Stochastic Quantisation and AdS/CFT

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Outline

Motivation

Recent progress in holographic Wilsonian renormalisation group (HWRG) has emphasised two aspects:

1. Flow equations have Hamilton-Jacobi form,
2. Complete description of HWRG flow necessarily requires multi-trace operators. Heemskerk-Polchinski, Nickel-Son, Faulkner-Liu-Rangamani

Stochastic Quantisation

- There has been a proposal of describing AdS/CFT in terms of Stochastic Quantisation (SQ) of a theory in one lower dimension. Akhmedov, Mansi-Mauri-Petkou
- It involves Hamilton-Jacobi set up with Fokker-Planck Hamiltonian.

Motivation Contd. . .

Questions

- Is it possible to relate SQ to HWRG in AdS/CFT correspondence?
- What is the relation between Fokker-Planck Hamiltonian and the HWRG Hamiltonian?
- What is the relation between the stochastic time and the radial direction in AdS?
- Is it possible to accommodate multi-trace operators in the SQ formulation?
- If yes, is it consistent with the SQ-HWRG dictionary?

Relating SQ to HWRG

The Proposal

- (The stochastic time) $t = r$ (the radial coordinate in AdS)
- (Classical action in SQ) $S_c \equiv 2\Gamma(\phi)$ (Classical effective action in AdS/CFT) Mansi-Mauri-Petkou
- (The Fokker-Planck Hamiltonian) $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$ (HWRG Hamiltonian in AdS/CFT).

Check

- We check our proposal by studying two examples, massless scalar field theory in AdS_2 and $U(1)$ gauge theory on AdS_4.
- In both cases, description of double trace operator deformation agrees between SQ and HWRG.

Relating SQ to HWRG Contd...

Relation

1. The boundary action S_B in AdS/CFT is given by

$$S_B = \int_{t_0}^{t} d\tilde{t} \int d^d x \, \mathcal{L}_{FP}(\phi(\tilde{t}, x)), \tag{1}$$

   $\mathcal{L}_{FP}$ is the Fokker-Planck Lagrangian density

2. The Langevin dynamics gives the relation between stochastic 2-point correlation functions and the double trace coupling in AdS/CFT

$$< \phi_q(t)\phi_{q'}(t) >_H^{-1} = < \phi_q(t)\phi_{q'}(t) >_S^{-1} - \frac{1}{2} \frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{q'}(t)}, \tag{2}$$

Lightning Review

- Consider a bulk action in the Euclidean AdS$_{d+1}$

$$S = \int_{r>\epsilon} dr d^d x \sqrt{g} \mathcal{L}(\phi, \partial\phi) + S_B[\phi, \epsilon], \tag{3}$$

  S_B is the boundary effective action and ϵ is radial cutoff.
- The AdS metric is

$$ds^2 = \frac{dr^2 + \sum_{i=1}^{d} dx_i dx_i}{r^2}. \tag{4}$$

- Canonical momentum is defined with boundary condition

$$\Pi_\phi = \sqrt{g} \frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} = \frac{\delta S_B}{\delta \phi(x)}. \tag{5}$$

Lightning Review

- Cutoff independence of the action implies

$$
\begin{aligned}
\partial_\epsilon S_B &= -\int_{r=\epsilon} d^d x \left(\frac{\delta S_B}{\delta \phi} \partial_r \phi - \mathcal{L}(\phi, \partial\phi) \right) \\
&= \int_{r=\epsilon} d^d x \, \mathcal{H}_{RG}\left(\frac{\delta S_B}{\delta \phi}, \phi\right),
\end{aligned}
\tag{6}
$$

  second equality follows from the Legendre transform.

- With a wavefunctional $\psi_H = \exp(-S_B)$, we can write

$$
\partial_\epsilon \psi_H = -\int_{r=\epsilon} d^d x \, \mathcal{H}_{RG}\left(-\frac{\delta}{\delta \phi}, \phi\right) \psi_H, \tag{7}
$$

  where we have assumed $\left(\frac{\delta S_B}{\delta \phi}\right)^2 >> \frac{\delta^2 S_B}{\delta \phi^2}$.

Basics

- Stochastic quantisation is a Hamiltonian description of Euclidean field theory evolving along fictitious stochastic time t. Damgaard-Huffel, Dijkgraaf-Orlando-Reffert
- The probability distribution $P(\phi, t)$ describes evolution of the system and reduces to Boltzmann measure at late time.
- N-point correlator is given by

$$< \phi(x_1)...\phi(x_N) >= \int D\phi \; P(\phi, t) \; \phi(x_1)...\phi(x_N). \tag{8}$$

Basics

The Langevin equation

$P(\phi, t)$ is determined using Langevin equation

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2}\frac{\delta S_c}{\delta \phi(x,t)} + \eta(x,t), \tag{9}$$

where $\eta(t)$ is Gaussian white noise, with property

$$\begin{aligned} < \eta_{i,q}(t) > &= 0 \\ < \eta_{i,q}(t)\eta_{j,q'}(t') > &= \delta_{ij}\delta^d(q-q')\delta(t-t'), \end{aligned} \tag{10}$$

with Gaussian weight

$$Z = \int D\eta(x,t) \exp(-\frac{1}{2}\int d^dxdt\, \eta^2(x,t)). \tag{11}$$

Langevin to Fokker-Planck

- The Fokker-Planck description is obtained by eliminating $\eta(x,t)$ in

$$Z = \int D\eta(x,t) \exp\left(-\frac{1}{2}\int d^d x dt\, \eta^2(x,t)\right), \qquad (12)$$

  using the Langevin equation.
- This gives

$$P(\phi,t) = \exp\left[-\frac{S_c(\phi(t))}{2} - \int_{t_0}^{t} d\tilde{t} \int d^d x \mathcal{L}_{FP}(\phi(\tilde{t},x))\right], \qquad (13)$$

  where the Fokker-Planck Lagrangian density is

$$\mathcal{L}_{FP} = \frac{1}{2}\left(\frac{\partial\phi(x)}{\partial t}\right)^2 + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)}. \qquad (14)$$

Probability distribution

- Equation satisfied by $P(\phi, t)$ can be put in a suggestive form by defining

$$\psi_S(\phi, t) \equiv P(\phi, t) \exp(\frac{S_c}{2}), \tag{15}$$

and then equation for $P(\phi, t)$ becomes the Schrödinger type equation for $\psi_S(\phi, t)$

$$\partial_t \psi_S(\phi, t) = -\int d^d x\, \mathcal{H}_{FP}(\frac{\delta}{\delta\phi}, \phi)\, \psi_S(\phi, t), \tag{16}$$

where,

$$\mathcal{H}_{FP} = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(x)} + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)}, \tag{17}$$

where $\mathcal{H}_{FP}$ is the Fokker-Planck Hamiltonian.

The Dictionary

SQ ⇔ HWRG

- We can now state the relation between SQ and HWRG.
  1. $t = r$
  2. $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$
  3. $\psi_S(\phi, t) \equiv P(\phi, t)\exp(\frac{S_c}{2}) = \psi_H(\phi, t) \equiv \exp(\frac{-S_B}{2})$

Implied relation

The second equality also implies a relation between the classical action S_c in SQ and the effective action Γ in AdS/CFT, namely

$$S_c = 2\,\Gamma \tag{18}$$

The Dictionary

2-point functions in SQ

- Langevin dynamics gives

$$< \phi_{q_1}(t_1)\phi_{q_2}(t_2) >_S = \int D\phi e^{-S_P(t)} \phi_{q_1}(t_1)\phi_{q_2}(t_2), \qquad (19)$$

- $P(\phi, t) \equiv \exp(-S_P(t)) = \exp(-\frac{1}{2}\int \mathcal{K}_q(t)\phi_q(t)\phi_{-q}(t)d^d q)$
  Latter expression is true only for a free theory.
- In a free theory

$$< \phi_{q_1}(t_1)\phi_{q_2}(t_2) >_S = \frac{1}{\mathcal{K}_q(t)}\delta^d(q_1 + q_2). \qquad (20)$$

The Dictionary

2-point functions in AdS/CFT

- According to relation **3**, $S_B = S_P - \frac{S_c}{2}$ and

$$< \phi_q(r)\phi_{q'}(r) >_H^{-1} = \frac{\delta^2 S_B}{\delta\phi_q(r)\delta\phi_{q'}(r)}. \tag{21}$$

- We thus have a relation

$$< \phi_{q_1}(t)\phi_{q_2}(t) >_H^{-1} = < \phi_{q_1}(t)\phi_{q_2}(t) >_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{-q}(t)}. \tag{22}$$

AdS/CFT

Scalar field action

- Action for massless scalar field in Euclidean AdS$_2$

$$S_{bulk} = \frac{1}{2}\int drd\tau\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{23}$$

  where $(g_{\tau\tau}, g_{rr}) = (r^{-2}, r^{-2})$.
- Notice,
  1. The action is invariant under Weyl rescaling of metric,
  2. AdS space is conformally flat.
- Therefore scalar field action can also be written as

$$S_{bulk} = \frac{1}{2}\int_{\mathbb{R}^2_+} drd\tau\partial_\mu\phi\partial_\mu\phi. \tag{24}$$

AdS/CFT

Double trace terms

- $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$ implies massless scalar field admits alternate quantisation in AdS_2.
- Assume the form of S_B

$$S_B = \Lambda(\epsilon) + \int \frac{d\omega}{2\pi} \sqrt{\gamma(\epsilon)} \mathcal{J}(\epsilon,\omega)\phi_{-\omega} - \frac{1}{2}\int \frac{d\omega}{2\pi} \sqrt{\gamma(\epsilon)} \mathcal{F}(\epsilon,\omega)\phi_{\omega}\phi_{-\omega}, \qquad (25)$$

where $\Lambda(\epsilon)$, $\mathcal{J}(\epsilon,\omega)$ and $\mathcal{F}(\epsilon,\omega)$ are unknown functions of radial cut-off ϵ.

- $\mathcal{F}(\epsilon,\omega)$ is the double trace coupling.

HWRG

- Holographic Hamilton-Jacobi equation is, Faulkner et al.

$$\partial_\epsilon S_B = -\frac{1}{2}\int_{r=\epsilon} d\omega \left(\left(\frac{\delta S_B}{\delta \phi_\omega}\right)\left(\frac{\delta S_B}{\delta \phi_{-\omega}}\right) - \omega^2 \phi_\omega \phi_{-\omega} \right). \tag{26}$$

- Substituting S_B into the holographic H-J equation

$$\partial_\epsilon \Lambda(\epsilon) = -\frac{1}{2}\int_\epsilon \frac{d\omega}{(2\pi)^2} J(\epsilon,\omega) J(\epsilon,-\omega), \tag{27}$$

$$\partial_\epsilon J(\epsilon,-\omega) = \frac{1}{2\pi} J(\epsilon,\omega) f(\epsilon,-\omega), \tag{28}$$

$$\partial_\epsilon f(\epsilon,\omega) = \frac{1}{2\pi} f(\epsilon,-\omega) f(\epsilon,\omega) - 2\pi\omega^2, \tag{29}$$

where, $J(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{J}(\epsilon,\omega)$ and $f(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{F}(\epsilon,\omega)$

HWRG

- Solution to these equation is Faulkner et al.

$$\begin{aligned} f(\epsilon,\omega) &= -2\pi\frac{\Pi_\omega(\epsilon)}{\phi_{-\omega}(\epsilon)}, \quad J(\epsilon,\omega) = -\frac{\beta_\omega}{\phi_\omega(\epsilon)}, \quad (30) \\ \text{and} \quad \partial_\epsilon\Lambda(\epsilon) &= -\frac{1}{2}\int_{r=\epsilon}\frac{d\omega}{(2\pi)^2}\frac{\beta_\omega\beta_{-\omega}}{\phi_\omega(\epsilon)\phi_{-\omega}(\epsilon)}, \end{aligned}$$

  where Π_ω is momentum conjugate to ϕ_ω and β_ω is independent of ϵ.
- General solution to ϕ equation of motion is

$$\phi_\omega(r) = \phi_\omega^{(0)}\cosh(|\omega|r) + \frac{\phi_\omega^{(1)}}{|\omega|}\sinh(|\omega|r). \qquad (31)$$

HWRG

- Consider only double trace coupling term, then

$$S_B(r) = \frac{1}{2}\int d\omega|\omega|\left(\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega \cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega \sinh(|\omega|r)}\right)\phi_\omega\phi_{-\omega}, \tag{32}$$

$$\mathcal{F}(r,\omega) = -2\pi|\omega|r\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega \cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega \sinh(|\omega|r)}. \tag{33}$$

Flows

- As $r \to 0$ (UV), $\mathcal{F}(r,\omega) \to 0$ for $\tilde{\phi} = 0$ and $\mathcal{F}(r,\omega) \to -2\pi$ for $\tilde{\phi} = \infty$. (Two Fixed points in UV)
- As $r \to \infty$ (IR), $\mathcal{F}(r,\omega) \to -\infty$ unless $\tilde{\phi} = -1$, then $\mathcal{F}(r,\omega) \to \infty$. (Two different fixed points in IR)

Stochastic Quantisation

Fokker-Planck action

- Recall $S_c = 2\,\Gamma$, and Γ is Legendre transform of on-shell action,

$$S_{cl} = \int_{-\infty}^{\infty} d\omega |\omega| \phi_\omega \phi_{-\omega}. \tag{34}$$

- The Fokker-Planck Lagrangian density is

$$\mathcal{L}_{FP} = \frac{1}{2}\dot{\phi}_\omega \dot{\phi}_{-\omega} + \frac{1}{2}\omega^2 \phi_\omega \phi_{-\omega}, \tag{35}$$

where, dot indicates derivative w.r.to stochastic time t.
- General solution to eq. of motion derived from $\mathcal{L}_{FP}$ is

$$\phi_\omega(t) = a_{1,\omega} \cosh(|\omega| t) + a_{2,\omega} \sinh(|\omega| t). \tag{36}$$

Stochastic Quantisation

Fokker-Planck action

- Consider a boundary condition that at time t we want the solution $\phi_\omega(\tilde{t} = t) = \phi_\omega(t)$ then

$$\phi_\omega(\tilde{t}) = \phi_\omega(t) \frac{\cosh(|\omega|\tilde{t}) + a_\omega \sinh(|\omega|\tilde{t})}{\cosh(|\omega|t) + a_\omega \sinh(|\omega|t)}. \tag{37}$$

- Substituting this in the Fokker-Planck action gives

$$S_{FP} = \frac{1}{2} \int d\omega |\omega| \phi_\omega(t) \phi_{-\omega}(t) \left(\frac{\sinh(|\omega|t) + a_\omega \cosh(|\omega|t)}{\cosh(|\omega|t) + a_\omega \sinh(|\omega|t)} \right). \tag{38}$$

- This action is identical to $S_B(r)$, with t replaced by r.
- It therefore has same fixed point structure.

SQ $\Leftrightarrow$ HWRG

The Proposal

- (The stochastic time) $t = r$ (the radial coordinate in AdS)
- (Classical action in SQ) $S_c \equiv 2\Gamma(\phi)$ (Classical effective action in AdS/CFT)
- (The Fokker-Planck Hamiltonian) $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$ (HWRG Hamiltonian in AdS/CFT).

Limitations

- We have checked our proposal for Weyl invariant actions only.
- We have only studied free theories in the bulk. It would be interesting to extend it at least to interacting Weyl invariant theories.