Introduction to Witt vectors, δ-rings, and prisms

Perfectoid Spaces
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<u>Goal</u>: Formula-free def. of W..... to which we will take a formula-based approach

1. <u>p-derivations</u> + <u>δ-rings</u>

p = prime (fixed)
R = ring

<u>Def</u>: A ring endo $\varphi : R \to R$ is a <u>Frobenius</u> <u>lift</u> if
$\forall x \in R$, $\varphi(x) \equiv x^p \mod pR$

<u>E.g.</u>: $\mathbb{Z}[x]$, $\varphi : x \mapsto x^p + p \cdot$ anything
global class field theory
crystalline cohomology
Adams operations ψ^p

Goal: leads inevitably to Witt vectors

* {Rings with Frob. lift} naturally forms a category
.... but <u>not</u> a good one!

Problem: "lift" has a hidden $\exists$:
$\forall x \, \exists x'$ s.t. $\varphi(x) = x^p + px'$.
x' is unique up to p-torsion — no control over it if R is not p-tor. free

<u>Ex</u>: ① Category doesn't have pullbacks, intersections of sub-objects?

<u>Solution</u>: Provide x' itself as part of the structure,
rather than the property of its mere existence.

i.e. want an operator $\delta : R \to R$ modelled on $\delta(x) = \dfrac{\varphi(x) - x^p}{p}$

Axioms? Write the ring-endo axioms for φ in terms of δ.

* $\varphi(x+y) = \varphi(x) + \varphi(y)$

$(x+y)^p + p\delta(x+y) \qquad x^p + p\delta(x) + y^p + p\delta(y)$

$\therefore \; p\delta(x+y) = p\delta(x) + p\delta(y) + x^p + y^p - (x+y)^p$ ⇝ equiv. to the additivity of φ

(i) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$ ⇝ equiv. only mod p-tor

* $\varphi(xy) = \varphi(x)\varphi(y)$

$(xy)^p + p\delta(xy) = (x^p + p\delta(x))(y^p + p\delta(y))$

(ii) $\delta(xy) = x^p\delta(y) + \delta(x)y^p + p\delta(x)\delta(y)$

(iii) $\delta(0) = 0$

(iv) $\delta(1) = 0$

"Leibniz rules" for δ under $+, \times, 0, 1$. Buium: $\delta = "\partial/\partial p"$

Def: A function $\delta : R \to R$ is a p-derivation if it satisfies (i)-(iv).
A δ-ring is a ring equipped with a p-derivation.

* $\{\text{p-der on } R\} \longrightarrow \{\text{Frob. lifts on } R\}$

$\delta \longmapsto \varphi$, where $\varphi(x) = x^p + p\delta(x)$

This is a bijection if R is p-tor free (but not in general!)

E.g. • Any p-tor-free ring with Frob. lift: $\mathbb{Z}[x]$, $\delta(x) =$ anything

• $R = K_0(\mathcal{C})$. $\delta = \lambda$-operation assoc. to $\frac{(x_1^p + x_2^p + \cdots) - (x_1 + x_2 + \cdots)^p}{p}$

Rmk: There is a formula-free def. of a δ-structure.

Ex: ② $R = \mathbb{F}_p$-alg: R admits a p-der. $\iff R = \{0\}$, whereas all such R have Frob. lifts!

③ Same for $\mathbb{Z}/p^n\mathbb{Z}$-algebras.

2. Witt vectors

Goal:
$$\begin{tikzcd} \delta\text{-rings} \arrow[d] \\ \text{Rings} \arrow[u, bend left] \arrow[u, bend right, "W"'] \end{tikzcd}$$

Compare:
$$\begin{tikzcd} \{\text{differential rings}\} \arrow[d] \\ \text{Rings} \arrow[u, bend left] \arrow[u, bend right, "\text{“}W^{\text{diff}}\text{”}"'] \end{tikzcd} \qquad \begin{tikzcd} C\text{-mod} \arrow[d] \\ \text{Ab} \arrow[u, bend left, "C\otimes -"] \arrow[u, bend right, "\mathrm{Hom}(C,-)"'] \end{tikzcd} \qquad \begin{tikzcd} G\text{-sets} \arrow[d] \\ \text{Sets} \arrow[u, bend left, "G\times -"] \arrow[u, bend right, "(-)^G"'] \end{tikzcd}$$

* Warm up with differential rings

$$W^{\text{diff}}(A) = \text{“divided power series”} = \Big\{ \sum_{n\geq 0} a_n \overbrace{\frac{t^n}{n!}}^{\text{formal place holder}} \;\Big|\; a_n \in A \Big\}$$

obvious ring str: $\dfrac{t^m}{m!}\cdot\dfrac{t^n}{n!} = \binom{m+n}{m}\dfrac{t^{m+n}}{(m+n)!}$

derivation: $d\left(\dfrac{t^n}{n!}\right) = \dfrac{t^{n-1}}{(n-1)!}$

Universal property:
$$\begin{tikzcd} d \circlearrowright R \arrow[rr, dashed, "\tilde{g}", "\exists!\ \text{diff}"'] \arrow[dr, "g"'] & & W^{\text{diff}}(A) \arrow[dl] \\ & A & \end{tikzcd}$$

$$\tilde{g}(r) = \text{“}g(\text{Taylor series of } r)\text{”} = \sum_{n\geq 0} g(d^{\circ n}(r))\,\frac{t^n}{n!}$$

Check: $\tilde{g}$ is diff. ring map lifting g, and is unique.

Alternative point of view:

$$W^{\text{diff}}(A) = A \times A \times \cdots$$
$$\sum a_n \frac{t^n}{n!} = (a_0, a_1, a_2, \ldots)$$
$$\frac{d}{dt} = \text{shift left}$$

Then $\tilde{g} : r \longmapsto (g(r), g(d(r)), g(d^{\circ 2}(r)), \ldots)$

We can see that $\tilde{g}$ is actually the unique <u>set</u> <u>map</u> lifting g + equivariant. So from this point of view, the ring str. on

$$W^{diff}(A) = A \times A \times \cdots$$

is forced to make $\tilde{g}$ a ring map.

In fact, the ring str. is a "purely syntactic" re-expression of the Leibniz rules for $d^{\circ n}$:

$$\sum a_i \frac{t^i}{i!} \sum b_j \frac{t^j}{j!} = \sum \binom{i+j}{i} a_i b_j \frac{t^{i+j}}{(i+j)!}$$

$$(a_0, a_1, \dots)\cdot(b_0, b_1, \dots) = \Big(\dots, \underbrace{\sum_{i+j=n} \binom{n}{i} a_i b_j}, \dots\Big)$$

$$d^{\circ n}(xy) = \underbrace{\sum_{i+j=n} \binom{n}{i} d^{\circ i}(x)\, d^{\circ j}(y)}$$

same

* back to W

$W(A) = A \times A \times \cdots$ ⇝ ring str. at the n^{th} component given by the Leibniz rules for $\delta^{\circ n}$ w.r.t. <u>both</u> $\underline{+}$ <u>and</u> $\underline{\times}$

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = \Big(a_0 + b_0,\ a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}, \dots\Big)$$

$$(a_0, a_1, \dots) \times (b_0, b_1, \dots) = (a_0 b_0,\ a_0^p b_1 + a_1 b_0^p + p a_1 b_1, \dots)$$

$$0 = (0, 0, 0, \dots)$$

$$1 = (1, 0, 0, \dots)$$

$$\delta : (a_0, a_1, a_2, \dots) \longmapsto (a_1, a_2, \dots)$$

$$\begin{tikzcd} \delta \circlearrowright R \arrow[rr, dashed, "\tilde{g}"] \arrow[dr, "g"'] & & W(A) \arrow[dl] \\ & A & \end{tikzcd}$$

$$\tilde{g}(r) = \big(g(r), g(\delta(r)), g(\delta^{\circ 2}(r)), \dots\big)$$

$\tilde{g}$ is the unique set map lifting g (and compat. with δ)
It is also a ring map by construction.

$\therefore$ W is the right adjoint!

Next time: Our W is canonically isom. to the usual Witt vector construction.
But not by the identity!

Ex: ④ Prove the poly $P_n^+(x_0, y_0, \dots, x_n, y_n)$ s.t. $\delta^{on}(x+y) = P_n^+(x, y, \dots, \delta^{on}(x), \delta^{on}(y))$ is unique. Similarly for multiplication.

3. Moduli interpretation

$R = \delta$-ring
$X = \mathrm{Spec}(R)$

$$\begin{tikzcd}
X(A) \arrow[d, dashed] \arrow[r, phantom, "="] & \mathrm{Hom}(R,A) \arrow[d, "\sim"] & g \arrow[d, mapsto] \\
X(W(A)) & \mathrm{Hom}_\delta(R, W(A)) \arrow[l, hook'] & \tilde{g}
\end{tikzcd}$$

* If a moduli space has a δ-structure, then the objects it classifies have a theory of canonical lifts.