Gromov–Witten Invariants and the Virasoro Conjecture

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Introduction

The theory known as pure topological gravity to physicists, and Gromov–Witten theory of a point to mathematicians, can be characterized in two different ways:

1. it solves the KdV hierarchy — this is a remarkable integrable system associated with isospectral deformation of the equation

$$-\frac{d^2}{dx^2} + \mathbf{u}(x),$$

   that is, a sequence of commuting nonlinear flows on the function $\mathbf{u}$;
2. it solves the Virasoro hierarchy, which is a sequence of linear (second-order) differential equations realizing the Lie algebra of vector fields on the line.

In the first lecture, we will discuss the relationship between these two sets of equations. In the following lecture(s), we will show how to generalize these ideas to Gromov–Witten theory with target a projective space (or more generally, certain Fano varieties).

Deligne-Mumford moduli spaces

For each genus $g \geq 0$ and number of marked points $n \geq 0$, the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed curves of genus g is a projective variety over $\mathbb{C}$ of dimension $3g - 3 + n$.

(It is empty if $2g - 2 + n \leq 0$. More precisely, $\overline{\mathcal{M}}_{g,n}$ is a smooth and proper Deligne–Mumford stack.)

Denote a pointed curve by $[C, z_1, \ldots, z_n]$. The tangent lines $T_{z_i}C$ assemble to form a line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$, with Chern class

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

The moduli space $\overline{\mathcal{M}}_{g,n}$ has a fundamental cycle

$$[\overline{\mathcal{M}}_{g,n}] \in H^{6g-6+2n}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Witten's conjecture concerns the intersection numbers

$$\langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} = [\overline{\mathcal{M}}_{g,n}] \cap \psi_1^{k_1} \dots \psi_n^{k_n} \in \mathbb{Q}.$$

For example, if $g > 0$,

$$\langle \tau_{3g-2} \rangle_{g,1} = \frac{1}{24^g g!}$$

and if $k_1 + \cdots + k_n = n - 3$,

$$\langle \tau_{k_1} \dots \tau_{k_n} \rangle_{0,n} = \frac{(n-3)!}{k_1! \dots k_n!}.$$

Witten introduced the sequence of generating functions in an infinite number of variables

$$F_g = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \dots k_n} \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} t_{k_1} \dots t_{k_n}.$$

Let $\partial_k = \partial/\partial t_k$. Let

$$F = \sum_{g=0}^{\infty} \hbar^g F_g.$$

Define $\mathbf{u} = \partial_0^2 F = t_0 + O(t^2 + \hbar)$.

Denote by $\mathbb{Q}\{\mathbf{u}\}$ the ring of differential polynomials

$$\mathbb{Q}[\hbar]\{\mathbf{u}\} = \mathbb{Q}[\hbar, \mathbf{u}, \mathbf{u}', \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(k)}, \dots].$$

We will think of $\mathbf{u}^{(k)}$ as the partial derivative $\partial_0^k \mathbf{u}$. Note that for the power series $\mathbf{u}$ associated to intersection numbers on $\overline{\mathcal{M}}_{g,n}$, we have

$$\mathbf{u}^{(k)} = \delta_{k1} + t_k + O(t^2 + \hbar).$$

Let ∂ be the derivation on $\mathbb{Q}[\hbar]\{\mathbf{u}\}$, defined on the generators by $\partial \mathbf{u}^{(k)} = \mathbf{u}^{(k+1)}$ and $\partial \hbar = 0$.

The Gelfand–Dikii polynomials

The KdV hierarchy is a series of equations for $\mathbf{u}$. Let D be the third-order linear differential operator

$$D = \frac{\hbar}{8}\partial^3 + \mathbf{u}\partial + \frac{1}{2}\mathbf{u}'.$$

The Gelfand–Dikii polynomials are defined by the recursion

$$DR_m = \left(m + \tfrac{1}{2}\right)\partial R_{m+1},$$

with initial condition $R_0(\mathbf{u}) = 1$, and such that of $R_k(0) = 0$ for $k > 0$. For example,

$$\begin{aligned} R_1 &= \mathbf{u}, \\ R_2 &= \tfrac{\hbar}{12}\mathbf{u}^{(2)} + \tfrac{1}{2}\mathbf{u}^2, \\ R_3 &= \tfrac{\hbar^2}{240}\mathbf{u}^{(4)} + \tfrac{\hbar}{12}\mathbf{u}\mathbf{u}^{(2)} + \tfrac{\hbar}{24}\left(\mathbf{u}'\right)^2 + \tfrac{1}{6}\mathbf{u}^3. \end{aligned}$$

This recursion has a unique solution for each $m \geq 0$, though this is not obvious.

The KdV conjecture

Recall the generating function for the intersection numbers on the moduli spaces $\overline{\mathcal{M}}_{g,n}$

$$\mathbf{u} = \partial_0^2 F = \sum_{g=0}^{\infty} \hbar^g \partial_0^2 F_g.$$

Witten's KdV conjecture states that for $k \geq 0$,

$$\partial_k \mathbf{u} = \partial R_{k+1}(\mathbf{u}).$$

The first of these equations says that

$$(\partial_0 - \partial)\mathbf{u} = 0,$$

allowing us to identify the variable x with the formal variable t_0. The second equation says that

$$\partial_1 \mathbf{u} = \tfrac{\hbar}{12}\mathbf{u}^{(3)} + \mathbf{u}\mathbf{u}'.$$

This is the KdV (Korteweg–de Vries) equation, with $t_0 = x$ (spatial variable) and $t_1 = t$ (time variable).

Extend the derivation ∂_k to $\mathbb{Q}[\hbar]\{\mathbf{u}\}$ by $\partial_k \mathbf{u}^{(j)} = \partial^{j+1} R_{k+1}(\mathbf{u})$ and $\partial_k \hbar = 0$. The Gelfand–Dikii polynomials satisfy integrability conditions

$$\partial_j R_{k+1} = \partial_k R_{j+1},$$

without which the KdV conjecture would not make sense.

The KdV conjecture was proved in 1992 by Kontsevich. Several other proofs have been found since then: a particularly beautiful proof is due to Mirzakhani.

The KdV conjecture in genus 0

In the theory of integrable systems, setting $\hbar = 0$ is known as the **dispersionless limit**. Denote the dispersionless limit of $\mathbf{u}$ by $u = \partial_0^2 F_0$. In this limit, the Gelfand–Dikii polynomials simplify to

$$R_k(u) = \tfrac{1}{k!} u^k,$$

and the KdV conjecture becomes

$$\partial_k u = \tfrac{1}{k!} u^k \partial_0 u.$$

Integrating this formula once with respect to t_0, it becomes

$$\partial_k \partial_0 F_0 = \tfrac{1}{k!} u^k.$$

The string equation

Suppose that $2g - 2 + n > 0$, so that $\overline{\mathcal{M}}_{g,n}$ is not empty. Knudsen and Mumford constructed a flat morphism

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

defined on irreducible stable curves by forgetting the last marked point

$$[C, z_1, \ldots, z_{n+1}] \mapsto [C, z_1, \ldots, z_n].$$

When C is reducible, forgetting the marked point z_{n+1} may yield a pointed curve that is no longer stable. In this case, one stabilizes $[C, z_1, \ldots, z_n]$ by contracting to a point any rational component which has continuous automorphisms. There are three cases: a rational component with two double points, or a rational component with a single double point and either zero or one marked points.

Knudsen and Mumford prove that the fibre of the resulting map is naturally isomorphic to C. By considering this fibration, Witten proved the string equation: if $2g - 2 + n > 0$, then

$$\langle \tau_0 \tau_{k_1} \ldots \tau_{k_n} \rangle_{g,n+1} = \sum_{i=1}^{n} \langle \tau_{k_1} \ldots \tau_{k_i - 1} \ldots \tau_{k_n} \rangle_{g,n}.$$

In terms of the generating functions F_g, this is equivalent to the series of equations

$$\partial_0 F_g - \sum_{k=0}^{\infty} t_{k+1} \partial_k F_g = \begin{cases} \frac{1}{2} t_0^2, & g = 0, \\ 0, & g > 0. \end{cases}$$

Let Z be the formal expression

$$Z = \exp(\hbar^{-1} F).$$

This is a generating function for intesection numbers on possibly disconnected stable pointed curves: if $\overline{\mathcal{M}}^*_{g,n}$ is the moduli space of possible disconnected stable curves of Euler characteristic $2 - 2g$ with n marked points (g may be positive or negative), and

$$\langle \tau_{k_1} \dots \tau_{k_n} \rangle^*_{g,n} = [\overline{\mathcal{M}}^*_{g,n}] \cap \psi_1^{k_1} \dots \psi_n^{k_n} \in \mathbb{Q},$$

we have

$$Z = \sum_{g=-\infty}^{\infty} \hbar^{g-1} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \dots k_n} \langle \tau_{k_1} \dots \tau_{k_n} \rangle^*_{g,n} t_{k_1} \dots t_{k_n}.$$

The string equation may be written $L_{-1} Z = 0$, where L_{-1} is the differential operator

$$-\partial_0 + \sum_{k=0}^{\infty} t_{k+1} \partial_k + \tfrac{1}{2\hbar} t_0^2.$$

Applying the differential operator ∂_0^2 to the string equation, we obtain the string equation for $\mathbf{u}$:

$$\partial_0 \mathbf{u} = 1 + \sum_{k=0}^{\infty} t_{k+1} \partial_k \mathbf{u}.$$

It is easily seen that the KdV conjecture together with the string equation uniquely characterize $\mathbf{u}$. Kontsevich proves that a solution exists using matrix integrals.

Another proof of the existence of a solution was given by Kac and Schwarz, using the relationship between solutions of the KdV hierarchy and representation theory of loop groups.

Let $\mathcal{V}_N$ be the space of $N \times N$ Hermitian matrices, and given a positive-definite Hermitian matrix Λ, let $d\mu_\Lambda$ be the probability measure on $\mathcal{V}_N$ with density

$$d\mu_\Lambda = \frac{1}{c_\Lambda} \exp\left(-\tfrac{1}{2} \operatorname{Tr}(\Lambda M^2)\right).$$

Kontsevich shows that the matrix integral

$$Z_N(\Lambda) = \int_{\mathcal{V}_N} \exp\left(\tfrac{i}{6} \operatorname{Tr}(M^3)\right) d\mu_\Lambda$$

depends on Λ only through the variables

$$t_k = -(2k-1)!! \operatorname{Tr}\left(\Lambda^{-2k-1}\right), \quad k < N/2,$$

and that

$$\lim_{N\to\infty} Z_N(t_m) = Z.$$

Using this representation, he shows that Z satisfies the KdV hierarchy, as well as the string equation.

The Virasoro conjecture

Dijkgraaf, Verlinde and Verlinde found an alternative formulation of the KdV conjecture. Witten used the Knudsen–Mumford fibration $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ to prove the **dilaton** equation: if $2g - 2 + n > 0$,

$$\langle \tau_1 \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n+1} = (2g - 2 + n) \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n}.$$

Multiplying by $\frac{3}{2}$ and using the formula for the dimension of $\overline{\mathcal{M}}_{g,n}$, we see that

$$\tfrac{3}{2} \langle \tau_1 \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n+1} = \left(k_1 + \dots + k_n + \tfrac{1}{2} n \right) \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n}.$$

We may rewrite this as a differential equation $L_0 Z = 0$, where

$$L_0 = -\tfrac{3}{2} \partial_1 + \sum_{k=0}^{\infty} \left(k + \tfrac{1}{2} \right) t_k \partial_k + \tfrac{1}{16}.$$

(The constant reflects the intersection number $\langle \tau_1 \rangle_{1,1} = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$.)

For $n > 0$, let L_n be the differential operator

$$L_n = -\frac{\Gamma(n+\frac{5}{2})}{\Gamma(\frac{3}{2})}\partial_{n+1} + \sum_{k=0}^{\infty} \frac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} t_k \partial_{k+n}$$
$$+ \frac{\hbar}{2} \sum_{k=-n}^{-1} (-1)^k \frac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} \partial_{-k-1}\partial_{k+n}.$$

This formula is easier to understand in terms of the **free field**

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})} z^{k+\frac{1}{2}} t_k - \hbar \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} z^{-k-\frac{1}{2}} \partial_k - \tfrac{4}{3} z^{3/2}.$$

For $n \geq -1$, we have

$$[L_n, \phi(z)] = -z^{n+1} \frac{\partial \phi(z)}{\partial z}.$$

It follows that $[L_m, L_n] - (m-n)L_{m+n}$ is a constant. This constant is obviously equal to 0 unless $m + n = 0$, in which case a more careful calculation shows that it is zero even in this case (that is, $[L_1, L_{-1}] = 2L_0$).

The Virasoro conjecture states that $L_n Z = 0$ for $n \geq -1$. Since the Lie algebra spanned by the operators $\{L_n\}_{n\geq -1}$ is generated by L_2 and L_{-1}, and we already know that $L_{-1}Z = 0$ by the string equation, the Virasoro conjecture could also be written simply as $L_2 Z = 0$. This trick is of no use in understanding or proving the conjecture, so we will not mention it again.

Like the KdV conjecture, the Virasoro conjecture completely determines the generating function Z. We may rewrite the Virasoro conjecture in terms of the functions F_g as follows: for $n \geq 0$ and $g \geq 0$,

$$\begin{aligned} \tfrac{\Gamma(n+\frac{5}{2})}{\Gamma(\frac{3}{2})} \partial_{n+1} F_g &= \sum_{k=0}^{\infty} \tfrac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} t_k \partial_{k+n} F_g \\ &\quad + \frac{1}{2} \sum_{k=-n}^{-1} (-1)^k \tfrac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} \partial_{-k-1} \partial_{k+n} F_{g-1} \\ &\quad + \frac{1}{2} \sum_{h=0}^{g-1} \sum_{k=-n}^{-1} (-1)^k \tfrac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} \partial_{-k-1} F_h \partial_{k+n} F_{g-h-1} + \delta_{n,0} \tfrac{1}{16}. \end{aligned}$$

Theorem

If $Z = \exp(\hbar^{-1}F)$ satisfies the string equation $L_{-1}Z = 0$, then the KdV and Virasoro conjectures for Z are equivalent.

The proof is obtained by combining two lemmas.

Lemma

If $L_{-1}f = \partial f = 0$, then f is a constant.

Lemma

Let $z_k = Z^{-1}L_kZ$. Then if the KdV conjecture holds, we have

$$D\partial z_k = \partial^2 z_{k+1}.$$