Graph entropy and the zero-error capacity of channels

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Based on joint work with Siddharth Bhandari (TIFR), Venkat Guruswami (CMU) and Marco Dalai (Brescia)

The binary difference problem

Fix N. What is the minimum n such that we can find N **different** sequences in $\{0,1\}^n$?

Answer: $\lceil \log_2 N \rceil$

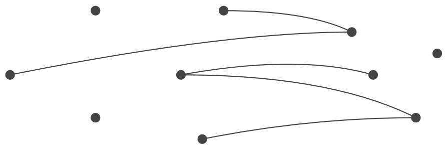
What is the maximum number of sequences in $\{0,1\}^n$ such that every **two** of them are **different**?

Answer: 2^n

Hansel's lemma about graph covering

Suppose $[M] = \{1, 2, \ldots, M\}$, K_M = the complete graph on $[M]$, and

- G_i ($i \in I$) a finite sequence of bipartite graphs on $[M]$
- τ_i fraction of non-isolated vertices in G_i



$\tau_i = 7/10$

Lemma (Hansel '64)

$$\bigcup_{i\in I} G_i = K_M \implies \sum_{i\in I} \tau_i \geq \log_2(M).$$

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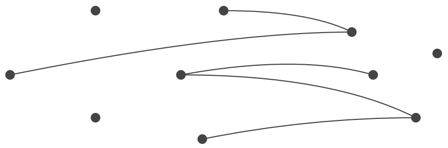

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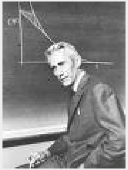
Entropy of a random variable

Definition (Shannon entropy)



Claude Shannon (1916-2001)

Let $\mathbf{X}$ be a random variable taking values in the set $[n] = \{1, 2, \ldots, n\}$. Then, its entropy is given by

$$H[\mathbf{X}] = -\sum_{i=1}^{n} p_i \log_2 p_i.$$

- $H[\mathbf{X}]$ measures the uncertainity in $\mathbf{X}$.
- $H[\mathbf{X}]$ is a function of the distribution of $\mathbf{X}$, not the actual values it takes.
- $H[\mathbf{X}] \leq \log_2 n$.

Pragmatics

Alice

Observes **X**

Sends Bob a message **M**

$\Leftarrow$

$\Rightarrow$

Bob

Recovers **X** from **M**

Goal

- Alice and Bob exchange bits.
- Bob must recover **X** exactly.
- Goal: minimize the (expected) total number of bits transmitted.

Transmission cost

Let $T[\mathbf{X}]$ denote the minimum (expected) cost of transmitting **X**.

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Entropy and transmission

Theorem

$$H[\mathbf{X}] \;\leq\; T[\mathbf{X}] \;\leq\; H[\mathbf{X}] + 1\,.$$

Long years ago . . . 1948

Shannon's source coding theorem

Let p be a probability distribution on $[n]$. For $\epsilon > 0$ and positive integer k, let

$$N(k,\epsilon) = \min_{A \subset [n]^k : p^k(A) \geq 1-\epsilon} |A|.$$

Theorem (Shannon)

For all $\epsilon \in (0,1)$, $\lim_{k \to \infty} \frac{1}{k} \log_2 |N(k,\epsilon)| = H(p)$.

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. . . not wholly or in full measure, but very substantially!

Conditional entropy

Definition

$(\mathbf{X}, \mathbf{Y})$: a pair of random variables with some joint distribution.

$$H[\mathbf{Y} \mid \mathbf{X}] = \sum_i p_X(i) H[\mathbf{Y}_i]$$

Fact

- *Conditioning reduces uncertainity:* $H[\mathbf{Y} \mid \mathbf{X}] \leq H[\mathbf{Y}]$.
- $H[\mathbf{XY}] = H[\mathbf{X}] + H[\mathbf{Y} \mid \mathbf{X}]$.

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The noisy channel

Specification

Input alphabet: $[m]$

Output alphabet: $[n]$

Characteristics: $\Pr[\text{output} = j \mid \text{input} = i] = p_{j|i}$.

Code of conduct

Encoding: $\{0,1\}^k \to [m]^t$

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Error: $\Pr[\text{input} \neq \text{output}] \leq \epsilon$.

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Capacity

Input to the channel: $\mathbf{X} \in [m]$
Ouput of the channel: $\mathbf{Y} \in [n]$.

Definition (Capacity of a channel E)

$$C(E) = \max_{X} H[\mathbf{X}] + H[\mathbf{Y}] - H[\mathbf{XY}].$$

Jaane kya toone kahi . . . jaane kya meine suni

Theorem (Shannon)

Let C be the capacity of the channel. Then, for all $\epsilon > 0$ and all k, there exist encoders and decoders such that

Encoding rate: $\frac{k}{t} \geq C - \epsilon$.

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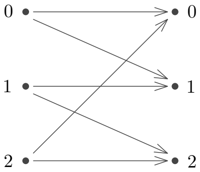
The 3/2 channel: trifference



- Any two input symbols are confusable (share a common output).
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  - Zero-error capacity 0.
- However, the three inputs have no common output.

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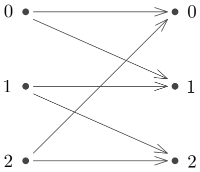

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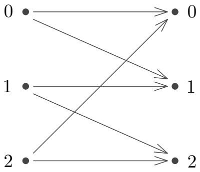

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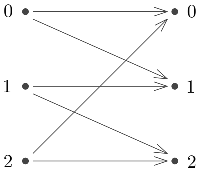

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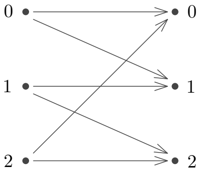

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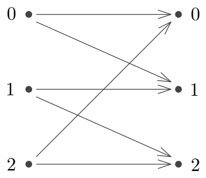

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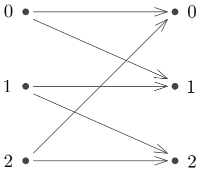

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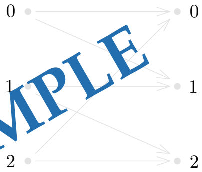

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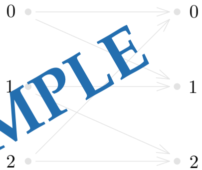

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VERY DIFFICULT

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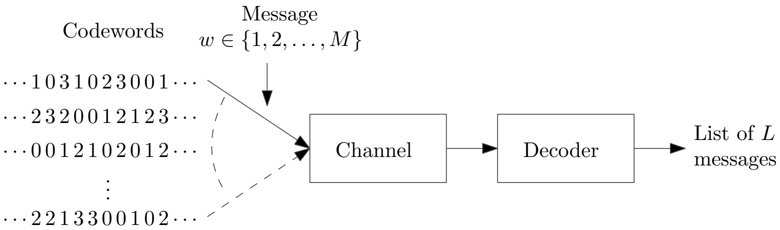
Zeror-error capacity with list decoding



- The decoder outputs a *list* of L messages
- Zero-error code: the correct message is always in the list
  - $\Leftrightarrow$ No $L+1$ codewords are compatible with any output sequence
- Zero-error capacity with list size L: the maximum R such that there is such a zero-error code of size 2^{nR}

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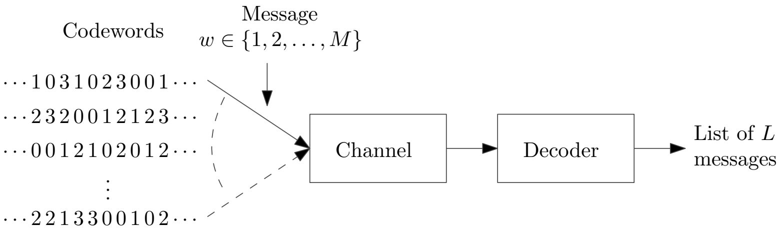

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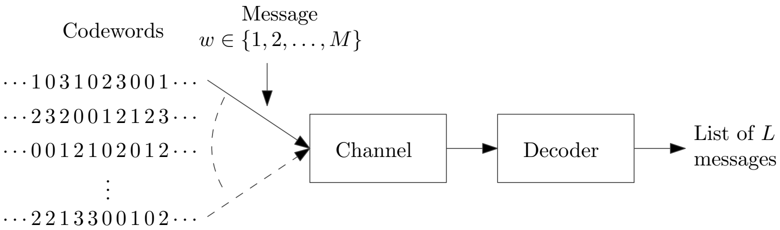

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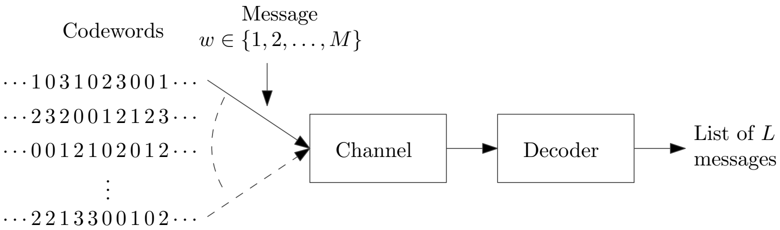

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Ternary encoding

Let $R_3(n)$ be the size of the largest subset $C \subseteq \{0,1,2\}^n$ such that every **three** of them are **trifferent**?

$$\textbf{trifferent} \equiv \forall \text{ (distinct) } x, y, z \in C \quad \exists i : \{x_i, y_i, z_i\} = \{0,1,2\}$$

$$\begin{array}{lcccccccc} x & = & 0 & 0 & 0 & 2 & 1 & 2 & \cdots \\ y & = & 0 & 1 & 1 & 1 & 2 & 1 & \cdots \\ z & = & 1 & 1 & 2 & 2 & 1 & 2 & \cdots \end{array}$$

Answer: $2^{0.212n} \leq R_3(n) \leq 2\left(\frac{3}{2}\right)^n \approx 2^{0.545n}$ (Körner & Marton '88)

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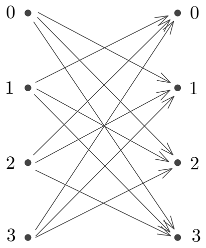

- The four inputs have no common output
  - We can build 4-tuples of codewords which cannot be confused

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  - Upper bounds for zero-error capacity with $L = 3$?

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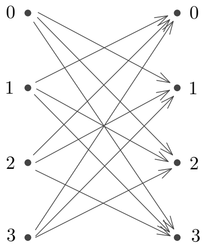

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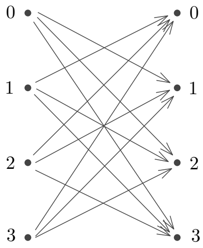

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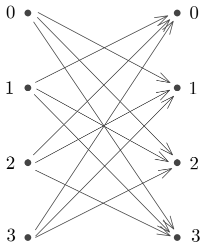

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Alphabet size 4

Problem

Let $R_4(n)$ be the size of the largest code $C \subseteq \{0,1,2,3\}^n$ that satisfies

$$\forall \text{ (distinct)} w, x, y, z \in C \quad \exists i : \{w_i, x_i, y_i, z_i\} = \{0,1,2,3\}$$

- Standard random coding: $R_4(n) \geq 2^{0.0473n}$
- Simple upper bound $R_4(n) \leq 3 \cdot (\frac{4}{3})^n \approx 2^{0.415n}$
- Fredman-Komlós '84: $R_4(n) \leq 2^{\frac{3}{8}(1+o(1))n} \approx 2^{0.375n}$
- Arikan '94: $R_4(n) \leq 2^{0.3512(1+o(1))n}$, using Plotkin the bound

  can be improved to about $2^{0.3276n}$, using the linear programming bound

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Let $R_4(n)$ be the size of the largest code $C \subseteq \{0,1,2,3\}^n$ that satisfies

$$\forall\ \text{(distinct)} w,x,y,z \in C \quad \exists i : \{w_i, x_i, y_i, z_i\} = \{0,1,2,3\}$$

- Standard random coding: $R_4(n) \geq 2^{0.0473n}$
- Simple upper bound $R_4(n) \leq 3 \cdot (\frac{4}{3})^n \approx 2^{0.415n}$
- Fredman-Komlós '84: $R_4(n) \leq 2^{\frac{3}{8}(1+o(1))n} \approx 2^{0.375n}$
- Arikan '94: $R_4(n) \leq 2^{0.3512(1+o(1))n}$, using Plotkin the bound

  ... can be improved to about $2^{0.3276n}$, using the linear programming bound

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Today

$$R_4(n) \leq 2^{\frac{6}{19}(1+o(1))n} \approx 2^{0.3158n},$$

that is,

$$\log \frac{R_4(n)}{n} \leq \frac{6}{19}(1+o(1)) \approx 0.3158.$$

(Joint work with Venkat Guruswami and Marco Dalai '17)

The framework (due to Fredman and Komlós)

Let $C \subseteq \{0,1,2,3\}^n$ be a 4-hash code.

Step 1: Fix distinct $x, y \in C$.

Step 2: For $i \in [n]$ and code word $z \in C \setminus \{x, y\}$

$\Longrightarrow$ if $x_i = y_i$, set z_i to $\star$;

$\Longrightarrow$ if $z_i \in \{x_i, y_i\}$, set z_i to $\star$;

$\Longrightarrow$ otherwise, let $\{a_i, b_i\} = \{0,1,2,3\} \setminus \{x_i, y_i\}$;
if $z_i = a_i$ replace it by 0, and if $z_i = b_i$, replace it by 1.

Step 3: Apply Hansel's lemma to the resulting code over $\{0, 1, \star\}$.

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The picture

$$\begin{bmatrix} 3 & 1 & 3 & 0 & 0 & 1 & \dots \\ 0 & 1 & 1 & 2 & 3 & 0 & \dots \\ 3 & 2 & 2 & 2 & 0 & 3 & \dots \\ 2 & 3 & 1 & 3 & 2 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 1 & 3 & 1 & 1 & 0 & 1 & \dots \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & 3 & 0 & 0 & 1 & \dots \\ 0 & 1 & 1 & 2 & 3 & 0 & \dots \\ \star & \star & 1 & \star & \star & 1 & \dots \\ 1 & \star & \star & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \star & \star & 0 & \star & \star & \dots \end{bmatrix}$$

The picture

$$
\begin{bmatrix}
3 & 1 & 3 & 0 & 0 & 1 & \dots \\
0 & 1 & 1 & 2 & 3 & 0 & \dots \\
3 & 2 & 2 & 2 & 0 & 3 & \dots \\
2 & 3 & 1 & 3 & 2 & 2 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\
1 & 3 & 1 & 1 & 0 & 1 & \dots
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
3 & 1 & 3 & 0 & 0 & 1 & \dots \\
0 & 1 & 1 & 2 & 3 & 0 & \dots \\
\star & \star & 1 & \star & \star & 1 & \dots \\
1 & \star & \star & 1 & 1 & 0 & \dots \\
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The Fredman-Komlós bound

Recall that $C \subseteq \{0,1,2,3\}^n$ be a 4-hash code.

- Pick distinct $x, y \in C$ uniformly at random.
- Let $f[0], f[1], f[2], f[3]$ be the frequencies of the symbols in i-th coordinate. Then,

$$\mathbb{E}[\tau(i)] \approx \sum_{a \neq b} f[a]\, f[b]\, (1 - f[a] - f[b]) \leq \frac{3}{8}.$$

- Use Hansel's lemma to obtain

$$\log_2(|C| - 2) \leq \sum_i \tau(i) \leq \left(\frac{3}{8}\right) n.$$

- Conclusion: $|C| \leq 2^{((3/8)+o(1))n}$.

The Arikan bound

- Pick two distinct $x, y \in C$ that agree on many coordinates.
(Coding theory says: high rate $\implies$ small distance, say Δn.)
- Use an *ad hoc* argument to argue that all four symbols appear in each coordinate with frequency at least f_0 (otherwise, the code must be small anyway).
- Use Hansel's lemma to conclude that

$$\log_2(|C| - 2) \leq \sum_i \tau(i) \leq \Delta(1 - 2f_0)n.$$

- Using bounds on distance in terms of rate gives $|C| \leq 2^{0.3512n}$ (using the Plotkin bound) and $|C| \leq 2^{0.3276n}$ (using the linear programming bound).

Looking deeper into the Plotkin bound

Recall that $C \subseteq \{0,1,2,3\}^n$ is a 4-hash code.

Main idea

Generate x, y more carefully, in a way reminiscent of the proof of the Plotkin bound.

- Fix $s \leq n$. Later we will set $s \approx \left(\frac{1}{2}\right) \log_2 |C|$.
- Pick $x \in C$ uniformly at random; let $w = x[1, \ldots, s]$ be the prefix of x of length s.
- Pick $y \in C$ distinct from x uniformly at random conditioned on $y[1, \ldots, s] = w$.
- This helps us derive a stronger conclusion from Hansel's lemma.

Analysis

- $\tau(i) = 0$ if $i = 1, 2, \ldots, s$.
- For $i = s+1, s+2, \ldots, n$, the quantity $\tau(i)$ is a random variable depending on our choice of x and y. We will show that $\mathbb{E}[\tau(i)] \leq \frac{3}{8}$.
- Use Hansel's lemma to conclude that

$$\log_2(|C| - 2) \leq \left(\frac{3}{8}\right)(n - s).$$

- Set $s \approx (\frac{1}{2}) \log_2 |C|$ and obtain

$$|C| \leq 2^{(6/19)n}.$$

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In expectation

For each prefix $w \in \{0,1,2,3\}^s$, let C_w be the subcode

$$C_w = \{x \in C : x[1,\ldots,s] = w\}.$$

For $i > s$, let $f_{i,w}[a]$ be the frequency of the symbol a in the i-th coordinate in the subcode C_w; let $\tau(i|w) = \sum_{a \neq b} f_{i,w}[a] f_{i,w}[b](1 - f_i[a] - f_i[b])$. Then,

$$\mathbb{E}[f_{i,W}] = f_i \qquad \text{and} \qquad E[\tau(i)] = \mathbb{E}_W[\phi(f_{i,W}, f_i)],$$

where $\phi(f', f) = \sum_{a \neq b} f'[a] f'[b](1 - f[a] - f[b])$.

Claim

Suppose $\mathbb{E}[f_W] = f$ and each component of f is at most $\frac{1}{2}$. Then, $\mathbb{E}_W[\phi(f_W, f)] \leq \phi(f, f) \leq \frac{3}{8}$.

Must ϕ meet our expectation?

Claim

Suppose $\mathbb{E}[f_W] = f$ *and each component of* f *is at most* $\frac{1}{2}$*. Then,*

$$\mathop{\mathbb{E}}_{W}[\phi(f_W, f)] \le \phi(f, f) \le \frac{3}{8}.$$

- Let $\Delta_w = f_w - f$; then, $\mathbb{E}_W[\phi(f_W, f)] = \phi(f, f) - \mathbb{E}_W[\Delta_W^t N \Delta_W]$, where

$$N = -\begin{pmatrix} 0 & f[2]+f[3] & f[1]+f[3] & f[1]+f[2] \\ f[2]+f[3] & 0 & f[0]+f[3] & f[0]+f[2] \\ f[1]+f[3] & f[0]+f[3] & 0 & f[0]+f[1] \\ f[1]+f[2] & f[0]+f[2] & f[0]+f[1] & 0 \end{pmatrix}.$$

- Now, $\mathbf{1} \cdot \Delta_w = 0$. In the subspace orthogonal to $\mathbf{1}$, the matrix N is positive semidefinite (because of our assumption on f).

What about the $q/(q-1)$ channel?

Let $R_{q,L}(n)$ be the size of the largest subset $C \subseteq \{0, 1, 2, \ldots, q-1\}^n$ such that every $L+1$ of them are **qifferent**?

$$\textbf{qifferent} \equiv \forall \text{ (distinct) } x_1, x_2, \ldots, x_{L+1} \in C \quad \exists i \in [L+1] : \\ \{x_1[i], x_2[i], \ldots, x_{L+1}[i]\} = \{0, 1, 2, \ldots, q-1\}.$$

Theorem (with Siddharth Bhandari '18)

For every $\epsilon < 1/6$, for all large q, if $L \leq \varepsilon q \ln q$, then

$$\frac{\log R_{q,L}(n)}{n} = O(\exp(-q^{1-6\epsilon}/8)).$$

Earlier, it was known that the rate of the code is exponentially small if the list size is bounded by $1.58q$ (Chakraborty, Radhakrishnan, Raghunathan and Sasatte '06).

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