 3:

Let δ be the triangular derivation of $K[x,y,z]$ defined by

$$\delta(x) = -2y, \quad \delta(y) = z, \quad \delta(z) = 0.$$

Then $w = y^2 + 2xz \in K[x,y,z]^{\delta}$ and $\exp(w\delta)$ is equal to the Nagata automorphism.

How wild is the Nagata automorphism:

Stably tame automorphisms

The automorphism $\varphi \in \mathrm{Aut}(K[X_n])$ is *stably tame* if for some $m > 0$ it becomes tame, extended to an automorphism of $K[X_{n+m}]$ by $\varphi(x_{n+i}) = x_{n+i}$, $i = 1, \ldots, m$.

Theorem

(Martha Smith, 1989)

Let $\delta(x_j)$ be a triangular derivation of $K[X_n]$, i.e., $\delta(x_j) \in K[X_{j-1}]$, and let $w \in K[X_n]^{\delta}$. Extend the automorphism $\varphi = \exp(w\delta)$ of $K[X_n]$ to an automorphism of $K[X_{n+1}]$ by $\varphi(x_{n+1}) = x_{n+1}$. Then φ becomes tame on $K[X_{n+1}]$.

Corollary

The Nagata automorphism is stably tame.

There is no room to work in the three dimensional space.

In the four dimensional space the Nagata automorphism is tame.

The work of Shestakov and Umirbaev

(The Russian Doklady – 2002; J. Amer. Math. Soc. – 2004)
In characteristic 0, Shestakov and Umirbaev developed an algorithm which decides whether an automorphism of the polynomial algebra $K[x, y, z]$ is tame.

Remarkable

Quite often Commutative Algebra serves as a model for the development of Noncommutative Algebra. *Shestakov and Umirbaev use "very noncommutative" methods to solve a problem in Commutative Algebra.*

Let $L(X_n)$ is the free Lie algebra with basis as a vector space

$$U = \{u_i \mid i = 1, 2, \ldots\}$$

The free Poisson algebra is the polynomial algebra in the commuting variables U with an additional binary operation (Poisson bracket) which satisfies the Leibniz rule and such that

$$[u_i, u_j] = \sum_{k=1}^{m} \alpha_{ij}^k u_k, \quad \alpha_{ij}^k \in K,$$

where $[u_i, u_j]$ is the commutator in $L(X_n)$.

Idea of the method of Shestakov and Umirbaev.

Let $f, g \in K[X_n]$ be algebraically independent, let their homogeneous components of maximal degree be algebraically dependent, but $\overline{f} \notin K[\overline{g}]$ and $\overline{g} \notin K[\overline{f}]$. The idea is to control the minimal degree of the nonconstant polynomials in the subalgebra $K[f, g]$ generated in $K[X_n]$ by f and g. *To estimate this minimal degree, Shestakov and Umirbaev use free Poisson algebras.*

Estimate of Shestakov and Umirbaev.

Let $f,g\in K[X_n]$, $h(u,v)\in K[u,v]$.
$m_1=\deg(f)$, $m_2=\deg(g)$, $m_1\nmid m_2$, $m_2\nmid m_1$,

$$p=\frac{m_1}{(m_1,m_2)},s=\frac{m_2}{(m_1,m_2)},$$

$$N=N(f,g)=\frac{m_1m_2}{(m_1,m_2)}-m_1-m_2+\deg([f,g]),$$

$$[f,g]=\sum_{1\leq i<j\leq n}[x_i,x_j]\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}-\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right).$$

If $\deg_u h(u,v)=pq+r$, $0\leq r<p$, then

$$\deg(h(f,g))\geq qN+m_2r.$$

If $\deg_v h(u,v)=sq_1+r_1$, $0\leq r_1<s$, then

$$\deg(h(f,g))\geq q_1N+m_1r_1.$$

Application – Proof of Jung-van der Kulk theorem in characteristic 0.

Let $\varphi \in \mathrm{Aut}K[x,y]$, $f = \varphi(x)$, $g = \varphi(y)$ and let the degrees of f and g do not divide each other. Then $x = h(f,g) \in K[f,g]$ and

$$\frac{m_1 m_2}{(m_1, m_2)} \geq m_1 + m_2, \quad 1 = \deg(x) \geq N(f,g) \geq \deg([f,g]) \geq 2,$$

which is a contradiction. Hence m_1 divides m_2 or m_2 divides m_1 and the proof follows.

Other estimates

Makar-Limanov and J.-T. Yu (2008): Estimates based on the lemma of Bergman (1978) on radicals for the Malcev-Neumann algebra of formal power series.

Main applications:

An algorithm which decides whether an automorphism of $K[x,y,z]$ is tame.

The Nagata automorphism is wild.

Theorem

If an automorphism of $(K[z])[x,y]$ is wild, then it is wild also as an automorphism of $K[x,y,z]$.

Drensky and J.-T. Yu - 2001

An algorithm which decides whether an endomorphism of $(K[z])[x, y]$ is a tame automorphism of $(K[z])[x, y]$.
(EASY)

$$\Downarrow$$

Many new examples of wild automorphisms of $K[x, y, z]$.
(DIFFICULT)

Theorem

(Drensky, van den Essen, and Stefanov – 2000)
Let δ be a locally nilpotent derivation of the algebra $K[X_n, Y_m]$ and let

$$\delta(x_j) = \sum_{k=1}^{n} a_{kj}(Y_m)x_k + b_j(Y_m),\ j = 1, \ldots, n,$$

$$\delta(y_i) = 0,\ i = 1, \ldots, m,$$

where $a_{kj}(Y_m)$ and $b_j(Y_m)$ do not depend on X_n. Let $w \in K[X_n, Y_m]^{\delta}$. Then $\varphi = \exp(w\delta)$ is a stably tame automorphism and its extension becomes tame if we add one variable when $n \geq 3$ and two variables when $n = 2$.
The proof uses the method of Martha Smith together with the Suslin theorem that *if $n \geq 3$ then every matrix in $SL_n(K[Y_m])$ is a product of elementary matrices.*

Freudenburg – 1998

The algebras of constants of most of the locally nilpotent derivations δ on $K[X_n]$, which we know, contain a coordinate (an image of x_1 under an automorphism of $K[X_n]$). Hence if we change properly the variables, one of the new variables will be in the kernel of δ and will be fixed by the automorphism $\exp(\delta)$.

Example of a locally nilpotent derivation on $K[x, y, z]$, which does not fix a coordinate.

$$\delta(f) = \begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial f}{\partial x} \\ \frac{\partial w}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial f}{\partial y} \\ \frac{\partial w}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial f}{\partial z} \end{pmatrix},$$

$$w = y^2 + xz, \quad v = zw^2 + 2x^2yw - x^5.$$

Proof of local nilpotency

$w, v \in \ker(\delta)$,

$$\delta(x) = -2wr, \quad \delta(y) = -6x^2 r - v, \quad r = x^3 - wy, \quad \delta(r) = wv.$$

Hence, $\delta^2(r) = 0$, $\delta^3(x) = 0$, $\delta^k(y) = 0$, $k \gg 0$,

$$zw^2 = v - 2x^2yw + x^5 = v - x^2yw + x^2r \in K[w, v, x, y, r],$$

$\delta^n(z) = 0$, $n \gg 0$.

Problem

Is the automorphism $\exp(\delta)$ *wild?*
Answer: YES, It is wild. (talk of D. Wright, Levico Terme – 2012).

Anick automorphism

The endomorphism of the algebra $K\langle x, y, z\rangle$

$$\omega(x) = x + z(xz - zy), \quad \omega(y) = y + (xz - zy)z, \quad \omega(z) = z.$$

is an automorphism. *The conjecture of Anick is that it is wild.* It induces a tame automorphism of $K[x, y, z]$. Additionally, it fixes z, and $\omega(x)$ and $\omega(y)$ are linear in x and y.

Drensky and J.-T. Yu - 2005

If the polynomial $f(X_n, z) \in K\langle X_n, z\rangle$ is linear with respect to X_n,

$$f(X_n, z) = \sum_{i=1}^{n} \sum_{j,k\geq 0} \alpha_{ijk} z^j x_i z^k, \quad \alpha_{ijk} \in K,$$

then we define "partial derivatives"

$$\frac{\partial f}{\partial x_i} = \sum_{j,k\geq 0} \alpha_{ijk} z_1^j z_2^k \in K[z_1, z_2].$$

If φ is an endomorphism of the algebra $K\langle X_n, z\rangle$, which fixes z and is linear in X_n, then we define its Jacobian matrix

$$J(\varphi) = \begin{pmatrix} \frac{\partial \varphi(x_1)}{\partial x_1} & \dots & \frac{\partial \varphi(x_n)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi(x_1)}{\partial x_n} & \dots & \frac{\partial \varphi(x_n)}{\partial x_n} \end{pmatrix}.$$

Theorem

(i) The linear in x and y automorphism φ of the algebra $K\langle x, y, z\rangle$ which fixes z, is tame (in the class of automorphisms fixing z) if and only if its Jacobian matrix belongs to the group $GE_2(K[z_1, z_2])$ generated by the elementary and diagonal matrices.
(ii) Every linear in x and y automorphism φ of $K\langle x, y, z\rangle$ which fixes z, induces a tame automorphism of the polynomial algebra $K[x, y, z]$.
(iii) If $n > 2$, then every linear in X_n automorphism φ of $K\langle X_d, z\rangle$ which fixes z, is tame.

Example – the Anick automorphism

$$\omega(x) = x + z(xz - zy), \quad \omega(y) = y + (xz - zy)z, \quad \omega(z) = z,$$

$$J(\omega) = \begin{pmatrix} 1 + z_1 z_2 & z_2^2 \\ -z_1^2 & 1 - z_1 z_2 \end{pmatrix}.$$

It is well known that this matrix cannot be presented as a product of elementary matrices. Hence, *the Anick automorphism is wild in the class of automorphisms of $K\langle x, y, z\rangle$ fixing z.*

Umirbaev – 2006-2007

(i) Umirbaev found a system of generators and defining relations of the group of tame automorphisms of the polynomial algebra $K[x, y, z]$.
(ii) A linear in x and y automorphism φ of the algebra $K\langle x, y, z\rangle$ which fixes z, is tame if and only if it it tame in the class of automorphisms fixing z.
(iii) **Corollary.** *The Anick automorphism of the algebra $K\langle x, y, z\rangle$ is wild.*

Problems

(i) Do the algebras $K[X_n]$ and $K\langle X_n\rangle$ have wild automorphisms for $n > 3$?
(ii) Do the algebras $K[x, y, z]$ and $K\langle x, y, z\rangle$ have wild automorphisms when the field K is of positive characteristic?
(iii) Can every automorphism of $K[X_n]$, $n > 2$, be lifted to an automorphism of $K\langle X_n\rangle$?
(iv) If δ is a locally nilpotent derivation of $K[X_n]$, $n > 2$, $\mathrm{char}(K) = 0$, is it possible to lift it to a locally nilpotent derivation of $K\langle X_n\rangle$?
(v) Is every automorphism of $K[X_n]$ and $K\langle X_n\rangle$, $n > 2$, stably tame?

Berson, van ven Essen, and Wright

(arXiv: 10 versions in 2007-2012,
journal version: Adv. Math. 2012)
All automorphisms of $K[x, y, z]$ which fix z are stably tame ($\mathrm{char}(K) = 0$).

Belov-Kanel, J.-T. Yu – 2011

A wild automorphism of $(K[z])[x,y]$ (K any field) cannot be lifted to an automorphism of $K\langle x,y,z\rangle$ which fixes z.
Corollary. (i) The Nagata automorphism cannot be lifted to an automorphism of $K\langle x,y,z\rangle$ which fixes z.
(ii) The derivation related to the Nagata automorphism cannot be lifted to a locally nilpotent derivation of $K\langle x,y,z\rangle$ with z in the kernel.

Belov-Kanel, J.-T. Yu – 2012

Every automorphism fixing z of the free associative algebra $K\langle x,y,z\rangle$ is stably tame and becomes tame if adding one new variable.