Multiplex Networks, Optimization and Spectra

Sarika Jalan

Complex Systems Lab

http://www.iiti.ac.in/~sarika

IIT Indore

Sanjiv K. Dwivedi, Murilo Baptista and SJ (arXiv)

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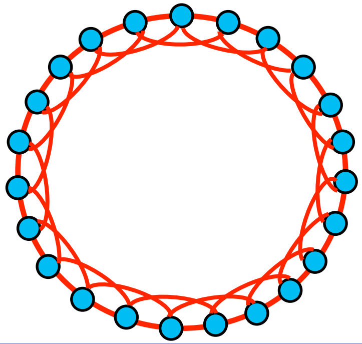
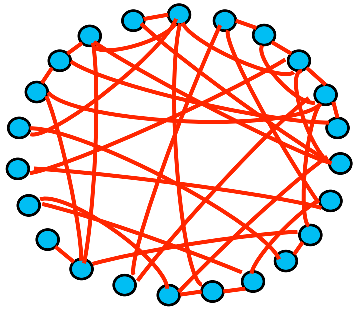
COMPLEX SYSTEM ⇨ NETWORKS (GRAPH)

def :- A graph (or network) is a pair of sets, $G = \{P,E\}$, where P is a set of N nodes, $p1, p2 ..pN$ and E is a set of edges that connect two elements of P





Random graphs: first studied by Erd”os - R’enyi

in series of papers 1959-1961

Universal Small-world and Scalefree phenomena

-Watts and Strogatz, Nature 1998, Albert and Barabasi, Science 1999

Network Science deals with ***collective phenomena*** arising from the ***interaction*** of many components.

“The system should be considered as a whole.”

Examples : Long-range correlations occurring at classical/quantum phase transition, self-organized behavior of neurons, cooperative phenomena in ant colonies or human cities, epidemic spreading

Overview of networks: Interesting universal structural properties

Important nodes (hubs)

↓

Responsible for scale-free nature or Power law

➢Robust against accidental failure but vulnerable to the coordinated attacks

(A. –L Barabsi et. al., 'Statistical mechanics of complex networks' Rev of Mod Phy 2002)

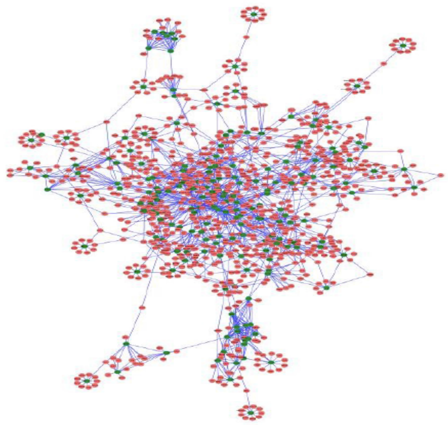
➢Small diameter, high clustering coefficient, (dis)assortativity

(S. Strogtaz "Exploring complex networks" Nature 2001; Boccaletti et. al. Phys Rep 2006, Costa et. al., Advances in Phy 2007)

➢Modular structure: Betweenness centrality measure implies nodes connecting different communities are important

(Series of papers by Newmann and others 'Hierarchical structure and the prediction of missing links in networks', Nature 2008)

➢ Adjacency matrix of real-world networks



$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & 1 & 0 & 1 & 1 & \dots \\ \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- ➢ Elements are 0's and 1's
- ➢ Real world networks are sparse ➔ less 1's (order of N)
- ➢ They have some structure ➔ matrix is not random

Eigenvalues of the underlying Adjacency matrix are called the spectra of network

$$\{\lambda_i\},\ i = 1, \dots, N$$

Spectra of Networks:

- Graph spectra provide information about basic topological properties of underlying network

  Chung, Spectral Graph Theory (AMS 1997)

- Multiplicities of eigenvalues, particularly at 0 and 1 provide understanding to structural patterns and evolution

  - Chung et. al PNAS (2003); SJ* et. al., Chaos (2016) ; Marrec and SJ*, arXiv (2016)

- Ecological Systems: *Stability is determined by eigenvalues of interaction matrix*

  R. M. May, Stability and complexity in model ecosystem (Princeton Univ. press 1974)

- Graph partition/community detection: First two eigenvalues of corresponding eigenvectors

  Series of papers by M. Newmann, PNAS (2006)

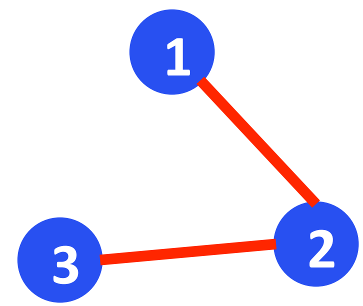
- Random matrix analysis of real-world networks

  - Rai, A. Vipin Menon and SJ*, Nature Scientific Reports (2014)
  - Agarawal, Sarkar, Dwivedi and SJ*, Physica A (2014)

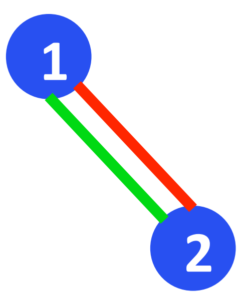
- Ratio of the first nonzero and the largest eigenvalues of the corresponding Laplacian matrix relates with synchronizability

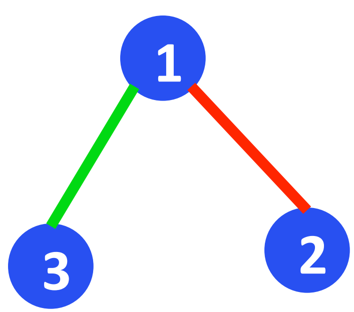
  - Pecora and Carroll, Phys. Rev. Lett (1998)

Network Science has so far focused on “one type of interactions” among individual



There may exist more than one type of interaction among the same units

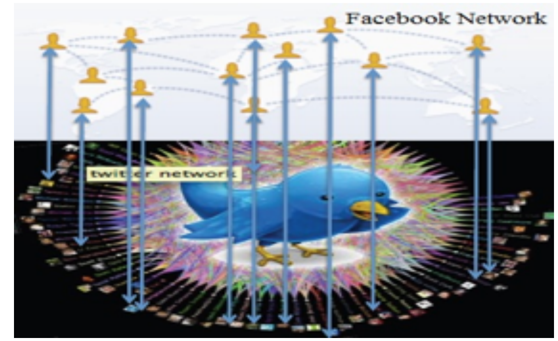




Multiplex Networks

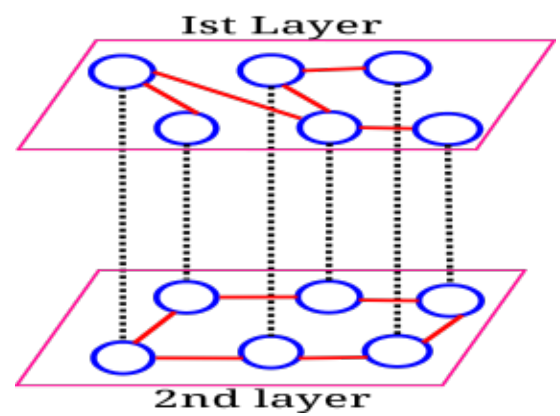
Multiplex networks

✓ A multiplex network is a set of N nodes interacting in M layers, each reflecting a distinct type of interaction linking the same pair of nodes



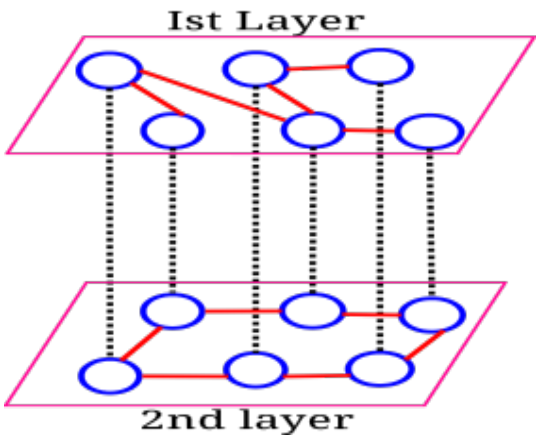
✓ In human brain, different regions can be seen connected by functional and structural neural networks [Bullmore, and Sporns, Nat. Rev. Neurosci 10, 186 (2010)]

✓ Transport network: Different layers can be Air, train and bus transportation networks [Boccaletti et al. Phys. Rep. (2014)]



✓ In social networks people may be connected because of belonging to the same family, being friend or work collaborators [M. Szell et al., PNAS (2010); [Camellia Sarkar, Alok Yadav and SJ* (2016)]

Ignoring impact of multiplexity (layer 2) may result in wrong prediction for the behavior of a system (layer 1)



A strike of the bus service may result in overloading the rail and air traffic routes

✓ Cluster synchronization in multiplex networks, Aradhana Singh and SJ*, EPL 113 30002 (2016)

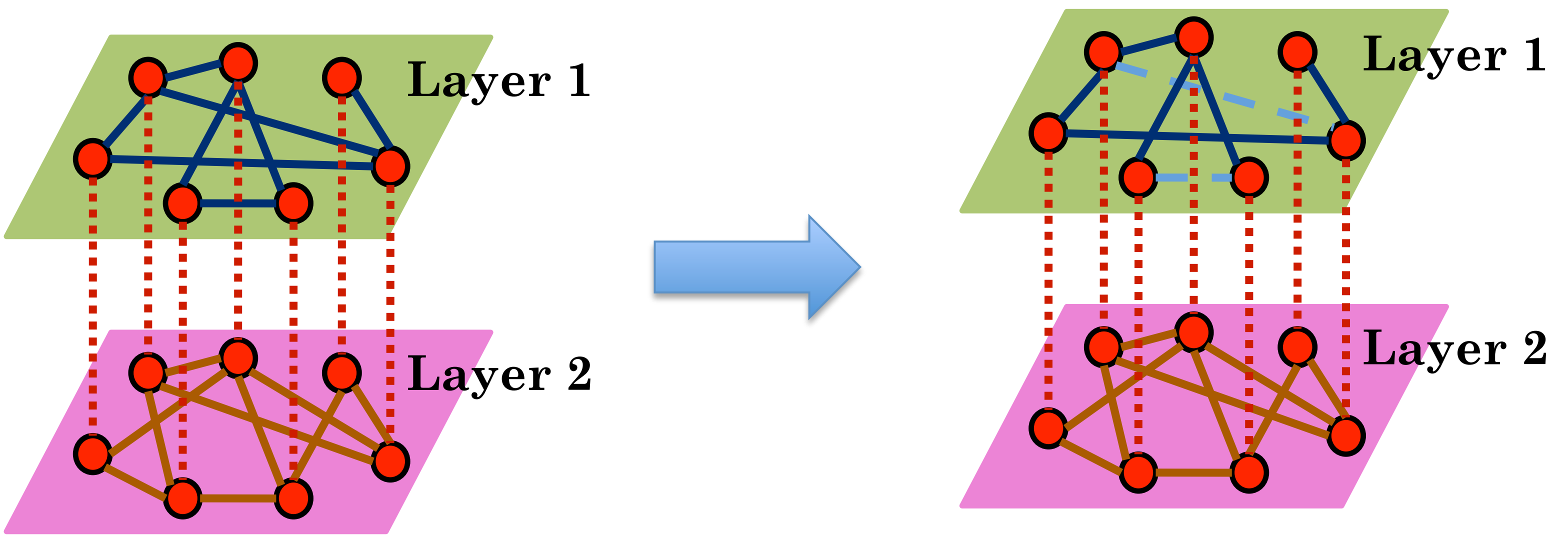
Multiplex networks with different layers of interactions

$$A = \begin{pmatrix} A_1 & I & I & . & . & . \\ I & A_2 & I & . & . & . \\ I & I & A_3 & . & . & . \\ . & I & I & A_4 & I & . \\ . & . & I & I & A_5 & I \end{pmatrix}$$

Eigenvalues of the underlying Adjacency matrix are spectra of multiplex network

Aim: To create the most synchronizable System represented by Multiplex networks

Restriction: only one layer can be altered or in our control



Synchronizability Measure

❖ Adjacency matrix of Multiplex network : $M^i = \begin{bmatrix} A^i & D_x I \\ D_x I & B^i \end{bmatrix}$

❖ Laplacian of this multiplex network:

$$L_{jk} = \begin{cases} d_j, & if\ j \neq k \\ -M_{jk}, & otherwise \end{cases}$$

where d_i is degree of j^{th} node in the multiplex network

Synchronizability of network is defined in terms of eigenvalues of multiplex network Laplacian matrix

$$R = \frac{\lambda_{max}}{\lambda_2}$$

Lower R value : better synchronizability

Theoretical Framework:

- For give *N* (size of the system) and *M* (building cost), if we can enumerate all the possible configurations (layer 1), the network corresponding to the lowest R will be our desired one

- Enumerating all the network configurations for a given *N* and *M* is computationally exhaustive

- We formulate this problem through an optimization technique

Our optimization aims at minimizing R, and thus, maximizing synchronizability

Optimization Technique

- Take an initial multiplex network with a given set of parameters

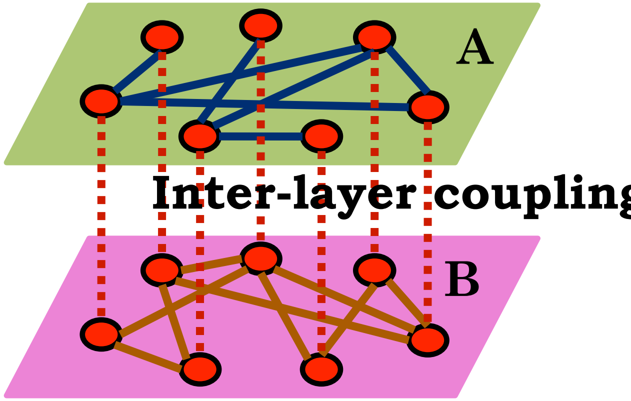
- Calculate the eigenvalue ratio of the corresponding Laplacian matrix of the initial multiplex network

Rewiring is performed only in one layer by keeping the second layer's architecture fixed

- The initial multiplex network is replaced by the rewired multiplex network if the latter is more synchronizable

How “only one” layer governs the synchronizability of The entire multiplex network

Changing network architecture of one layer affects dynamics of other layers due to inter-layer couplings



- Factors affecting eigenvalues of Laplacian matrix :

i. intra-layer coupling strength going through evolution
ii. inter-layer coupling strength

- The fixed layer introduces a limit on the efficiency of optimization

Single Network Framework

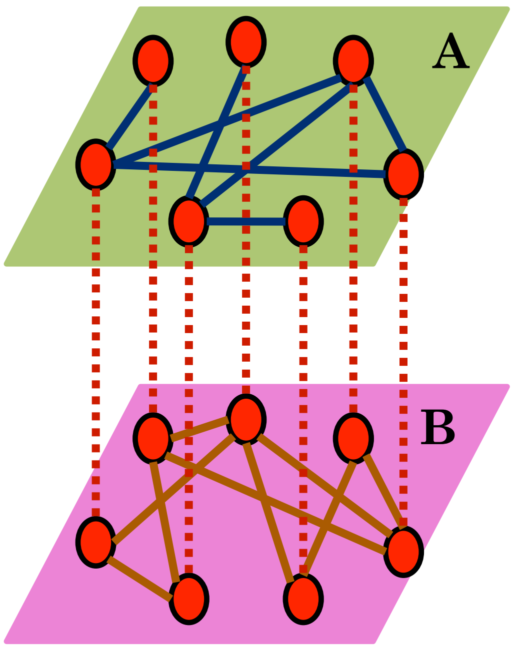
- Factors supporting synchronization:
  - Homogeneous degree distribution
  - Smaller diameter

L. Donetti, et al., Phys. Rev. Lett. 95, 188701 (2005)

Multiplex Networks:

- The layer going through evolution attempts to achieve above features
- The fixed layer applies limitation

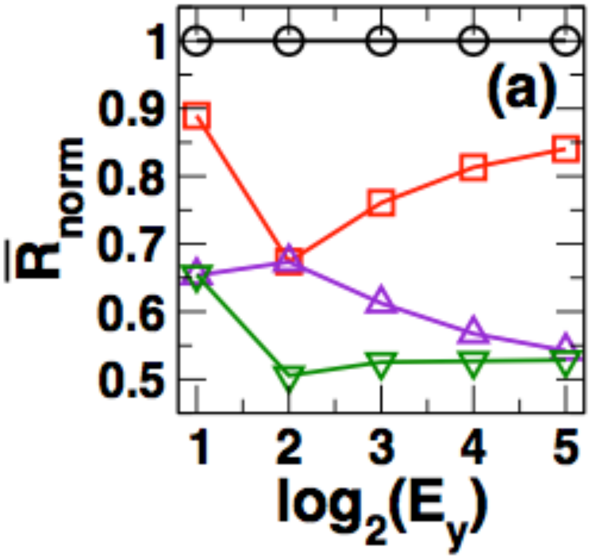
- Additional parameter: *impact of one type of interaction on another type (through inter-layer coupling strength)*



Case I:

$$M = \begin{bmatrix} A & D_x I \\ D_x I^T & E_y B \end{bmatrix}$$

- Inter-layer coupling strength is weak (circle)
- The layer with weaker intra-layer coupling strengths is rewired and layer with stronger intra-layer coupling strengths $E_y > 1$ is fixed



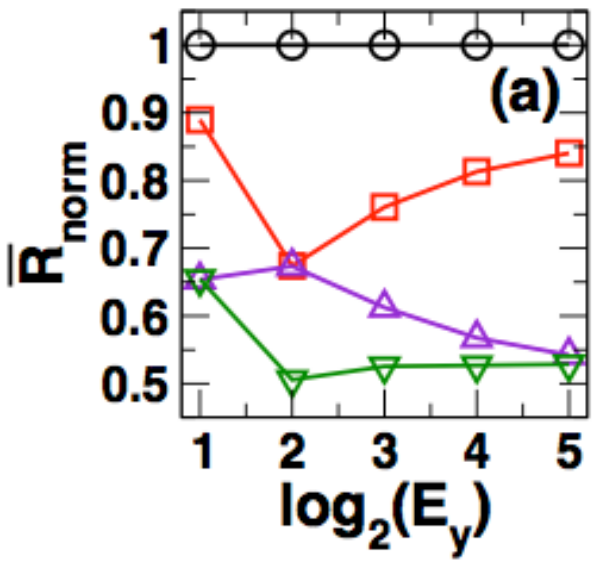
Optimization does not work

The fixed layer introduces a limit on the efficiency of optimization by governing eigenvalues spectra of the entire network

Case II:

- Inter-layer coupling strength is strong, i.e. D_x **>> 1** (square)
- The layer with *weaker intra-layer coupling strengths is rewired* and layer with stronger intra-layer coupling strengths is fixed *(square symbol)*

$$M = \begin{bmatrix} A & D_x I \\ D_x I^T & E_y B \end{bmatrix}$$



Optimization works !!

The fixed layer introduces a limit on the efficiency of optimization by governing eigenvalues spectra of the entire network: ***inter-layer couplings dominates***

✓There is a trade-off between inter-layer coupling and the largest and first non zero eigenvalue

✓Using S. Ribalta et. al. (2013) we get:

(1) For $D_x \ll 1$

$$R \approx \frac{\max_{\alpha}\left[\lambda_{\max}(L^{\alpha}) + D_x\right]}{2D_x}$$

(2) For $D_x \gg 1$

$$R \approx \frac{2D_x + \sqrt{2}\lambda_{\max}(L^{\mathrm{AV}})}{\lambda_2(L^{\mathrm{AV}})}$$

Where L^{α} is Laplacian of α th layer. L^{AV} is the average Laplacian of two layers

✓R is a decreasing function of D_x for weaker inter-layer coupling strength, and an increasing function for the stronger strength

Case I:

(1) For $D_x \ll 1$ $R \approx \frac{\max_{\alpha}\lfloor \lambda_{\max}(L^{\alpha}) + D_x \rfloor}{2D_x}$

- D_x is fixed
- $\lambda_{\max}$ is governed by the fixed layer having stronger intra-layer coupling strength

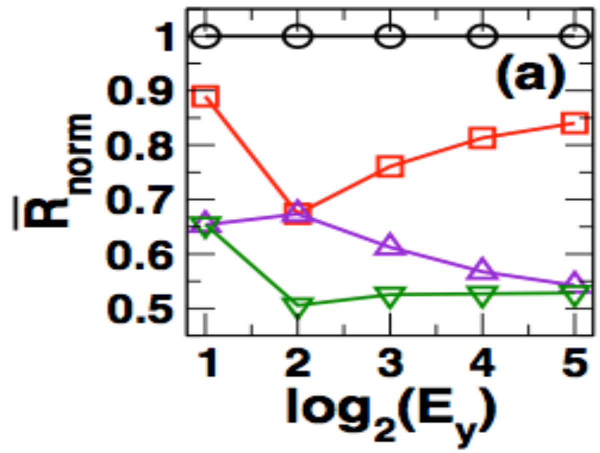
Since R is dominated by $\lambda_{\max}$ *resulting in efficiency of optimization*

$$R_{norm} = 1$$

Case II:

(2) For $D_x \gg 1$

$$R \approx \frac{2D_x + \sqrt{2}\lambda_{\max}(L^{\mathrm{AV}})}{\lambda_2(L^{\mathrm{AV}})}$$



✓ Structural changes caused by the evolution process are capable of steering towards larger values λ_2, resulting smaller R values

<u>*optimization is successful*</u>

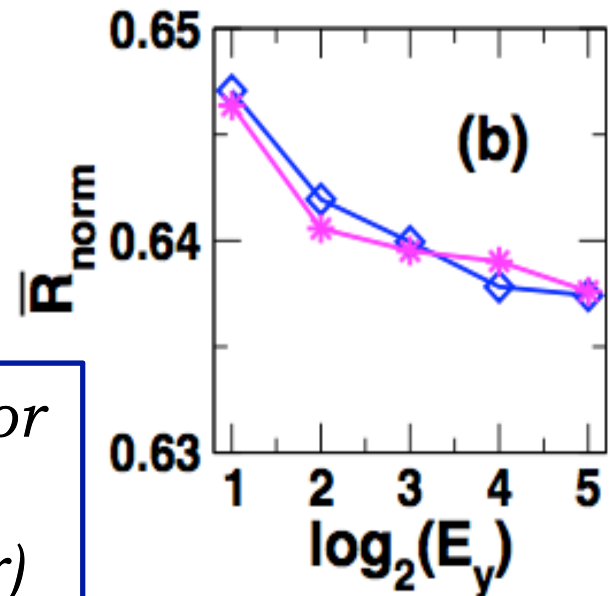
✓ For further increase in D_x, above Eq. holds even better for the R values, and there is an increase in the efficiency of the optimization

✓ For very large intra-layer coupling of the fixed layer, contribution of the fixed layer becomes large in L^{AV} resulting in decrease in the efficiency

Single vs. both layers rewiring:

<u>Weak Inter-layer coupling:</u>

- For the single layer evolution rewiring is done in the layer with stronger intra-layer couplings



Efficiency of optimization is same for single layer rewiring (square) and when both layers are rewired (star)

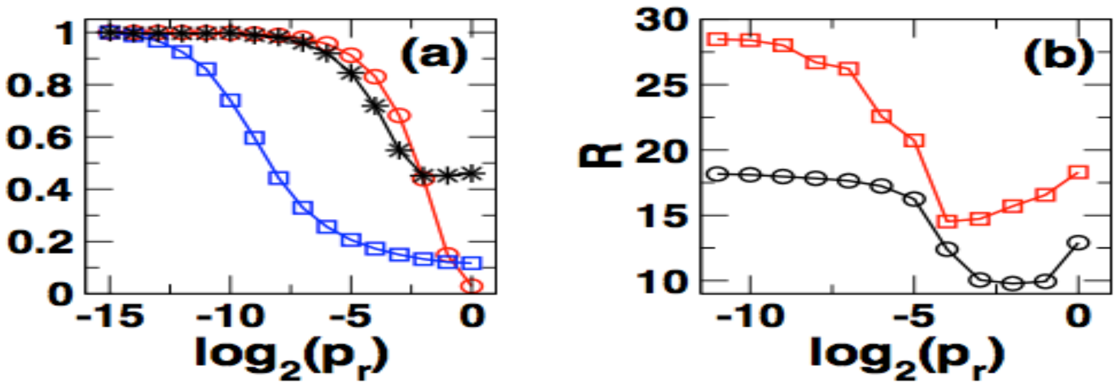
For both the cases, R depends on λ_{max} which is only determined by the layer having the stronger intra-layer coupling strength

- We show that synchronizability of whole *multiplex network* can be optimized by only *rewiring a single network layer*

Optimization complexity can be drastically optimized

Change in Network Architecture of *Fixed Layer*

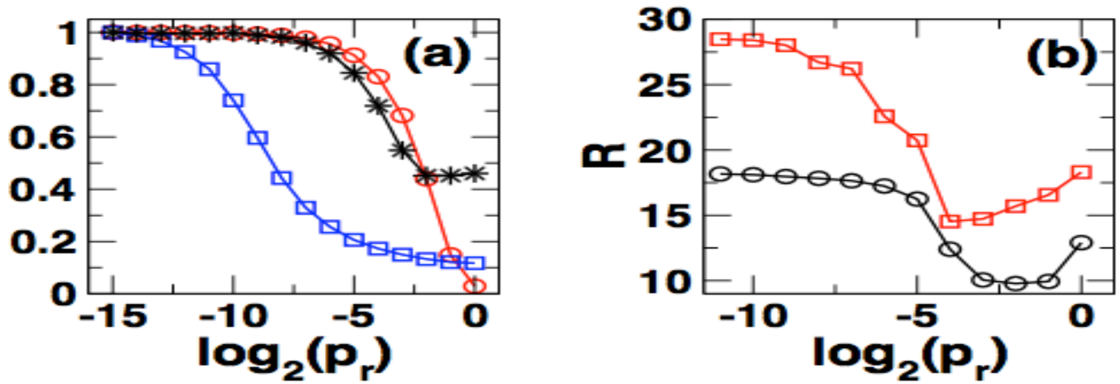
Fixed layer being represented by small-world
(high clustering coefficient and small characteristics path length)



Interplay between the degree homogeneity of the fixed layer and the layer going through the optimization

- ❑for small p_r, initial and optimized multiplex networks have same synchronizability
- ❑For larger values, synchronizability increases and attains maximum value (the lowest R value) at a rewiring probability much higher than the critical parameter

Continued…

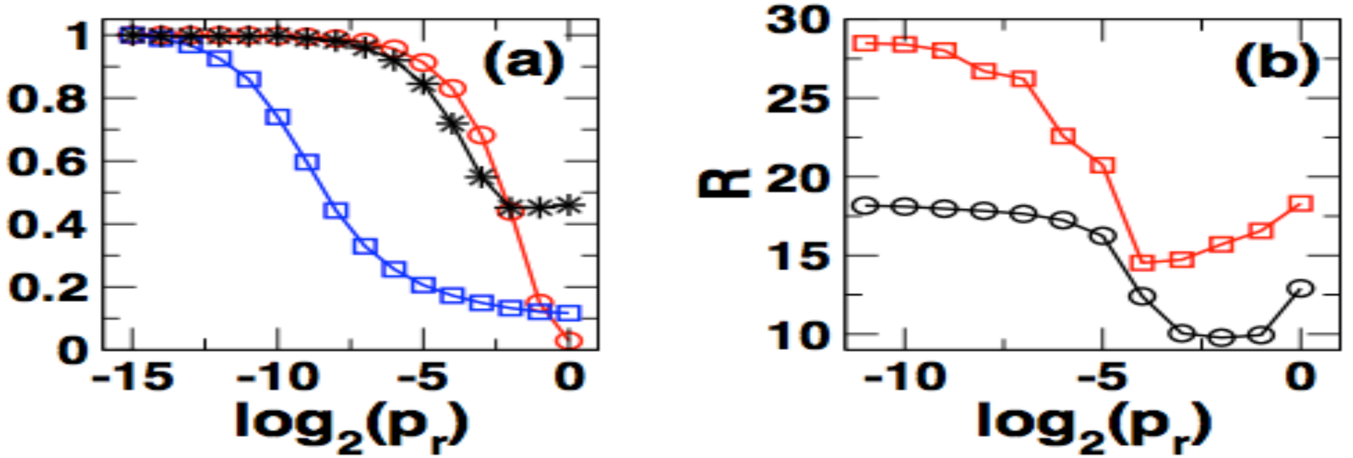


- Before the SW transition, diameter of fixed layer is large:
  poor synchronizability of the entire multiplex network

- Above the SW transition, as long as the fixed layer has small degree heterogeneity, the optimized multiplex networks posses following factors
  - (1) small degree homogeneity for both the fixed layer and the layer experiencing the rewiring
  - (2) smaller value of the average path length and the diameter of the entire multiplex networks.

  better synchronizability

Continued…



- For the fixed layer with p_r = 1 or close to 1
  - ✓Small diameter and average path length
  - ✓High degree heterogeneity of the fixed layer which does not get balanced by the rewiring of another layer

<u>*poor synchronizability of the optimized network*</u>

GIST

- There are several pathways to improve synchronizability of multiplex networks
  - by altering parameters such as those that promote integration of the layers (increasing the inter- layer coupling strength)

- Optimization of a single layer can achieve a network that is roughly as capable to synchronize as the network where all the layers are evolved

- Results are particularly relevant to improve synchronization of systems where only one layer is accessible or when one wants to optimize a system in a very cost effective fashion

Importance of Optimization:

- Optimization of complex networks is behind the success of technological as well as natural adaptive processes
- The brain learns by rewiring its synaptic connections. Deep learning machines changes internal structures of its neural network to optimize its logical outputs
- It is a current scientific challenge to understand natural optimization processes in order to reproduce it
- The difficulty lies on the fact that optimization complexity increases exponentially by the size of the system
- We show that synchronizability of a whole multiplex network can be optimized by only rewiring a single network layer

Optimization complexity can be drastically optimized

Future Aspects

- Networks theory has proven its aptness in providing insights into controllability at a fundamental level

- In traditional approaches, external inputs are imposed to affect the dynamics of few nodes causing control of the entire system

  Y. Y. Liu et. al. Nature 473, 167 (2011)

- Our work refines the concept of controllability: by addition of a new system (one layer) we can change the dynamical evolution of the entire system (multiplex) to a desired behavior

- It complements works on controllability by creating more synchronous evolved networks that could be more controllable ??

Network Spectra + Optimization + Multilayer Framework

- Optimization of Synchronizability by Rewiring one Layer,
  Dwivedi, Baptista and SJ* (preprint)
- Evolution of Correlated Multiplexity in Multiplex networks
  Dwivedi and SJ*, Phys. Rev. E 95, 022309 (2017)

- Cluster synchronization in multiplex networks: Impact of change in properties of one layer on dynamical evolution of coupled units in other layers
  SJ et. al., EPL 113 30002 (2016)
  SJ et. al., Phys. Rev E 94, 062202 (2016)
  Singh, Ghosh, SJ* and J. Kurths, EPL 111, 30010 (2015)

- Existence of various structural patterns in real world systems: Eigenvalues and optimization
  SJ et. al., EPL 112, 48003 (2015)
  Dwivedi, Sarkar and SJ*, EPL 111 10005 (2015)

- Optimized evolution of networks for principle eigenvector localization
  -Pradhan, Yadav, SJ*, arXiv:1701.03576

http://www.iiti.ac.in/~sarika

Complex Systems Lab@IITI

Research Scholars/Post-docs

Aradhana Singh

Aparna Rai

Sanjiv Dwivedi

Alok Yadav

Camellia Sarkar

Pramod Shinde

Saptarshi Ghosh

Anil Kasnia

Interns:

Loic Marrec (Uni of Paris)

Parth Chholak (IIT Bombay)

Madhoolika Bisht (MIT Manipal)

Ankit Mishra (IIT Indore)

Collaborators:

- Jurgen Kurths, PIK Berlin
- Anna Zakharova, TU Berlin
- Murilo S. Baptista (University of Aberdeen)
- Boccaletti, CNR Institute
- Himanshu Sinha, Dept of Biotechnology, IITM

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