Lectures on the Radon transform and the wave front set

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1 Definition and basic properties of the wave front set

It is an elementary fact that a compactly supported function or distribution is infinitely differentiable if and only if its Fourier transform decays as $\mathcal{O}(|\xi|^{-m})$ as $|\xi| \to \infty$ for every m. The following statement is an immediate consequence of this fact.

Proposition 1. *A distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ is equal to a C^∞ function in some neighborhood of the point $x^0 \in \mathbf{R}^n$ if and only if there exists a function $\psi \in C_c^\infty$ with $\psi(x^0) \neq 0$ such that*

$$|\widehat{\psi f}(\xi)| \leq C_m(1 + |\xi|)^{-m}, \quad m = 1, 2, \dots. \tag{1.1}$$

The singular support of a function or distribution f, denoted $\operatorname{sing\,supp} f$, is defined as the complement of the largest open set in which the function is C^∞. The proposition can therefore be expressed as follows: the singular support of f is equal to the complement of the set of x^0 for which there exists $\psi \in C_c^\infty$ with $\psi(x^0) \neq 0$ such that (1.1) holds. This observation makes it possible to introduce a more precise description of singularities by restricting the set of directions in which $\widehat{\psi f}(\xi)$ must decay fast.

Definition. The wave front set, $WF(f)$, of f is the complement of the set of $(x^0, \xi^0) \in T^*(\mathbf{R}^n) \setminus 0$ with the following property: there exists a function $\psi \in C_c^\infty$ with $\psi(x^0) \neq 0$, a conic neighborhood Γ of ξ^0, and constants C_m such that

$$|\widehat{\psi f}(\xi)| \leq C_m(1 + |\xi|)^{-m}, \quad m = 1, 2, \dots, \quad \xi \in \Gamma. \tag{1.2}$$

Example 1. Let $f(x_1, x_2) = 1$ for $x_2 > 0$ and $f(x_1, x_2) = 0$ for $x_2 < 0$. Then

$$WF(f) \subset \{(x_1, 0; 0, \xi_2);\ x_1 \in \mathbf{R}, \xi_2 \neq 0\}. \tag{1.3}$$

Later we shall see that there is in fact equality in (1.3).

Example 2. More generally, let $f(x_1, x_2) = h(x_2)$ for some function $h \in L^1_{\text{loc}}(\mathbf{R})$. Then

$$WF(f) \subset \{(x_1, x_2; 0, \xi_2);\ x_2 \in \operatorname{sing\,supp} h, \xi_2 \neq 0\}. \tag{1.4}$$

It is obvious that the wave front set is always conic in the ξ-variable, which means that $(x, \xi) \in WF(f)$ if and only if $(x, \lambda\xi) \in WF(f)$ for every $\lambda > 0$. It is also obvious that $(x, \xi) \notin WF(f)$ for all $\xi \neq 0$, if f is equal to a C^∞ function in some neighborhood of x. This is the same as saying that the projection of $WF(f)$ onto the first component is contained in the singular support, i.e.,

$$\pi_X(WF(f)) \subset \operatorname{sing\,supp}(f).$$

Here π_X denotes the projection $T^*(\mathbf{R}^n) \setminus 0 \ni (x, \xi) \mapsto x \in \mathbf{R}^n$. To prove the opposite inclusion we need the following basic lemma.

Lemma 1. *Let* Γ *be an open cone and assume* $\varphi \in C^\infty(\mathbf{R}^n)$*,* $f \in \mathcal{E}'(\mathbf{R}^n)$*, and*

$$|\widehat{f}(\xi)| \leq C_m(1+|\xi|)^{-m}, \quad m = 1, 2, \dots, \quad \xi \in \Gamma.$$

Let Γ_0 *be a cone whose conic closure is contained in* Γ*. Then*

$$|\widehat{\varphi f}(\xi)| \leq C'_m(1+|\xi|)^{-m}, \quad m = 1, 2, \dots, \quad \xi \in \Gamma_0.$$

Proof. We may assume $\varphi \in C_c^\infty$. By Fourier's inversion formula

$$\widehat{\varphi f}(\xi) = \frac{1}{(2\pi)^n} \int \widehat{f}(\xi - \eta)\widehat{\varphi}(\eta) d\eta = \int_{|\eta| > \varepsilon|\xi|} + \int_{|\eta| < \varepsilon|\xi|}.$$

Choose $\varepsilon > 0$ so small that

$$\xi \in \Gamma_0 \quad \text{and} \quad |\eta| < \varepsilon|\xi| \qquad \text{implies} \qquad \xi - \eta \in \Gamma.$$

Let us first assume that $f \in L^1_c(\mathbf{R}^n)$. Choose M so that $\sup|\widehat{f}| \leq M$ and $\sup|\widehat{\varphi}| \leq M$. Then, since f satisfies (1.2)

$$\Big|\int_{|\eta|<\varepsilon|\xi|} \dots \Big| \leq \sup|\widehat{\varphi}| \int_{|\eta|<\varepsilon|\xi|} C_m(1+|\xi-\eta|)^{-m} d\eta = \mathcal{O}(|\xi|^{-m+n}) \quad \text{as } |\xi| \to \infty \text{ with } \xi \in \Gamma_0$$

for all m. And since $\varphi \in C_c^\infty$

$$\Big|\int_{|\eta|>\varepsilon|\xi|} \dots \Big| \leq \sup|\widehat{f}| \int_{|\eta|>\varepsilon|\xi|} |\widehat{\varphi}(\eta)| d\eta = \mathcal{O}(|\xi|^{-m}) \quad \text{as } |\xi| \to \infty \text{ for all } m.$$

To consider the general case assume that f is a distribution of order r so that $|\widehat{f}(\xi)| \leq M(1+|\xi|)^r$. The term $\int_{|\eta|<\varepsilon|\xi|}$ can be estimated as before. And

$$\Big|\int_{|\eta|>\varepsilon|\xi|} \dots \Big| \leq M \int_{|\eta|>\varepsilon|\xi|} (1+|\xi-\eta|)^r C_p(1+|\eta|)^{-p} d\eta$$

for any p. If $\varepsilon \leq 1$ and $|\eta| > \varepsilon|\xi|$ we have $|\xi - \eta| \leq 2|\eta|/\varepsilon$, so

$$\Big|\int_{|\eta|>\varepsilon|\xi|} \dots \Big| \leq MC_p \int_{|\eta|>\varepsilon|\xi|} (1+2|\eta|/\varepsilon)^r(1+|\eta|)^{-p} d\eta,$$

which is $\mathcal{O}(|\xi|^{-m})$ as $|\xi| \to \infty$ if $p > r+n+m$. The proof is complete.

Proposition 2. *Let* $\psi \in C^\infty(\mathbf{R}^n)$ *and* $f \in \mathcal{D}'(\mathbf{R}^n)$*. Then*

$$WF(\psi f) \subset WF(f).$$

Proof. Assume $(x^0, \xi^0) \notin WF(f)$. By the assumption there exists $\varphi \in C_c^\infty$ with $\varphi(x^0) \neq 0$ and a conic neighbourhood Γ of ξ^0 such that (1.2) holds. Let Γ_0 be a conic neighbourhood of ξ^0 whose conic closure is contained in Γ. Applying Lemma 1 to ψ and φf proves the assertion.

Corollary 1. The singular support of f is *equal* to the projection of $WF(f)$ onto the first component, i.e.,

$$\pi_X(WF(f)) = \operatorname{sing\,supp}(f).$$

Proof. It remain to prove the inclusion $\supset$. Assume $x^0 \notin \pi_X(WF(f))$, i.e., for every $\xi^0 \neq 0$ we have $(x^0, \xi^0) \notin WF(f)$. This means that for every ξ^0 with $|\xi^0| = 1$ there exists $\psi \in C_c^\infty$ with $\psi(x^0) \neq 0$, a conic neighborhood Γ of ξ^0, and constants C_m such that (1.2) holds. Note that the function ψ may depend on ξ^0. By the Borel-Lebesgue Lemma there exists a fine subset of those cones that covers the unit sphere. Let ψ_ν, $\nu = 1, \ldots, p$, be the corresponding ψ functions and let φ be the product of all ψ_ν. Then $\varphi(x^0) \neq 0$ and it follows from Lemma 1 that $\widehat{\varphi f}(\xi)$ satisfies (1.1) in all of $\mathbf{R}^n$.

Corollary 2. Assume that for every $\varepsilon > 0$ there exists $\phi \in C^\infty$ with support in $B_\varepsilon(x^0) = \{x;\, |x - x^0| < \varepsilon\}$ such that $\widehat{\phi g}(\lambda\xi^0)$ does not decay as $\mathcal{O}(|\lambda|^{-m})$ as $\lambda \to \infty$ for every m. Then $(x^0, \xi^0) \in WF(g)$.

Proof. Assume the assertion were false, i.e. that $(x^0, \xi^0) \notin WF(g)$. This means that there exists $\psi \in C_c^\infty$ with $\psi(x^0) \neq 0$ such that $\widehat{\psi g}$ satisfies (1.2). Choose $\varepsilon > 0$ and δ such that $|\psi(x)| \geq \delta > 0$ for $x \in B_\varepsilon(x^0)$. According to the assumption we can take ϕ with support in $B_\varepsilon(x^0)$ such that $\widehat{\phi g}(\lambda\xi^0)$ does not decay as $\mathcal{O}(|\lambda|^{-m})$ as $\lambda \to \infty$ for every m. Applying Lemma 1 with $f = \psi g$ and $\varphi = \phi/\psi$ we obtain a contradiction, which completes the proof.

Remark. Essentially the same argument proves the following stronger statement.

Corollary 2′. Assume that for every $\varepsilon > 0$ and every conic neighborhood Γ of ξ^0 there exists $\phi \in C^\infty$ with support in $B_\varepsilon(x^0)$ such that $\widehat{\phi g}(\xi)$ does not decay as $\mathcal{O}(|\xi|^{-m})$ for every m as $|\xi| \to \infty$ with $\xi \in \Gamma$. Then $(x^0, \xi^0) \in WF(g)$.

Exercise. Show that equality holds in (1.3).

Exercise. Let $f(x_1, x_2) = 1$ if $x_1 > 0$ and $x_2 > 0$, $f(x_1, x_2) = 0$ for all other x. Prove that

$$\begin{aligned} WF(f) =& \{(x_1, 0; 0, \xi_2);\, x_1 > 0,\, \xi_2 \neq 0\} \cup \{(0, x_2; \xi_1, 0);\, x_2 > 0,\, \xi_1 \neq 0\} \\ & \cup \{(0, 0; \xi_1, \xi_2);\, (\xi_1, \xi_2) \neq (0, 0)\}. \end{aligned}$$

Exercise. Let $f \in \mathcal{D}'(\mathbf{R}^n)$ be a smooth density on the hypersurface $x_n = 0$, i.e., let $f(x) = f(x', x_n) = g(x')\delta_0(x_n)$ for some $g \in C^\infty(\mathbf{R}^{n-1})$. Prove that

$$WF(f) = \{(x', 0; 0, \xi_n);\, x' \in \operatorname{supp} g,\, \xi_n \neq 0\}.$$

There exist functions whose wave front set contains (x, ξ) but not $(x, -\xi)$. An example is the inverse Fourier transform of the Heaviside function $H(\xi)$. The latter is defined as the characteristic function for the positive half-axis. The distribution $\mathrm{vp}(1/x)$ is defined by

$$\langle \mathrm{vp}(1/x), \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x} dx \quad \text{for all } \varphi \in C_c^\infty(\mathbf{R}).$$

Here is one way to compute the inverse Fourier transform of $H(\xi)$. Observe that $u_\varepsilon(x) = e^{-\varepsilon x} H(x) \to H(x)$ as $\varepsilon \to 0$ and compute

$$\widehat{u_\varepsilon}(\xi) = \int_0^\infty e^{-\varepsilon x} e^{-ix\xi} dx = \frac{1}{\varepsilon + i\xi} = \frac{\varepsilon - i\xi}{\varepsilon^2 + \xi^2} \to \pi\delta_0 - i\ \mathrm{vp}\ \frac{1}{\xi} \quad \text{as} \quad \varepsilon \to 0.$$

Thus $\widehat{H}(\xi) = \pi\delta_0(\xi) - i \text{ vp}(1/\xi)$. Using Fourier's inversion formula $\widehat{\widehat{g}}(x) = 2\pi g(-x)$ we conclude that the Fourier transform of $\pi\delta_0(x) + i \text{ vp}(1/x)$ is equal to $2\pi H(\xi)$.

Exercise. Set $f(x) = \pi\delta_0(x) + i \text{ vp}(1/x)$. Prove that $(0, -1) \notin WF(f)$ and that $(0, 1) \in WF(f)$.

Hint. Use the argument of Lemma 1 to show that $(0, -1) \notin WF(f)$.

On the other hand, if f is real-valued, then $WF(f)$ is symmetric in ξ in the sense that $(x, \xi) \in WF(f)$ if and only if $(x, -\xi) \in WF(f)$. This is an immediate consequence of the fact that the Fourier transform of a real-valued function f is even, $\widehat{f}(\xi) = \widehat{f}(-\xi)$.

We next prove that the wave front set behaves as it should under coordinate transformations. Consider first affine transformations. Since the effect of a translation is trivial, it is enough to consider linear coordinate transformations. If A is a non-singular transformation and we set $\widetilde{f}(y) = f(Ay)$ we note that for $f \in L^1_c(\mathbf{R}^n)$

$$\begin{aligned} \widehat{\widetilde{f}}(\eta) &= \int e^{-iy\cdot\eta} f(Ay)dy = \int e^{-iA^{-1}x\cdot\eta} f(x)|\det A|^{-1}dx \\ &= |\det A|^{-1} \int e^{-ix\cdot A^{*-1}\eta} f(x)dx = |\det A|^{-1}\widehat{f}(A^{*-1}\eta) = |\det A|^{-1}\widehat{f}(\xi), \end{aligned} \tag{1.5}$$

if $\eta = A^*\xi$. Here we have used the fact that $A^{-1*} = A^{*-1}$. Applying (1.5) to ψf with $\psi(x^0) \neq 0$ and ψ supported near x^0, we see that

$$\begin{aligned} &(x^0, \xi^0) \in WF(f) \Longleftrightarrow (y^0, \eta^0) \in WF(\widetilde{f}), \\ &\text{if} \quad x^0 = Ay^0 \quad \text{and} \quad \eta^0 = A^*\xi^0. \end{aligned} \tag{1.6}$$

We shall need an extension of (1.6) to the case when A is a mapping from $\mathbf{R}^n$ onto $\mathbf{R}^m$ and $n \geq m$. Let N be the kernel of A. Then A can be factored $A = A_0 \circ \pi$, where π is the projection $\mathbf{R}^n \mapsto A/N$ and A_0 is non-singular. It is therefore sufficient to consider the case when A is a projection. Choosing suitable coordinates we can assume that A is the projection $x = (x', x'') \mapsto x'$. Then $\widetilde{f}(x) = (f \circ A)(x) = f(x')$. It is an easy exercise to verify that in this case

$$(x, \xi) = (x', x''; \xi', \xi'') \in WF(\widetilde{f})$$

if and only if $(x', \xi') \in WF(f)$ and $\xi'' = 0$. Since $A^*\xi' = (\xi', 0)$ this agrees with (1.6). Observing that (1.6) is valid for the composition of two operators $A = A_1A_2$ if it is valid for A_1 and A_2 we have proved (1.6) for the general case.

Example 3. The wave front set of the function $f(x) = |x_1 - x_2|$ is equal to

$$\{(x, \xi);\, x_1 = x_2,\ \xi_1 = -\xi_2,\ \xi_1 \neq 0\}.$$

More generally, the wave front set of the function $|x - y|^\alpha$, $\alpha \neq 0$, $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$, is equal to

$$\{(x, y; \xi, \eta);\, x = y,\, \xi = -\eta,\ \xi \neq 0\}.$$

It is important to avoid thinking of ξ as a vector in x-space. Instead, we can think of a direction in ξ-space as an (oriented) hyperplane through the origin in the tangent space to $\mathbf{R}^n$ at x^0, which can be identified with a hyperplane in $\mathbf{R}^n$ itself. Because a ray in the cotangent space $T^*_{x^0}(\mathbf{R}^n)$ is uniquely determined by its zero-set, which is a hyperplane through the origin in the tangent space $T_{x^0}(\mathbf{R}^n)$. If L is a hyperplane in $T_{y^0}(\mathbf{R}^n)$, then A transforms L into the hyperplane $\widetilde{L} \subset X$ consisting of all Av for $v \in L$. Let η^0 be conormal to L and choose ξ^0 so that $A^*\xi^0 = \eta^0$. Then ξ^0 is conormal to $\widetilde{L}$, because (write for a moment $\langle \xi, v\rangle$ instead of $\xi \cdot v$)

$$\langle \xi^0, Av\rangle = 0 \Longleftrightarrow \langle A^*\xi^0, v\rangle = \langle \eta^0, v\rangle = 0 \quad \text{for all } v \in L,$$

which agrees with (1.6).

Next, let Ψ be a diffeomorphism of a neighborhood of y^0 onto a neighborhood of $x^0 = \Psi(y^0)$ and set $\widetilde{f}(y) = f(\Psi(y))$. We claim that, in analogy with (1.6),

$$\begin{aligned} &(x^0, \xi^0) \in WF(f) \Longleftrightarrow (y^0, \eta^0) \in WF(\widetilde{f}), \\ &\text{if} \quad x^0 = \Psi(y^0) \quad \text{and} \quad \eta^0 = \Psi'(y^0)^*\xi^0. \end{aligned} \tag{1.7}$$

Here $\Psi'(y^0)$ denotes the linear transformation $\mathbf{R}^n \to \mathbf{R}^n$ that is given by the Jacobian matrix $\partial_{y_j}\Psi_i$ at the point y^0 and $\Psi'(y^0)^*$ denotes the adjoint of $\Psi'(y^0)$.

Sketch of proof. It is clear that we may assume that $x^0 = y^0 = 0$, and by (1.6) we may assume that $\Psi'(y^0)$ is equal to the identity matrix. Thus $\Psi(y) - y = \mathcal{O}(|y|^2)$ as $|y| \to 0$, and hence with $\Phi(x) = \Psi^{-1}(x)$

$$\Phi(x) = x + h(x),$$

where $h(x) = \mathcal{O}(|x|^2)$ as $|x| \to 0$. Instead of (1.5) we now have

$$\begin{aligned} \widehat{\widetilde{f}}(\eta) &= \int e^{-iy\cdot\eta} f(\Psi(y))dy = \int e^{-i(x+h(x))\cdot\eta} f(x)|\det \Phi'(x)|dx \\ &= \int e^{-ix\cdot\eta}\varphi_\eta(x) f(x)dx, \end{aligned} \tag{1.8}$$

where $\varphi_\eta(x) = e^{-ih(x)\cdot\eta}|\det\Phi'(x)|$. It is enough to prove that $\widehat{\widetilde{f}}(\eta)$ decays rapidly in a conic neighborhood of $\eta^0 = \xi^0$, if $\widehat{f}(\xi)$ decays rapidly in some conic neighborhood of ξ^0. To prove this we argue as in the proof of Lemma 1 with $\varphi(x)$ replaced by $\varphi_\eta(x)$, which depends on the parameter η. The only new element relative to the proof of Lemma 1 is that we need to prove that

$$\int_{|\theta|>\varepsilon|\eta|} |\widehat{\varphi_\eta}(\theta)|d\theta \le C_m(1+|\eta|)^{-m} \quad \text{for every } m. \tag{1.9}$$

We will sketch the proof of this fact in the next lemma.

Lemma 2. *Assume that $h(x)$ is a C^∞ function from $\mathbf{R}^n$ to $\mathbf{R}^n$ that satisfies $h(x) = \mathcal{O}(|x|^2)$ as $|x| \to 0$. Let $\psi(x)$ be a C^∞ function supported in the ball $|x| \le \delta$ and set*

$$\varphi_\eta(x) = e^{ih(x)\cdot\eta}\psi(x).$$

Then (1.9) *holds if δ is small enough.*

Sketch of proof. We have

$$\widehat{\varphi_\eta}(\theta) = \int_{\mathbf{R}^n} e^{-i(x\cdot\theta - h(x)\cdot\eta)}\psi(x)dx. \tag{1.10}$$

We are going to make the substitution $z = \gamma_{\theta,\eta}(x) = x - \theta\big(h(x)\cdot\eta\big)/|\theta|^2$ in the integral (1.10). Since $h(x) = \mathcal{O}(|x|^2)$ and $|\eta|/|\theta| \le 1/\varepsilon$ in the integrand (1.9), it is easily seen that $\gamma_{\theta,\eta}(x)$ is invertible in $|x| < \delta$ if δ is small enough. Denote its inverse by $x = \rho_{\theta,\eta}(z)$. It follows that

$$\widehat{\varphi_\eta}(\theta) = \int_{\mathbf{R}^n} e^{-iz\cdot\theta}\widetilde{\psi}(z)dz = \widehat{\widetilde{\psi}}(\theta)$$

with

$$\widetilde{\psi}(z) = \psi(\rho_{\theta,\eta}(z))|\det \rho'_{\theta,\eta}(z)|.$$

The estimate (1.10) now follows, after we have verified that all derivatives of $\widetilde{\psi}(z)$ are bounded uniformly with respect to the parameters θ and η in the region $|\eta| \le |\theta|/\varepsilon$. And this in turn follows from the fact that all derivatives of $\gamma_{\theta,\eta}(x)$ and $\rho_{\theta,\eta}$ are uniformly bounded for such θ and η. The proof is complete.

Example 5. Let f be the characteristic function for the unit disk in $\mathbf{R}^2$. Then

$$WF(f) = \{(x,\xi);\, |x| = 1,\, \xi = \lambda x, \lambda \ne 0\}.$$

Example 6. More generally, let $h \in L^1_{\rm loc}(\mathbf{R})$ (or $h \in \mathcal{D}'(\mathbf{R})$) and let $F \in C^\infty(\mathbf{R}^n)$ with $\nabla F = (\partial_{x_1}F,\ldots,\partial_{x_n}F) \ne (0,\ldots,0)$ for all x. Then $WF\big(h(F(x))\big) \subset \{(x,\xi);\, F(x) \in \operatorname{supp} h,\ \xi = \lambda\nabla F(x),\ \lambda \ne 0\}$. In other words, $WF\big(h(F(x))\big)$ is contained in the union of the conormal manifolds to all level surfaces $F(x) = c$ and $c \in \operatorname{supp} h$.

Finally we shall investigate the effect on the wave front set of integration over a family of submanifolds. Set $e_n = (0,\ldots,0,1)$ and assume $(x^0, \pm e_n) \notin WF(f)$. By the definition of the wave front set there exists $\psi \in C_c^\infty$ and a conic neighbourhood Γ of $\pm e_n$ such that $\psi(x^0) \ne 0$ and $|\widehat{\psi f}(\xi)| = \mathcal{O}(|\xi|^{-m})$ as $|\xi| \to \infty$ in Γ for every m. In particular

$$\widehat{\psi f}(0,\ldots,0,\xi_n) = \mathcal{O}(|\xi_n|^{-m}) \quad \text{as } |\xi_n| \to \infty \text{ for every } m. \tag{1.11}$$

But this implies that the function

$$x_n \mapsto \int_{\mathbf{R}^{n-1}} \psi(x', x_n)f(x', x_n)dx' \qquad \text{is } C^\infty, \tag{1.12}$$

because the left hand side of (1.11) is the 1-dimensional Fourier transform of the function (1.12). This simple observation together with a partition of unity proves the following. Assume $f \in L^1_c(\mathbf{R}^n)$ and that $(x, \pm e_n) \notin WF(f)$ for every x. Then the function

$$x_n \mapsto \int_{\mathbf{R}^{n-1}} f(x', x_n)dx' \qquad \text{is } C^\infty. \tag{1.13}$$

The same assertion holds for distributions $f \in \mathcal{E}'(\mathbf{R}^n)$ if the distribution (1.11) is defined as $\varphi \mapsto \langle f, \varphi\rangle$ for $\varphi(x_n)$ in $C_c^\infty(\mathbf{R})$. Here $\varphi \mapsto \langle f, \varphi\rangle$ should be understood as the action of the distribution $f \in \mathcal{E}'(\mathbf{R}^n)$ on the test function $\varphi(x_n)$ considered as a function of (x', x_n) that is independent of x'.

We can express this fact in terms of the Radon transform as follows.

Proposition 3. *Let $f \in \mathcal{E}'(\mathbf{R}^n)$ and assume $(x, \pm\omega) \notin WF(f)$ for every $x \in \operatorname{supp} f$. Then $p \mapsto Rf(\omega, p)$ is a C^∞ function.*

By a similar argument we can prove the analogous statement for integration over submanifolds with codimension greater than 1. Denote again the elements of $\mathbf{R}^n = \mathbf{R}^{p+q}$ by $x = (x', x'')$ and $\xi = (\xi', \xi'')$. Let f be compactly supported and assume $(x', x''; 0, \xi'') \notin WF(f)$ for every x and every $\xi'' \neq 0$. Then $F(x'') = \int_{\mathbf{R}^p} f(x', x'')dx'$ is in $C^\infty(\mathbf{R}^q)$. More generally,

$$\text{if } (x', x''; 0, \xi'') \notin WF(f) \text{ for all } x'', \text{ then } (x'', \xi'') \notin WF(F)\ . \tag{1.14}$$

By change of variables we get a similar statement for integration over families of general submanifolds. We state it only for the case of codimension 1. Let $\Phi(x)$ be a smooth real-valued function on a neighbourhood of $\operatorname{supp} f$ with gradient $\nabla\Phi(x) \neq 0$. Then $S_t = \{x;\, \Phi(x) = t\}$ is a 1-parametric family of smooth hypersurfaces. Let $d\mu_t$ be a smooth measure on S_t that depends smoothly on t (in the sense that $d\mu_t = a(x, t)ds$ where $a(x, t)$ is smooth and ds is surface measure on S_t).

Proposition 4. *Assume $N^*(S_t) \cap WF(f) = \emptyset$. Then*

$$t \mapsto \int_{S_t} f\, d\mu_t$$

is a C^∞ function.

Proof. Choose coordinates so that $\Phi(x) = x_n$. In those coordinates we have $d\mu_t = a(x', t)dx'$ for some smooth function $a(x', x_n)$. By (1.7) $WF(f)$ contains no element of the form $(x, \pm e_n)$. By Proposition 2 the same is true of $a(x)f(x)$. The assertion now follows from (1.13).

Exercise. Let f be the characteristic function for the first quadrant in $\mathbf{R}^2$. Use (1.14) to give a new proof of the fact that $(0, 0; \xi_1, \xi_2) \in WF(f)$ for every $\xi \neq (0, 0)$.

2 Estimates for $WF(Rf)$

Let $X = \mathbf{R}^2$ and let Y be the manifold of lines in $\mathbf{R}^2$. The Radon transform

$$Rf(L) = \int_L f\, ds, \quad L \in Y, \tag{2.1}$$

is a linear map from $C_c(X)$ into $C_c(Y)$. We shall study how singularities in f are mapped to singularities in Rf by the map R. In particular, given that f is smooth except for a singularity at x^0, we may ask where Rf can have singularities. I turns out that it is not much we can say. The singularity at x^0 may give rise to a singularity in Rf at any line passing through x^0. And conversely, a singularity in Rf at L^0 may be caused by a singularity in f at an arbitrary point x in L^0.

Therefore it may appear remarkable that we get a $1-1$ correspondence between singularities in f and singularities in Rf if we describe singularities in terms of the wave front set. And this correspondence is described by the wave front set of the Schwartz kernel of the operator R

To specify cotangent vectors above Y it is useful to have coordinates in Y. Let us for a moment consider the 2-dimensional case. Let $L(y) = L(y_1, y_2)$ be the line defined by the equation

$$x_2 = y_1 x_1 + y_2. \tag{2.2}$$

All lines except the vertical lines $x_1 = c$ can be represented in this way. Set

$$F(x, y) = -x_2 + y_1 x_1 + y_2.$$

If we replace the arclength measure ds by dx_1, which has no influence on singularities, we can now write the Radon transform (2.1)

$$Rf(y) = \int_{\mathbf{R}} f(x_1, y_1 x_1 + y_2) dx_1. \tag{2.3}$$

This expression can be written

$$Rf(y) = \int_{\mathbf{R}^2} K(y, x) f(x) dx = \int_{\mathbf{R}} \Big(\int_{\mathbf{R}} K(y, x) f(x) dx_2 \Big) dx_1, \tag{2.4}$$

where $K(y, x)$ is the smooth density on the hypersurface $F(x, y) = x_2 - y_1 x_1 - y_2 = 0$ defined by

$$K(y, x) = \delta_0(x_2 - y_1 x_1 - y_2) = \delta_0(F(x, y)).$$

We may regard the iterated integral in (2.4) as the definition of the double integral. Indeed, by the definition of the measure $\delta_0(x_2 - a)$ the inner integral will be evaluated to

$$f(x_1, y_1 x_1 + y_2),$$

and after performing the outer integral we obtain (2.3). The wave front set of $\delta_0(F(x, y))$ is equal (see Example 6) to the set of all conormals to the hypersurface $F(x, y) = 0$. The gradient of F can be written

$$\nabla_{(x,y)} F = (-y_1, 1; -x_1, -1).$$

Hence an arbitrary non-vanishing conormal at (x, y) to the hypersurface

$$Z = \{(x, y);\ F(x, y) = 0\}$$

can be written

$$(\xi, \eta) = \lambda(y_1, -1; x_1, 1)$$

for some $\lambda \neq 0$. Thus the conormal manifold to Z, which we denote by $N^*(Z)$, consists of all the elements

$$N^*(Z) = \{(x, y, \xi, \eta);\ F(x, y) = 0,\ \xi = \lambda(y_1, -1),\ \eta = \lambda(x_1, 1),\ \lambda \neq 0\}.$$

$N^*(Z)$ is a 4-dimensional submanifold of the 8-dimensional space $T^*(X \times Y)$. It is easy to see that there is a natural isomorphism between $T^*(X \times Y)$ and $T^*(X) \times T^*(Y)$. Thus we can reorder the elements and describe $N^*(Z)$ as a subset of $T^*(X) \times T^*(Y)$ as follows

$$N^*(Z) = \{(x, \xi; y, \eta);\ F(x, y) = 0,\ \xi = \lambda(y_1, -1),\ \eta = \lambda(x_1, 1),\ \lambda \neq 0\}.$$

This is a very important object, and it is usually denoted by Λ, or Λ_R if we want to emphasise that Λ is associated to the operator R. For reasons that will be explained later we shall have to change sign of the η vector and introduce the set

$$\Lambda' = \{(x,\xi;y,-\eta);\ F(x,y)=0,\ \xi=\lambda(y_1,-1),\ \eta=\lambda(x_1,1),\ \lambda\neq 0\}. \tag{2.5}$$

Being a subset of the product space $T^*(X)\times T^*(Y)$ it is of course a *relation* in $T^*(X)\times T^*(Y)$. However, this relation is actually $1-1$, if we identify cotangent vectors ξ and θ whenever $\theta=\lambda\xi$ for some $\lambda>0$. This means that we can view Λ' as a *function* or *mapping* from $T^*(X)$ into $T^*(Y)$, a mapping that has an inverse.

Exercise. Prove that the relation Λ' is one to one.

A mapping, or more generally a relation, acts on subsets in an obvious way, which in this case means the following. If E is a subset of $T^*(X)$, then $\Lambda'\circ E$ is defined as

$$\Lambda'\circ E = \{(y,\eta)\in T^*(Y);\ (x,\xi;y,\eta)\in\Lambda',\ (x,\xi)\in E\}.$$

Theorem 1. *The following inclusion is valid*

$$WF(Rf)\subset\Lambda'\circ WF(f). \tag{2.6}$$

In the next section we shall prove that $WF(Rf)$ is in fact *equal* to $\Lambda'\circ WF(f)$.

Lemma 3. *Assume* $f\in C_c(\mathbf{R}^2)$ *and set* $g(y)=\int_{\mathbf{R}} f(x_1,y_1x_1+y_2)dx_1$. *Then*

$$(0,0;0,\pm1)\notin WF(f) \tag{2.7}$$

$$\textit{implies}\qquad (0,0;0,\pm1)\notin WF(g). \tag{2.8}$$

Sketch of proof. By assumption there exists a function $\varphi\in C_c^\infty$ with $\varphi(0,0)\neq 0$ and a $\delta_1>0$ such that

$$|\widehat{\varphi f}(\xi)|\le C_m(1+|\xi|)^{-m},\quad \text{if}\quad |\xi_1|<\delta_1|\xi_2|,\quad m=1,2,\dots. \tag{2.9}$$

We have to prove that there exists a function $\psi\in C_c^\infty$ with $\psi(0,0)\neq 0$ and $\delta_2>0$ such that

$$|\widehat{\psi g}(\eta)|\le C_m(1+|\eta|)^{-m},\quad \text{if}\quad |\eta_1|<\delta_2|\eta_2|,\quad m=1,2,\dots. \tag{2.10}$$

The proof consists of a succession of three simple observations.

1. Assume first that $f=0$ for $|x_1|<\varepsilon$. By Example 2 we know that the function $(y_1,y_2)\mapsto f(x_1,y_1x_1+y_2)$ can only have wave fronts parallel to $(x_1,1)$ for each fixed x_1. Hence the Fourier transform of $y\mapsto\psi(y)f(x_1,y_1x_1+y_2)$ must satisfy the estimates (2.10) if $\delta_2<\varepsilon<\delta_1$ for arbitrary $\psi\in C_c^\infty$. Integrating with respect to x_1 we see that $\widehat{\psi g}(\eta)$ must satisfy the same estimates. Splitting $f=f_0+f_1$ by means of a partition of unity, where $f_1=0$ for $|x_1|<\varepsilon/2$ and $f_0=0$ for $|x_1|>\varepsilon$ we conclude that it is enough to consider f_0, which we will denote by f from now on.

2. If we choose ψ with support in $|y|<\varepsilon/2\le 1/2$ it is clear that $|y_1x_1+y_2|\le\varepsilon(\varepsilon/2)+\varepsilon/2\le\varepsilon$ in the integral defining $\psi(y)g(y)$. Hence we may assume that $\operatorname{supp} f\subset\{\max|x_\nu|\le\varepsilon\}\subset\{|x|\le 2\varepsilon\}$.

3. Take $\psi(y)$ of the form $\psi(y_1)\psi(y_2)$ and consider first

$$\int e^{-i\eta_2 y_2}\psi(y_2)g(y_1,y_2)dy_2 = \int e^{-i\eta_2 y_2}\psi(y_2)\int f(x_1, y_1x_1+y_2)dx_1\,dy_2. \tag{2.11}$$

Change order of integration and then make the translation $y_1x_1 + y_2 \mapsto y_2$ in the y_2 integral. This gives

$$\int \Big(\int e^{-i\eta_2(y_2-y_1x_1)}\psi(y_2-y_1x_1)f(x_1,y_2)dy_2\Big)dx_1.$$

Changing the variable y_2 to x_2 gives the more natural looking expression

$$\int\int e^{-i\eta_2(x_2-y_1x_1)}\psi(x_2-y_1x_1)f(x_1,x_2)dx_2\,dx_1.$$

Writing the exponent $\eta_2(x_2 - y_1x_1) = \eta_2(x_1, x_2)\cdot(-y_1, 1)$ we see that for fixed y_1 the last expression can be written

$$\widehat{\psi_{y_1} f}(\eta_2(-y_1,1)), \tag{2.12}$$

where we have written $\psi_{y_1}(x_1, x_2) = \psi(x_2 - y_1x_1)$. The fact that ψ_{y_1} depends on y_1 will cause no problem. By the assumption (2.7) the expression (2.12) is $\mathcal{O}(|\eta_2|^{-m})$ for every m as $|\eta_2| \to \infty$, if $|y_1| < \delta_1$. Multiplying (2.11) by $\psi(y_1)e^{-iy_1\eta_1}$ and integrating with respect to y_1 we obtain the same estimates for $\widehat{\psi g}(\eta)$, that is, we obtain (2.10) with $\delta_2 = \delta_1$, provided $\psi(y_1)$ is supported in $|y_1| < \delta_1$.

Proof of Theorem 1. Let (x^0, ξ^0) be an arbitrary element of $T^*(X)$. Choose coordinates in $X = \mathbf{R}^2$ so that $x^0 = (0,0)$ and ξ^0 is parallel to $(0,1)$. Choose coordinates (y_1, y_2) as above on the subset of non-vertical lines in Y. We have to prove that $(y^0, \eta^0) \notin WF(Rf)$, if $(y^0,\eta^0) = \Lambda' \circ (x^0, \xi^0)$. Next let us compute (y^0, η^0). We are going to use (2.5). Since $F(x^0, y^0) = 0$ and $x^0 = (0,0)$, we must have $y_2^0 = 0$. Since ξ^0 is parallel to $(0,1)$ it follows from (2.5) that $y_1^0 = 0$. Moreover, since $x_1^0 = 0$ we see from (2.5) that η^0 must be parallel to $(0,1)$. Hence the assertion of the theorem follows from Lemma 3.

It is interesting to describe the relation Λ in geometric terms. To do this we think of Y as the 2-dimensional manifold of all lines in the plane X. Then Z is the 3-dimensional submanifold of $X \times Y$ that consists of all pairs (x, L) such that $x \in L$. The manifold Λ is the 4-dimensional conormal bundle of Z, a subset of the 8-dimensional manifold $T^*(X\times Y) \approx T^*(X)\times T^*(Y)$. Since we don't have natural coordinates on Y it appears hard to "see" the conormal vectors η above the lines $L \in Y$. Here is a way to do this, at least if we agree to ignore the distinction between η and $-\eta$.

In any vector space V, the *direction* of a vector $\eta \in V^*$ is determined by the zero set of η, which is a hyperplane through the origin in V. In our case $V = T_p(M)$ and $V^* = T_p^*(M)$, where $p \in M$, and M is X or Y. So, to a direction η in $T_p^*(M)$ corresponds a hyperplane in $T_p(M)$. But if the manifold M is not a vector space, we cannot talk about hyperplanes in M. Instead, a hyperplane in $T_p(M)$ can be specified by a piece of hypersurface through p in M. Thus a cotangent vector in $T_p^*(M)$ is determined up to $\pm$ by a piece of a hypersurface in M that contains p. If we insist on distinguishing between η and $-\eta$, we need just specify an orientation of the hypersurface. In particular, a cotangent vector above a line L^0 in the space Y of all lines in $\mathbf{R}^2$ can be defined as the conormal to the curve $\Sigma_x \subset Y$ consisting av all lines near L^0 that contain the point $x \in L^0$.

Let us now describe the correspondence $(x, \xi) \sim (y, \eta)$ defined by our Λ' in geometric terms. We will consider the n-dimensional case, since this causes no additional difficulties. Thus $X = \mathbf{R}^n$ and Y is the n-dimensional manifold of hyperplanes L in X. Let $(x^0, \xi^0) \in T^*(X)$. We shall describe up to $\pm$ the element $(L^0, \eta^0) \in T^*(Y)$ that is coupled to (x^0, ξ^0) by Λ'. First of all L^0 is the hyperplane through x^0 that is conormal to ξ^0. Moreover, I claim that η^0 (up to $\pm$) is the element of $T^*_{L^0}(Y)$ that is defined by the surface Σ_{x^0} consisting of all hyperplanes (near L^0) that contain x^0. In other words, η^0 is the (unique up to multiplication by scalars) conormal to Σ_{x^0}.

Exercise. Given a line L^0 in the manifold Y of all lines in $\mathbf{R}^2$. Prove that all cotangent vectors $\eta \in T^*_{L^0}(Y)$ with one exception (up to multiplication by non-zero scalars) can be defined as conormals to some surface $\Sigma_x \subset Y$ for $x \in L^0$.

Exercise. Prove that the geometric description of Λ' given above is correct, i.e., that it agrees with (2.5).

Corollary to Theorem 1. Let $f \in C_c(\mathbf{R}^n)$ and assume $Rf(L) = 0$ for all hyperplanes L in some neighborhood of the hyperplane L^0. Then

$$WF(f) \cap N^*(L^0) = \emptyset,$$

or more explicitly

$$(x, \xi) \notin WF(f) \quad \text{for every } (x, \xi) \text{ with } x \in L^0 \text{ and } \xi \text{ conormal to } L^0.$$

Proof. We have seen that every $(x, \xi) \in N^*(L^0)$ is coupled by Λ' to some cotangent vector $\eta \in T^*_{L^0} \setminus 0$. But the assumption implies that $(L^0, \eta) \notin WF(Rf)$ for every $\eta \in T^*_{L^0} \setminus 0$. The assertion now follows from Theorem 1.