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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Onset of of thermalisation in *turbulence*

Insights from the Burgers equation

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with **Divya Venkataraman, Proceedings of the Royal Society A 473 (2017)**

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Background

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- Numerical solutions of the Navier-Stokes equation, for *small enough* viscosities, consistent with turbulent flows.
  - Energy cascade.
  - Intermittency and Multiscaling.
  - $E(k) \sim k^{-5/3}$.
- Numerical solutions of the Burgers equation, *small enough* viscosities, consistent with theoretical results.
  - Energy cascade.
  - Bifractal.
  - $E(k) \sim k^{-2}$.

Background

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Fact

Numerical solutions of the Euler (Navier-Stokes with $\nu = 0$) or the inviscid Burgers equation (*entropy solutions*) very different from the *actual* solutions.

Reason

On the computer what we solve is not the infinite-dimensional partial differential equation but their finite-dimensional Galerkin-truncated *avatars*.

Consequence

Finite-dimensional, inviscid system which conserves energy and volume in phase space yielding thermalised solutions with

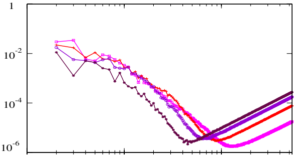
$$E(k) \sim k^{d-1}$$

Footnote

Viscosity saves the day when solving the viscous Navier-Stokes or Burgers equation.

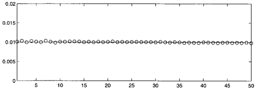
E. Hopf, Commun. Pure Appl. Math. (1950)
T. D. Lee, Q. J. Appl. Math. (1952)

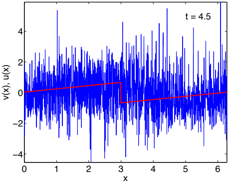
Consequence of Galerkin Truncation



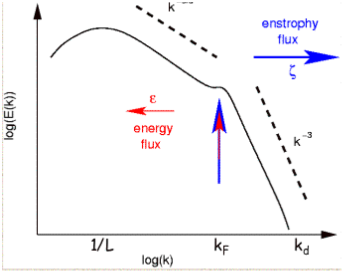
Cichowlas, Bonaiti, Debbash, and Bracket,
Phys. Rev. Lett. (2005)

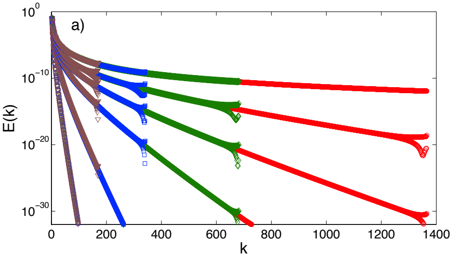
Majda & Timofyev,
Proc. Natl. Acad. Sci. (2000)





Ray, Frisch, Nazarenko, and Matsumoto,
Phys. Rev. E (2011)





Kraichnan, 1967

Bustamante & Brachet, Phys. Rev. E (2012)

Statistical Physics

Applied Mathematics

Galerkin Truncation: The Burgers Equation

- The inviscid Burgers equation:

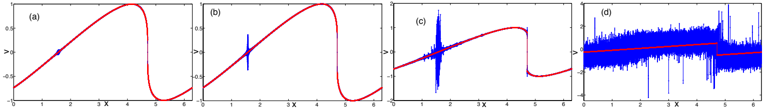
$$\partial_t u + \partial_x (u^2/2) = 0; \qquad u(x,0) = u_0(x).$$

- Define the Galerkin projector, via the truncation wavenumber $K_G \in \mathcal{Z}^+$:

$$\mathrm{P}_{K_\mathrm{G}} u(x) = \sum_{|k| \leq K_\mathrm{G}} \mathrm{e}^{\mathrm{i}kx} \hat{u}_k .$$

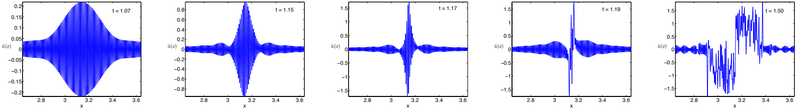
- The associated Galerkin-truncated Burgers equation

$$\partial_t v + \mathrm{P}_{K_\mathrm{G}} \partial_x (v^2/2) = 0; \qquad v_0 = \mathrm{P}_{K_\mathrm{G}} u_0 .$$



Ray, Frisch, Nazarenko, and Matsumoto, Phys. Rev. E (2011)
Banerjee & Ray, Phys. Rev. E (R) (2014)
Venkataraman & Ray, Proc. Roy. Soc. A (2017)

Harbinger of Thermalisation: Tygers



A resonant particle-wave interaction leads to thermalised solutions, asymptotically, for the truncated Euler or Burgers equation.

Question: Can we predict the time scale τ_c for the onset of thermalisation?

Answer: $\tau_c \sim {K_{\mathrm{G}}}^{\xi} \qquad \xi = -4/9$

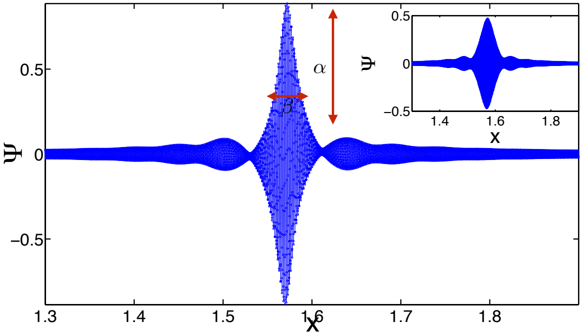
Blake, *Songs of Experience* (1794)
Ray, Frisch, Nazarenko, and Matsumoto, Phys. Rev. E (2011)
Banerjee & Ray, Phys. Rev. E (R) (2014)
Venkataraman & Ray, Proc. Roy. Soc. A (2017)

Outline of Theory

Ansatz

$$\alpha \sim \tau^{\gamma_\alpha} K_G^{\delta_\alpha}$$

$$\beta \sim \tau^{\gamma_\beta} K_G^{\delta_\beta}$$



Onset time: Coincident with bulge asymmetry

Bulge asymmetry: Finite Reynolds stress

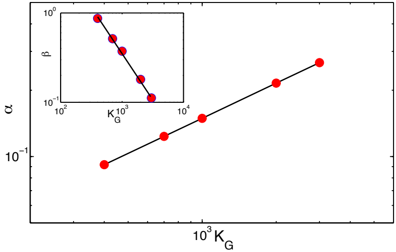
$$\alpha^2/\beta = \mathcal{O}(1) \implies \tau_c \sim {K_G}^{\xi}$$

$$\boxed{\xi = \frac{\delta_\beta - 2\delta_\alpha}{2\gamma_\alpha - \gamma_\beta}}$$

Venkataraman & Ray, Proc. Roy. Soc. A (2017)

Fixing the Exponents

- Resonance and phase mixing: $\gamma_\beta = \delta_\beta = -1$
- Energy conservation: $2\delta_\alpha + \delta_\beta = 0 \implies \delta_\alpha = 1/2$
- The cubic singularity of the entropy solution: $\gamma_\alpha = 7/4$

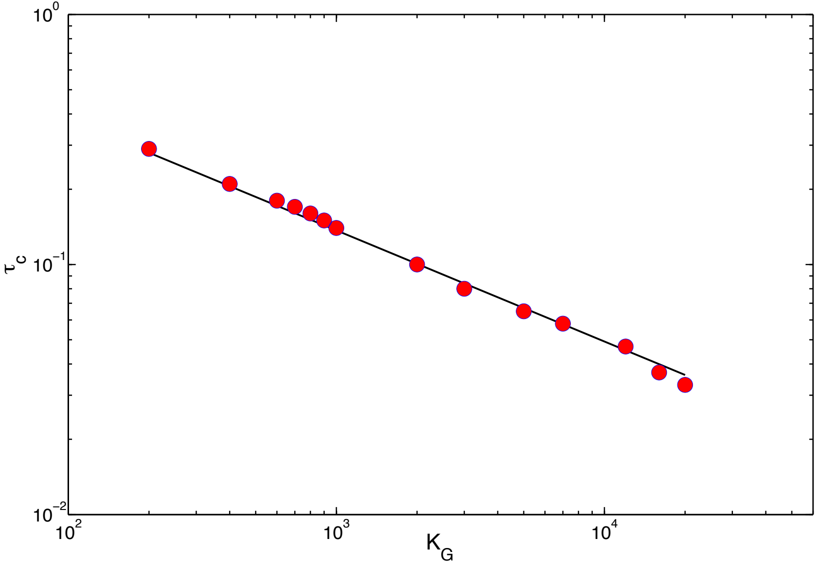


$$\xi = \frac{\delta_\beta - 2\delta_\alpha}{2\gamma_\alpha - \gamma_\beta} = -4/9$$

	$\alpha \sim \tau^{\gamma_\alpha} K_G{}^{\delta_\alpha}$		$\beta \sim \tau^{\gamma_\beta} K_G{}^{\delta_\beta}$		$\tau_c \sim K_G{}^{\xi}$
	γ_α	δ_α	γ_β	δ_β	ξ
Theory	7/4	1/2	-1	-1	$\xi = \frac{\delta_\beta - 2\delta_\alpha}{2\gamma_\alpha - \gamma_\beta}$ = -4/9
Simulations	1.74 $\pm$ 0.04	0.50 $\pm$ 0.01	-0.97 $\pm$ 0.08	-1.01 $\pm$ 0.02	-0.46 $\pm$ 0.07

Venkataraman & Ray, Proc. Roy. Soc. A (2017)

Onset of Thermalisation



Venkataraman & Ray, Proc. Roy. Soc. A (2017)

Summary

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- Implication
  - Numerical simulations for tracing complex singularities, in ideal equations of hydrodynamics, by using the method of the analyticity strip.
    - The temporal measurement of the distance, to the real domain, of the nearest singularity, is limited not only by computing power but also by the onset of thermalisation.
- Perspective
  - Does a similar scaling argument hold for the incompressible three-dimensional Euler equation?
  - The onset of thermalisation is accompanied by the generation of Fourier harmonics other than K_{G}. The precise mechanism of this is yet to be understood.
  - A systematic theory which explains the full transition to thermalised states, accompanied by *beating* effects.