Spectra and stable homotopy theory

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The basic idea is to start with the category of based topological spaces, and invert the suspension functor (denoted by Σ). Recall that for a based topological space X,

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Thus we have to formally introduce objects which are "$\Sigma^{-n}X$" and figure out the correct notion of homotopy classes of maps once we have introduced these new objects.

Spanier-Whitehead category

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The Spanier-Whitehead category $\mathcal{SW}$ is defined as :
Objects of $\mathcal{SW}$ = based topological spaces.
$Map_{\mathcal{SW}}(X, Y) = \{X, Y\}$.

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which is a direct limit of abelian groups (for $n \geq 2$).

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$\pi_0^s \cong \mathbb{Z}$ (degree), $\pi_1^s \cong \pi_4 S^3 \cong \mathbb{Z}/2$ (generated by the suspension of the Hopf map).

Definition of spectra

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The spectrum $\Sigma^{-n} X$ is defined as

$$(\Sigma^{-n}X)_m = \begin{cases} * & \text{if } m \leq n \\ \Sigma^{m-n}X & \text{if } m > n. \end{cases}$$

Homotopy classes of maps between spectra

A map between spectra X and Y is defined as a sequence of maps f_m from X_m to Y_m defined for $m \geq n$ that commutes with the structure maps. A homotopy between two such maps is a coherent homotopy that starts from a sufficiently high level. This defines $\{X, Y\}$ for spectra X and Y.

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The homotopy groups of spectra are defined as

$$\pi_k X = \{S^k, X\} = \varinjlim_n \pi_{n+k} X_n.$$

This is evidently defined even when $k < 0$.

Examples

For an abelian group A, the Eilenberg MacLane spaces $K(A, n)$ satisfy $\Omega K(A, n) \simeq K(A, n-1)$. Thus the spectrum HA defined by $HA_n = K(A, n)$ is an Ω-spectrum. This is called the Eilenberg MacLane spectrum associated to A.

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From Bott periodicity theorem, we have

$$\Omega^2 \mathbb{Z} \times BU \simeq \mathbb{Z} \times BU,$$

so that

$$KU_{2n} = \mathbb{Z} \times BU, \ KU_{2n+1} = \Omega \mathbb{Z} \times BU,$$

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Analogously using the Bott periodicity theorem for orthogonal groups : $\Omega^8\mathbb{Z} \times BO \simeq \mathbb{Z} \times BO$, one obtains an Ω-spectrum KO, called the real K-theory spectrum.

Spectra arising from generalized cohomology theories

A generalized cohomology theory is a sequence of contravariant functors E^n, $n \in \mathbb{Z}$ defined on CW complexes which satisfy the long exact sequence, excision, and wedge axioms. The **Brown Representability Theorem** implies that there are spaces E_n such that $\tilde{E}^n(X) \cong [X, E_n]$.

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Conversely, a spectrum E induces a cohomology theory by the assignment $\tilde{E}^n : X \mapsto \{\Sigma^n X, E\}$ for $n \in \mathbb{Z}$.

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The Thom spectrum of a vector bundle ξ is defined as $\Sigma^\infty X^\xi$. If ξ is a virtual bundle and X is a finite CW complex, then $\xi \simeq \eta - n$ where η is a vector bundle, and then the Thom spectrum is $\simeq \Sigma^{-n} X^\eta$.

Thom spectra II

We note that $BO = \varinjlim BO(n)$ classifies virtual bundles of dimension 0.

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For a map $f : X \to BO$ we define $X(n)$ as the pullback of X over $BO(n)$, and $f_n : X(n) \to BO(n)$ as the associated map. Then f_n defines an n-dimensional vector bundle over $X(n)$, and we define $X_n^f = X(n)^{f_n}$.

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One uses the fact that the universal n-plane bundle γ_n over $BO(n)$ pulls back under the usual map $BO(n-1) \to BO(n)$ to the bundle $\gamma_{n-1} \oplus \epsilon$. This allows us to construct structure maps $\Sigma X_n^f \to X_{n+1}^f$. Hence we obtain a spectrum X^f, which is called the Thom spectrum associated to the map f.

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Thom spectra and cobordism

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One may define the Thom spectra associated to maps $BSO \to BO$ (denoted MSO), $BU \to BO$ (denoted MU), $BSU \to BO$ (denoted MSU) , and so on. All of these carry identifications with certain cobordism classes as above. For details see Stong : Notes on Cobordism theory.

Smash products

The definition of smash products of spectra is quite complicated. There are two good models that people work with

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We recall its salient features. There is a smash product of spectra which is associative : $(X \wedge Y) \wedge Z \simeq X \wedge (Y \wedge Z)$, and commutative : $X \wedge Y \simeq Y \wedge X$. The smash product with suspension spectra is equivalent to the space level smash product with the space itself.

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For a space X, $\pi_n(E \wedge X) \cong \tilde{E}_n(X)$, the homology groups for the cohomology theory associated to E.

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Suppose M is a compact closed manifold and TM is its tangent bundle. Then,
Theorem (Atiyah duality) $DM \simeq M^{-TM}$.

THANK YOU