# Gromov–Witten Invariants and the Virasoro Conjecture. III

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Recall that $X$ is a projective algebraic manifold over $\mathbb{C}$ of dimension $\dim(X) = r$. $M \subset H_2^+(X, \mathbb{Z})$ is the monoid of homology classes represented by closed algebraic curves in $X$. The Novikov ring is the set of functions $\Lambda = \mathbb{Q}^M$, with convolution product.

Choose a homogenous basis $\{\gamma_a \in H^{p_a,q_a}(X, \mathbb{Q})\}_{a \in I}$ of the rational cohomology $H^*(X, \mathbb{Q})$, with $\gamma_0 = 1 \in H^0(X, \mathbb{Q})$. Let $\mu_a = p_a - \frac{r}{2}$.

Denote the non-degenerate Poincaré form by

$$\eta_{ab} = \int_X \gamma_a \cup \gamma_b,$$

and its inverse by $\eta^{ab}$.

The **large phase space** is the formal space with coordinate ring

$$\Lambda[[\tilde{t}_k^a \mid a \in I, k \geq 0]],$$

where $\tilde{t}_k^a$ is obtained from $t_k^a$ by the **dilaton shift**

$$\tilde{t}_k^a = t_k^a - \delta_{k1}\eta^{0a}.$$

The jet space is the formal space with coordinate ring

$$\Lambda\{u\} = \Lambda[[u]][u', u^{(2)}, \dots, u^{(k)}, \dots].$$

The coordinate ring $\Lambda\{u\}$ has an increasing filtration

$$F^k\Lambda\{u\} = \Lambda[[u]][u', u^{(2)}, \dots, u^{(k)}].$$

The topological recursion relations state that

$$\partial_{k_1,a_1} \dots \partial_{k_n,a_n} F_g \in F^{3g-2+n}\Lambda\{u\}.$$

The total potential of the theory is

$$Z = \exp(\hbar^{-1} F),$$

where

$$F = \sum_{g=0}^{\infty} \hbar^g F_g.$$

Let $R$ be multiplication by the Chern class $c_1(X)$:

$$R_a^b = \eta^{bc} \int_X c_1(X) \cup \gamma_a \cup \gamma_c.$$

The string equation is $L_{-1} Z = 0$, where

$$L_{-1} = \mathcal{L}_{-1} + \frac{1}{2\hbar} \eta_{ab} \tilde{t}_0^a \tilde{t}_0^b.$$

Here, $\mathcal{L}_{-1}$ is the vector field

$$\mathcal{L}_{-1} = \sum_{a \in I} \sum_{k=0}^{\infty} \tilde{t}_{k+1}^a \partial_{k,a}.$$

There is a morphism from the large phase space to the jet space, defined by

$$\partial^k u^a = \eta^{ab}(\partial_{0,0})^{k+1}\partial_{0,b}F_0.$$

The string equation implies that

$$\partial^k u^a = \delta_{k1}\eta^{0a} + t_k^a + O(t^2).$$

In other words, the large phase space is a completion of $\mathrm{Spec}(\Lambda\{u\})$ at the "dilaton" point

$$\tilde{t}_k^a = t_k^a - \delta_{k1}\eta^{0a}.$$

Roughly speaking, the KdV (or integrable systems) picture studies the coordinate ring $\Lambda\{u\}$, while the Virasoro conjecture studies its completion $\Lambda[[\tilde{t}_k^a]]$. Our task is to compare these pictures.

The Hori equation is $L_0 Z = 0$, where

$$L_0 = \mathcal{L}_0 + \frac{1}{2\hbar} R_{ab} \tilde{t}_0^a \tilde{t}_0^b + \frac{1}{48} \int_X \big((3-r)c_r(X) - 2c_1(X)c_{r-1}(X)\big).$$

Here, $\mathcal{L}_0$ is the vector field

$$\mathcal{L}_0 = \sum_{a \in I} \sum_{k=0}^{\infty} \big(k + \mu_a + \tfrac{1}{2}\big) \tilde{t}_k^a \partial_{k,a} + \sum_{a,b \in I} \sum_{k=0}^{\infty} R_a^b \tilde{t}_{k+1}^a \partial_{k,b}.$$

Consider the generating function

$$\sum_{i=0}^{k} s^i e_i(R,x,n) = (sR+x)(sR+x+1)\dots(sR+x+n)$$

Thus, for each $i$, $e_i(R,x,n)$ is a matrix whose coefficients are polynomials of degree $k-i+1$ in $x$.

We now introduce the differential operators $L_n$, $n>0$, that enter into the Virasoro conjecture. Unlike $L_{-1}$ and $L_0$, these are second order differential operators on the large phase space.

$$L_n = \mathcal{L}_n + \frac{\hbar}{2} \sum_{a,b\in I} \sum_{i=0}^{n+1} \sum_{k=i-n}^{-1} (-1)^k e_i\big(R, k+\mu_a+\tfrac{1}{2}\big)^{ab} \partial_{-k-1,a}\partial_{k+n-i,b}.$$

Here, $\mathcal{L}_n$ is the vector field

$$\mathcal{L}_n = \sum_{a,b\in I} \sum_{i=0}^{n+1} \sum_{k=0}^{\infty} e_i\big(R, k+\mu_a+\tfrac{1}{2}\big)_a^b \tilde{t}_k^a \partial_{k+n-i,b}$$

The Virasoro conjecture says that $L_n Z = 0$, $n > 0$.

The commutation relations $[L_n, L_m] = (n-m)L_{n+m}$ give a necessary consistency condition for this conjecture: for example, if $L_{n-1}Z = 0$, then

$$L_{-1}L_n Z = \big([L_n, L_{-1}] - (n+1)L_{n-1}\big)Z,$$

and so must vanish if $L_n Z$ is to vanish.

These commutation relations are proved using the free field

$$\phi^a(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})} z^{k+\mu_a+\frac{1}{2}} \tilde{t}_k^a - \hbar\eta^{ab} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} z^{-k+\mu_a-\frac{1}{2}} \partial_{k,b}.$$

For $n \geq -1$, we have

$$[L_n, \phi(z)] = -(z+R)^{n+1} \partial_z \phi(z).$$

The tricky part in proving the commutation relations is checking the constant term in the formula

$$[L_1, L_{-1}] = 2L_0.$$

This uses the Libgober-Wood formula, itself a consequence of the Riemann-Roch-Hirzebruch theorem (or an explicit calculation, in the case of projective spaces):

$$\sum_{a\in I}(-1)^{p_a+q_a}\mu_a^2 = \frac{1}{12}\int_X rc_r(X) + 2c_1(X)c_{r-1}(X).$$

We now begin the translation of the Virasoro conjecture into a statement in the ring $\Lambda\{u\}$. Introduce the generating function of two-point correlators in genus 0

$$\Theta(z)_a^b = \delta_a^b + \eta^{bc} \sum_{k=0}^{\infty} z^{k+1} \partial_{k,a} \partial_{0,c} F_0$$

The following fundamental formula is due to Dubvrovin.

**Lemma**

$$\Theta(z)\Theta(-z)^* = I$$

Proof.

The topological recursion relation in genus 0 shows that

$$\partial(\Theta(z)\Theta(-z)^*) = 0,$$

and the string equation shows that

$$\mathcal{L}_{-1}(\Theta(z)\Theta(-z)^*) = 0.$$

Thus $\Theta(z)\Theta(-z)^*$ is a constant. But at $t = 0$ it equals $I$, and the formula is proved. □

Let $\mathcal{M}$ be the matrix

$$\mathcal{M}^a_b = \eta^{bc}\partial_{0,a}\partial_{0,c}F_0,$$

and let $\mathcal{X} = \partial\mathcal{M}$. The following formula is again proved using the topological recursion relation and string equation in genus 0:

$$\Theta^{-1}(z)\partial\Theta(z) = \mathcal{X}.$$

At the point $t = 0$, the matrix $\mathcal{X}$ equals the identity: thus, it is invertible in a neighbourhood of this point.

The following theorem is a consequence of the topological recursion relation and string equation in genus 0.

### Theorem

*Let $\mathbf{t}(z)$ be the formal generating function given by the formula*

$$\mathbf{t}^a(z) = \eta^{ab} \sum_{k=0}^{\infty} z^k \partial_{k,b} F_0 + \sum_{k=0}^{\infty} (-z)^{-k-1} \tilde{t}_k^a.$$

*We have*

$$\mathbf{s}(z) = \Theta^*(z)\mathbf{t}(z) = \sum_{k=0}^{\infty} z^{-k-1} \left( (\mathcal{X}^{-1}\partial)^k \mathcal{X}^{-1} \right)^0.$$

We may use $\Theta(z)$ to convert derivatives in the coordinates $t_k$ to derivatives in the coordinates $u^{(k)}$. Let

$$\tau(z) = \sum_{k=0}^{\infty} z^k \partial_k,$$

and let

$$\sigma(z) = \sum_{k=0}^{\infty} z^k \sigma_k = \Theta^{-1}(z)\tau(z).$$

Observe that

$$\mathbf{t}(z) = z^{-1} \sum_{\ell=0}^{\infty} \partial^{\ell+1}\Theta(z)\frac{\partial}{\partial u^{(\ell)}},$$

and hence

$$\sigma(z) = z^{-1} \sum_{\ell=0}^{\infty} \Theta^{-1}(z)\partial^{\ell+1}\Theta(z)\frac{\partial}{\partial u^{(\ell)}}.$$

For example,

$$\sigma_0 = \sum_{\ell=0}^{\infty} \partial^\ell \mathcal{X} \frac{\partial}{\partial u^{(\ell)}}.$$

In general, $\sigma_\ell$ vanishes on $F^k\Lambda\{u\}$ if $\ell > k$; conversely, $F^k\Lambda\{u\}$ is the subalgebra of functions that are constant along the vector fields $\sigma_\ell$, $\ell > k$.

Let $\mathcal{U} = \mathcal{M} + [\mu, \mathcal{M}] + R$, and let

$$\delta_z = \partial + z^{-1}(\mu + \tfrac{1}{2}) + \mathcal{U}.$$

Although we have not emphasized this point of view in the lectures, $\mathcal{U}$ may be interpreted as multiplication in the quantum cohomology by the Euler vector field

$$E = \sum_{a \in I} \big( (1 - p_a) u^a + R_0^a \big) \partial_a .$$

Recall that this product is given by the formula

$$\eta(\partial_a *_t \partial_b, \partial_c) = \partial_{0,a} \partial_{0,b} \partial_{0,c} F_0 .$$

It is associative, commutative, and equals the cup-product at $t = 0$.

Most results on the Virasoro conjecture concern **semisimple** quantum cohomology: this is the situation where the matrix $\mathcal{U}$ is generically semisimple, with distinct eigenvalues. It follows from this condition that the product on the quantum cohomology is also semisimple: for example, this is the case for projective spaces, for which the relation $x^{m+1} = 0$ in $H^*(\mathbb{CP}^m)$ deforms to $x^{m+1} = q$.

In the semisimple case, the Virasoro conjecture follows from the fact that it holds for a point. The basic idea is to find a group action on tau-functions $Z$ such that

$$Z_X = g \cdot Z_{\mathrm{KdV}}^{\otimes N},$$

and then to identify the conjugate of the Virasoro operators in the KdV case by the group element $g$ with the Virasoro operators for $X$.

The following formula is straightforward, using the above definitions and Dubrovin's formula for $\Theta^{-1}(z)$.

**Theorem**

$$\mathcal{L}_n = \operatorname{Res}_{z=0}\big(\mathbf{s}(z), z\delta_z^{n+1}\sigma(z)\big)$$

The vanishing of the coefficients of $z^k$, $k \geq 0$, in $\mathbf{s}(z)$ now implies the Virasoro conjecture in genus 0 (for all compact Kähler, and even symplectic, manifolds!).

Now that we know that the Virasoro conjecture holds in genus 0, we can reformulate the Virasoro conjecture in higher genus in the ring $\Lambda\{u\}$.

Consider the expansion of the Virasoro conjecture

$$Z^{-1} L_n Z = \sum_{g=0}^{\infty} \hbar^{g-1} z_{n,g}.$$

If $n > 0$, we have $z_{n,0} = \mathcal{L}_n F_0 = 0$.

The case $g = 1$ is due to Liu.

**Theorem**

$$z_{n,1} = \mathcal{L}_n F_1 - \tfrac{1}{4} \sum_{k=0}^{n} \mathrm{Str}\big((\mu - \tfrac{1}{2})\mathcal{U}^k(\mu + \tfrac{1}{2})\mathcal{U}^{n-k}\big)$$

Note that this formula holds for $n = 0$ (the constant term of Hori's equation) and $n = -1$ as well.

Next, we have a formula for $z_{n,g}$, $g > 1$.

**Theorem**

$$z_{n,g} = \mathcal{L}_n F_g + \frac{1}{2} \operatorname{Res}_{z=0} \eta\big(\sigma(-z), z\delta_z^{n+1}\sigma(z)\big) F_{g-1} + \frac{1}{2} \sum_{h=1}^{g-1} \operatorname{Res}_{z=0} \eta\big(\sigma(-z)F_h, z\delta_z^{n+1}\sigma(z)F_{g-h}\big).$$

The following result is due to Dubrovin and Zhang.

### Lemma

*Suppose that $X$ has semisimple quantum cohomology. If $f \in F^k\Lambda\{u\}$ and $\mathcal{L}_n f = 0$ for all $n \geq -1$, then $f \in F^{k-1}\Lambda\{u\}$.*

It follows that if the quantum cohomology is semisimple and $Z$ satisfies the Virasoro conjecture, then the solution of the Virasoro conjecture is unique.

At first sight, this result is surprising, since it seems that the Virasoro conjecture doesn't impose enough constraints. But in combination with the topological recursion relations, and under the condition of semisimplicity, it does give sufficient information.