Operads and Ring structures on spectra

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If X is a loop space, that is, $X \simeq \Omega Y$ for some space Y, then it is a homotopy associative H-space but also it satisfies certain higher coherence rules for associativity.

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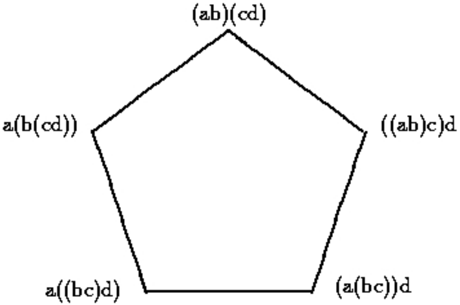

The 4-fold rule involves extending the diagram above to the interior of the pentagon in the space $Map(X^4, X)$.

Theorem (Stasheff) There are spaces K_n, called associahedra, homeomorphic to $\mathcal{D}^{n-2}$, such that the n-fold rule is equivalent to an extension of a map $\partial K_n \to Map(X^n, X)$ to all of K_n. If a connected space satisfies all the coherence rules then it is weakly equivalent to ΩY for some space Y.

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The spaces K_n fit together to form an operad.

Operads

An operad $\mathcal{O}$ is a sequence of Σ_n-spaces $\mathcal{O}(n)$ together with structure maps

$$\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

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We denote the structure maps by $\gamma_k(y; x_1, \cdots, x_k)$ for $y \in \mathcal{O}(k)$ and $x_i \in \mathcal{O}(n_i)$. The associativity condition may be written as

$$\gamma_k(y; \gamma_{s_1}(y_1; x_{1,1}, \cdots, x_{m_1,1}), \cdots, \gamma_{s_k}(y_k; x_{1,k}, \cdots, x_{m_k,k}))$$

$$= \gamma_{\sum s_i}(\gamma_k(y; y_1, \cdots, y_k); x_{1,1}, \cdots, x_{m_k,k}).$$

Example

For a topological space X, $\mathcal{E}_X(n) = Map(X^n, X)$ satisfies the above. For, having maps from X^{n_i} to X gives a map $X^{\sum n_i} \to X^k$ which on composition with a map from X^k to X gives a map $X^{\sum n_i} \to X$.

Action of operads

An operad $\mathcal{O}$ is said to act on a space X if we have a map of operads from $\mathcal{O} \to \mathcal{E}_X$. Equivalently, we must have maps

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Operads acting on objects may be defined in any symmetric monoidal category. In this talk we restrict ourselves to topological spaces.

Little n-cubes operad

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An operad whose n^{th}-space is Σ_n-equivariantly homotopic to Σ_n is called an A_∞-operad. An example of an A_∞-operad is $\mathcal{C}_1$.

Recognition principle

The space $\Omega^n Y \simeq Map_*(S^n, Y)$ is a $\mathcal{C}_n$-space. Given k based maps $f_1, \cdots, f_k$ from S^n to Y and an element $(c_1, \cdots, c_k)$ of $\mathcal{C}_n(k)$, we obtain a based map f from S^n to Y so that f maps c_i using the map f_i and sends the rest to the basepoint.

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Theorem (May) If X is a connected $\mathcal{C}_n$-space, then $X \simeq \Omega^n Y$ for some space Y.

For $n = \infty$, we have that X is an infinite loop space. That is, for each n there is a space Y_n such that $X \simeq \Omega^n Y$.

For an abelian group A, the Eilenberg MacLane spaces $K(A, n)$ satisfy $\Omega K(A, n) \simeq K(A, n-1)$. So each $K(A, n)$ is an infinite loop space. In fact, the spaces in an Ω-spectrum are all infinite loop spaces.

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A spectrum X is said to be connective if it has homotopy groups in only non-negative dimensions. The functors Σ^∞ and Ω^∞ induce an equivalence between connective spectra and (grouplike) E_∞-spaces.

The definition of ring spectra was classically a difficult problem. Much of this is due to the fact that the definition of an associative and commutative smash product of spectra is complicated. Intuitively, a ring spectrum R should be equipped with maps

$$\mu_{n,m} : R_m \wedge R_n \to R_{m+n}$$

which commutes with appropriate structure maps and satisfies an appropriate associativity condition. R is said to be commutative if in addition it satisfies a commutativity condition.

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Suppose $\mathcal{O}$ is an operad. An $\mathcal{O}$-ring spectrum R is one which is equipped with maps

$$\mathcal{O}(k)_+ \wedge R_{n_1} \wedge \cdots \wedge R_{n_k} \to R_{n_1+\cdots+n_k}$$

with appropriate coherence conditions.

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2. Thom spectra of infinite loop maps. Examples : MO, MU, etc.
3. Spectra arising from bipermutative categories. Examples : KO, KU, algebraic K-theory spectrum of a commutative ring.

Ring structures on Thom spectra

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Theorem (Lewis, May, Steinberger) Suppose $f : X \to BO$ is a map of $\mathcal{O}$-spaces. Then, the Thom spectrum X^f is an $\mathcal{O}$-ring spectrum.

The maps $BU \to BO$, $BSO \to BO$, etc are all infinite loop maps. Hence, they may be refined as maps of $\mathcal{E}$-spaces. One deduces as a consequence that the cobordism spectra such as MO, MU, MSO are all E_∞-ring spectra.

THANK YOU