Gromov–Witten Invariants and the Virasoro Conjecture. II

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Introduction

In this talk, we give an introduction to the Virasoro conjecture for Gromov-Witten invariants with target a projective manifold. This subject has been developed by Witten, Ruan, Kontsevich, Manin, Behrend, Fantechi, Tian, Givental, Dubrovin, Zhang, Teleman, Losev, Shadrin, and others.

We will use the language of Frobenius manifolds, developed by Dubrovin. Important special cases of this formalism were studied earlier by K. Saito.

The first talk was based on the review article

E. Getzler, *The Virasoro conjecture for Gromov-Witten invariants.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 147-176, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999. `arXiv:9812026`

This second talk in this series is based on the review article

E. Getzler, *The jet-space of a Frobenius manifold and higher-genus Gromov-Witten invariants.* Frobenius manifolds, 45-89, Aspects Math., E36, Friedr. Vieweg, Wiesbaden, 2004. `arXiv:0211338`

and the article

A. Buryak, H. Posthuma and S. Shadrin, *A polynomial bracket for the Dubrovin-Zhang hierarchies.* J. Differential Geom. **92** (2012), no. 1, 153-185. `arXiv:1009.5351`

The list of references in this article is particularly complete.

Let X be a projective algebraic manifold over $\mathbb{C}$ of dimension $\dim(X) = r$. (Some results may be extended to compact Kähler manifolds.)

Let $M \subset H_2^+(X, \mathbb{Z})$ be the monoid of homology classes represented by closed algebraic curves in X. For each $x \in M$, the set $\{y \in M \mid x \in y + M\}$ is finite. Hence the set of functions $\Lambda = \mathbb{Q}^M$ is a commutative ring, called the Novikov ring. We denote elements of Λ by

$$\sum_{\beta \in M} a_\beta q^\beta.$$

The ring Λ is graded by assigning to q^β degree

$$\deg(q^\beta) = \beta \cap c_1(X).$$

Moduli space of stable maps

The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$, $g \geq 0$, $n \geq 0$, $\beta \in M$, was introduced by Kontsevich, and constructed by Behrend and Manin. It is a proper Deligne-Mumford stack, but is only smooth in rare cases, such as when $g = 0$ and X a homogenous space.

We denote a stable map by

$$[f : C \to X, z_1, \ldots, z_n].$$

Here, C is a projective curve of arithmetic genus g with double point singularities and n marked smooth points, and f is an algebraic morphism of degree β. The stability condition says that f has no continuous automorphisms: this is a nontrivial condition only on irreducible components of C on which f is constant.

The virtual fundamental class

The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$ (Behrend and Fantechi) is an algebraic cycle, well-defined up to rational equivalence, on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. It has dimension

$$\dim[\overline{\mathcal{M}}_{g,n}(X,\beta)] = (1-g)(r-3) + \beta \cap c_1(X) + n.$$

Note that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is rarely smooth, and the virtual fundamental class is not always the fundamental class of the moduli space even when it is, if the space $\mathbb{H}^1(C, TC \to f^*TX)$ of obstructions is nonzero.

For example, when $\beta = 0$, we have

$$\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X,$$

and by the excess intersection formula,

$$[\overline{\mathcal{M}}_{g,n}(X,0)] = c_{rg}(\mathbb{E} \boxtimes TX) \cap [\overline{\mathcal{M}}_{g,n}] \times [X].$$

Here, $\mathbb{E}$ is the rank g vector bundle $\pi_*\omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ over $\overline{\mathcal{M}}_{g,n}$.

Gromov-Witten invariants

Choose a homogenous basis $\{\gamma_a \in H^{p_a,q_a}(X,\mathbb{Q})\}_{a\in I}$ of the rational cohomology $H^*(X,\mathbb{Q})$. A special role is played by $\gamma_0 = 1 \in H^0(X,\mathbb{Q})$, the identity element of the cohomology ring. Denote the non-degenerate Poincaré form by

$$\eta_{ab} = \int_X \gamma_a \cup \gamma_b,$$

and its inverse by η^{ab}.

The fibre of the line bundle $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ at the point $[f : C \to X, z_1, \ldots, z_n]$ is $T_{z_i}C$. Denote by

$$\Psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$$

the Chern class $c_1(\mathcal{L}_i)$.

Let $\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X^n$ be the evaluation map

$$\mathrm{ev}[f : C \to X, z_1, \ldots, z_n] = (f(z_1), \ldots, f(z_n))$$

Choose $k_1, \ldots, k_n \geq 0$ and $a_1, \ldots, a_n \in I$. The Gromov-Witten invariants are the intersection numbers

$$\langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \rangle_{g,n,\beta} = \mathrm{ev}^*(\gamma_{a_1} \boxtimes \cdots \boxtimes \gamma_{a_n}) \Psi_1^{k_1} \ldots \Psi_n^{k_n} \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)] \in \mathbb{Q}.$$

This vanishes unless

$$k_1 + \cdots + k_n + p_{a_1} + \cdots + p_{a_n} = (1-g)(r-3) + c_1(X) \cap \beta + n.$$

(There is a similar equation in which p_a is replaced by q_a.)

These numbers are *invariant* under deformation of X.

Summing over degree β, we obtain the Gromov-Witten invariant as an element of Λ:

$$\langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \rangle_{g,n} = \sum_{\beta \in M} \langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \rangle_{g,n,\beta} q^\beta \in \Lambda.$$

The **large phase space** is the formal space with coordinates $\{t_k^a \mid a \in I, k \geq 0\}$. We are interested in differential equations among the generating functions

$$F_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{a_1 \ldots a_n \\ k_1 \ldots k_n}} \langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n} \rangle_{g,n} t_{k_1}^{a_1} \ldots t_{k_n}^{a_n}$$

and

$$F = \sum_{g=0}^{\infty} \hbar^g F_g.$$

Let $\partial_{k,a} = \partial / \partial t_k^a$.

The analogue of the coordinate $\mathbf{u}$ of the KdV hierarchy is

$$\mathbf{u}^a = \eta^{ab}\partial_{0,0}\partial_{0,b}F = t_0^a + O(t^2 + \hbar).$$

We prefer to work with the genus 0 limit of these coordinates

$$u^a = \eta^{ab}\partial_{0,0}\partial_{0,b}F_0 = t_0^a + O(t^2).$$

The statement about the limiting value of u^a comes from the formula for genus 0 Gromov-Witten invariants in degree 0: since $\overline{\mathcal{M}}_{0,3}(X,0) = X$,

$$\langle \tau_0(\gamma_{a_1})\tau_0(\gamma_{a_2})\tau_0(\gamma_{a_3})\rangle_{0,3,0} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Denote by $\Lambda\{u\}$ the ring of differential polynomials

$$\Lambda\{u\} = \Lambda[[u]][u', u^{(2)}, \ldots, u^{(k)}, \ldots].$$

We will think of $u^{(k)}$ as the partial derivative $\partial_{0,0}^{k} u$. Let ∂ be the derivation on $\Lambda\{u\}$, defined on the generators by $\partial u^{(k)} = u^{(k+1)}$ and $\partial q^{\beta} = 0$.

The ring $\Lambda\{u\}$ has an increasing filtration

$$F^k \Lambda\{u\} = \Lambda[[u]][u', u^{(2)}, \ldots, u^{(k)}].$$

The topological recursion relations

If $2g-2+n>0$ and $N \geq 0$, there is a morphism

$$\overline{\mathcal{M}}_{g,n+N}(X,\beta) \to \overline{\mathcal{M}}_{g,n},$$

which forgets the map $f : C \to X$ and the marked points $(z_{n+1}, \ldots, z_{n+N})$, and contracts any unstable rational components of $[C, z_1, \ldots, z_n]$ to obtain a stable n-pointed curve.

For $1 \leq i \leq n$, we have Chern classes Ψ_i on $\overline{\mathcal{M}}_{g,n+N}(X,\beta)$ and ψ_i on $\overline{\mathcal{M}}_{g,n}$. It is not the case that ψ_i pulls back to Ψ_i. (Givental calls intersections with ψ_i **ancestors** and intersections with Ψ_i **descendents**.) But there is an explicit formula for their difference.

If $k_1 + \cdots + k_n > 2g-2+n$, we see that $\psi_1^{k_1} \ldots \psi_n^{k_n} = 0$. Pulling this relation back to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and rewriting in terms of the classes Ψ_i, we obtain relations among the Gromov-Witten invariants which in genus 0 are due to Witten, and in higher genus to Eguchi and Xiong.

Topological recursion relations

In genus 0, Witten's topological relations state that

$$\partial_{k_1,a_1}\partial_{k_2,a_2}\partial_{k_3,a_3}F_0 = \eta^{ab}\partial_{0,a}\partial_{k_1-1,a_1}F_0\partial_{0,b}\partial_{k_2,a_2}\partial_{k_3,a_3}F_0.$$

The topological recursion relations imply that

$$\partial_{k_1,a_1}\partial_{k_2,a_2}F_0 \in \Lambda[[u]],$$

and more generally, that

$$\partial_{k_1,a_1}\dots\partial_{k_n,a_n}F_0 \in F^{n-2}\Lambda\{u\}.$$

In genus $g > 0$, the topological recursion relations imply that

$$F_g \in F^{3g-2}\Lambda\{u\},$$

and more generally, that

$$\partial_{k_1,a_1}\dots\partial_{k_n,a_n}F_g \in F^{3g-2+n}\Lambda\{u\}.$$

The string equation

There is a flat morphism

$$\overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

defined by forgetting the last marked point and contracting to a point any unstable rational component of the resulting pointed map.

The generalization of Knudsen and Mumford's theorem holds: the fibre of the resulting map is naturally isomorphic to C. This yields the string equation: if $2g-2+n>0$, then

$$\langle \tau_{0,0}\tau_{k_1,a_1}\ldots\tau_{k_n,a_n}\rangle_{g,n+1} = \sum_{i=1}^{n}\langle \tau_{k_1,a_1}\ldots\tau_{k_i-1,a_i}\ldots\tau_{k_n,a_n}\rangle_{g,n}.$$

In terms of the generating functions F_g, this is equivalent to the series of equations

$$\partial_{0,0}F_g - \sum_{a\in I}\sum_{k=0}^{\infty} t_{k+1}^a \partial_{k,a}F_g = \begin{cases} \frac{1}{2}\eta_{ab}t_0^a t_0^b, & g=0,\\ 0, & g>0.\end{cases}$$

The string equation implies that

$$\partial^k u^a = \delta_{k1}\eta^{0a} + t_k^a + O(t^2).$$

In other words, the large phase space is a completion of $\mathrm{Spec}(\Lambda\{u\})$ at the "dilaton" point

$$\tilde{t}_k^a = t_k^a - \delta_{k1}\eta^{0a}.$$

Roughly speaking, the KdV (or integrable systems) picture studies the coordinate ring $\Lambda\{u\}$, while the Virasoro conjecture studies its completion $\Lambda[[\tilde{t}_k^a]]$. Our task is to compare these.

In terms of the formal expression (tau-function)

$$Z = \exp(\hbar^{-1} F),$$

the string equation may be written as a homogenous linear equation $L_{-1} Z = 0$, where L_{-1} is the differential operator

$$L_{-1} = \sum_{a \in I} \sum_{k=0}^{\infty} \tilde{t}^a_{k+1} \partial_{k,a} + \tfrac{1}{2\hbar} \eta_{ab} \tilde{t}^a_0 \tilde{t}^b_0.$$

Hori's equation

Introduce the matrix

$$R_a^b = \eta^{bc} \int_X c_1(X) \cup \gamma_a \cup \gamma_c .$$

Let L_0 be the operator

$$L_0 = \mathcal{L}_0 + \frac{1}{2\hbar} R_{ab} t_0^a t_0^b + \frac{1}{48} \int_X \big((3-r) c_r(X) - 2 c_1(X) c_{r-1}(X) \big),$$

where $\mathcal{L}_0$ is the vector field

$$\mathcal{L}_0 = \sum_{k=0}^{\infty} (k + p_a + \tfrac{1-r}{2}) \tilde{t}_k^a \partial_{k,a} + \sum_{k=0}^{\infty} R_a^b \tilde{t}_{k+1}^a \partial_{k,b} .$$

It may be proved that $\mathcal{L}_0$ induces a derivation of $\Lambda\{u\}$.

In his thesis, Hori proved that $[L_0, L_{-1}] = L_{-1}$, and that

$$L_0 Z = 0.$$

(He discusses projective spaces: the generalization to projective manifolds is due to Sheldon Katz.)

In terms of the potentials F_g, this becomes

$$\mathcal{L}_0 F_g = \begin{cases} \frac{1}{2} R_{ab} t_0^a t_0^b, & g = 0, \\ \frac{1}{48} \int_X \big((3-r)c_r(X) - 2c_1(X)c_{r-1}(X)\big), & g = 1, \\ 0, & g > 1. \end{cases}$$

The proof makes use of the equation for the dimension of the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]$ and two further ingredients:

1. the **dilaton equation**: if $2g-2+n>0$,

$$\langle \tau_{1,0}\tau_{k_1,a_1}\dots\tau_{k_n,a_n}\rangle_{g,n+1} = (2g-2+n)\langle \tau_{k_1,a_1}\dots\tau_{k_n,a_n}\rangle_{g,n}.$$

2. the **divisor equation**: if D is a divisor in X and

$$R_a^b(D) = \eta^{bc}\int_D \gamma_a \cup \gamma_c,$$

then

$$\begin{aligned}\langle \tau_0(D)\tau_{k_1,a_1}\dots\tau_{k_n,a_n}\rangle_{g,n+1} &= (D\cap\beta)\langle \tau_{k_1,a_1}\dots\tau_{k_n,a_n}\rangle_{g,n}\\ &\quad + \sum_{i=1}^{n} R_{a_i}^b\langle \tau_{k_1,a_1}\dots\tau_{k_i-1,b}\dots\tau_{k_n,a_n}\rangle_{g,n}.\end{aligned}$$

Euler vector field

Introduce the vector field in the coordinates u^a (i.e. derivation of $\Lambda[[u]]$)

$$E = \sum_{a \in I} \big((1 - p_a)u^a + R_0^a\big)\partial_a.$$

Hori's equation in genus 0 says that

$$\mathcal{L}_0 u^a + E u^a = 0.$$

We may interpret this as saying that the formal morphism from the large phase space to the Frobenius manifold $\operatorname{Spec}(\Lambda[[u]])$ is equivariant, with respect to the vector field $\mathcal{L}_0$ upstairs, and the vector field $-E$ downstairs.

The Virasoro conjecture

In the next lecture, we will introduce a sequence L_n, $n \geq 0$, of differential operators on the large phases space which satisfy

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m, n \geq -1.$$

As in the case where X is a point, this commutation relation is relatively straightfoward to prove if $m+n$ is nonzero. But proving that the constant terms on both sides of the relation

$$[L_1, L_{-1}] = 2L_0$$

agree turns out to be quite subtle, depending on a formula of Libgober and Wood, which is itself a consequence of the Riemann-Roch-Hirzebruch theorem:

$$\sum_{a \in I} (-1)^{p_a+q_a}(p_a - r/2)^2 = \frac{1}{12}\int_X rc_r(X) + 2c_1(X)c_{r-1}(X).$$

The Virasoro conjecture states that $L_n Z = 0$ for $n > 0$. A version of this conjecture holds in genus 0 for all symplectic manifolds, but in higher genus, it has only been proved for certain classes of projective manifolds (and their Kähler deformations), such as homogenous spaces, del Pezzo varieties, and K3 surfaces.

The conjecture was stated for projective spaces by Eguchi, Hori and Xiong, based on their physics proof for the projective line. It was generalized to projective manifolds by Sheldon Katz.

It was proved for homogenous spaces by Givental. Later, another proof was given by Teleman.

In degree $\beta = 0$, the Virasoro conjecture has the following surprising consequence, derived by G. and Pandharipande. Let

$$\lambda_g = c_g(\mathbb{E})$$

be the Euler class of the Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$. If $k_1 + \cdots + k_n = 2g + n - 3$,

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \cup \lambda_g = (-1)^{g-1} \frac{2^{2g} - 2}{2^{2g}} \frac{B_{2g}}{(2g)!} \frac{(2g+n-3)!}{k_1! \dots k_n!}$$