Random fiber bundle model: how loading paths matter

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Introduction

Strength of materials is an important issue in physics, engineering and industry.

Fracture and material stability have interested humanity for practical reasons ever since we started using tools.

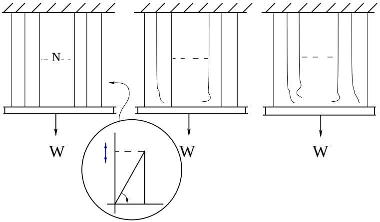
From Leonardo da Vinci to Galileo to modern days...... there have been attempts to understand the properties of fibers and beams

In a broader sense such systems can be,

- disordered solids under stress
- power-grid networks carrying currents
- roads carrying vehicular traffic
- computer circuitry

etc.

Fiber bundle model



A collection of elastic objects connected in parallel and clamped to a medium that transmits forces between the fibers.

Peirce, 1926

The elongation of a fiber is linearly related to the force it carries, up to a maximum value.

When this value is reached, the fiber fails by no longer being able to carry any force.

Random Fiber bundle model (RFBM):

Pradhan, Hansen and Chakrabarti, RMP 82, 499 (2010); Kawamura et al RMP 84, 839 (2012)

The threshold value is assigned from some initially chosen probability distribution and does not change thereafter.

RFBM: details

Let there be N fibers. Load placed $W = \sigma N$

The load may be placed entirely in the first step (sudden loading) or in parts (quasi-static loading)

As a load is placed, a fiber with threshold less than the load it carries will break

This load will be shared by others.

Subsequently more fibers break and the process continues.

End result: either a finite fraction remains unbroken or the whole system collapses

Two aspects to be addressed today:

- Redistribution scheme
- Loading process

In all cases, a critical load per fiber, above which all fibers fail exists.
Known results with uniform loading and uniform global redistribution of load:
$\sigma_c = 0.25$

(when the thresholds are ≤ 1 and naturally $\sigma \leq 1$)

While in some cases (e.g. solids under stress) such mechanisms are properties inherent to the system, in many other cases (e.g. power-grids, traffic controls) it is a matter of design that can be optimized to achieve maximum robustness.

Such robustness properties of networks under targeted or random attacks have received substantial attention in recent works.

$\sigma_c = 0.25$ with equal load sharing, uniform initial loading.

Question asked: Can one attain a larger σ_c by a different scheme of load sharing in systems which can be controlled?

First consider the case with uniform initial loading.

Sudden loading and a theoretical limit

First step: load $W = \sigma N$ is placed uniformly.

Let some fibers break but let the redistribution be such that the rest get load exactly equal to their threshold

Condition for maximum σ_m is then

$$\int_{\sigma_m}^{1} \sigma d\sigma = \sigma_m$$

Or,

$$\sigma_m = \sqrt{2} - 1 > 0.25$$

Thus a way to achieve the maximum is using a redistribution scheme such that each surviving fiber has load (after redistribution) equal to its strength f_i in a one step process.

Consistent with the natural expectation for a better scheme that the stronger fibers should get a higher share.

Effectively stronger fibers are those which have higher *difference* between the failure threshold and the load they currently carry.

Claim: wiser to put more load to fibers which have higher value of $f_i - \sigma_i$ (threshold - current load)

New general scheme proposed:

The load received by the i-th fiber having present stress σ_i and failure threshold f_i is

$$x_i = A(f_i - \sigma_i)^b w_e$$

w_e is the excess load to be redistributed in a given step; b is a parameter.

Using this, the exact value of σ_c for $b = 1$ can be calculated which is equal to σ_m.

Details of calculations:
The load conservation) condition reads

$$\sum_{i \in survivingfibers} \int_{\sigma}^{1} \left[A(f_i - \sigma)^b \sigma^2 N + \sigma \right] P(f_i) df_i = \sigma N, \qquad (1)$$

with σN being the applied load and $P(f) \propto 1/(1-\sigma)$
(the normalized distribution of thresholds of remaining fibers),

giving $A = (b+1)/(N(1-\sigma)^{b+1})$

Hence the load received by the *i*-th fiber in the first redistribution step, having failure threshold f_i is

$$x(f_i) = \frac{\sigma^2(b+1)}{(1-\sigma)^{b+1}}(f_i - \sigma)^b. \qquad (2)$$

The redistribution will occur only once if after the first redistribution all fibers have load lower than their respective thresholds i.e., $x(f_i) + \sigma < f_i$ for all surviving fibers.

For $b = 1$, this inequality leads to

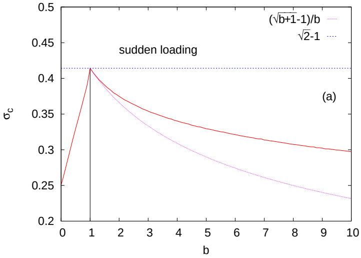
$$f_i \left[\frac{2\sigma^2}{(1-\sigma)^2} - 1 \right] > \sigma \left[\frac{2\sigma^2}{(1-\sigma)^2} - 1 \right], \tag{3}$$

which is satisfied up to a critical σ_c where $2\sigma_c^2/(1-\sigma_c)^2 = 1$ or, $\sigma_c = \sqrt{2} - 1$.

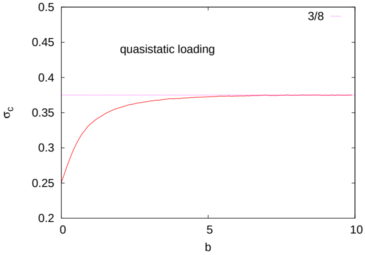
But this is the maximum possible sustainable load per fiber value, implying for $b = 1$, the maximum strength is achieved and that is a single step redistribution!

The case discussed is for sudden loading. One can also study the quasi-static loading case.

Results for $\sigma_c(b)$ for both cases



Sudden loading

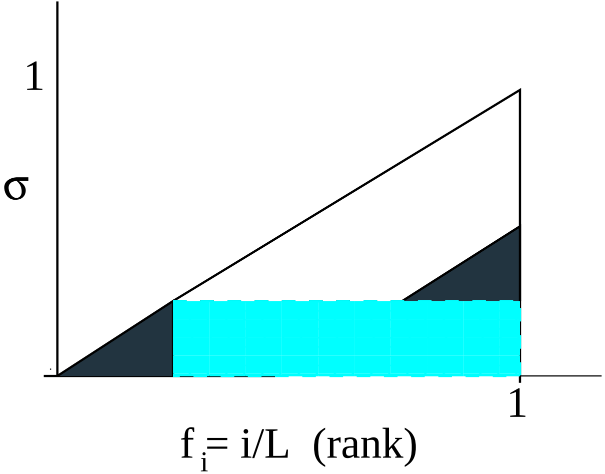


Quasi-static loading

Only for $b = 0$, results are independent of the loading schemes.

Can the saturation value for gradual loading be calculated? Yes!

Suppose $b >> 1$. Only the strongest of the fibers will share the extra load - with $f_i - \sigma_i$ identical for each.



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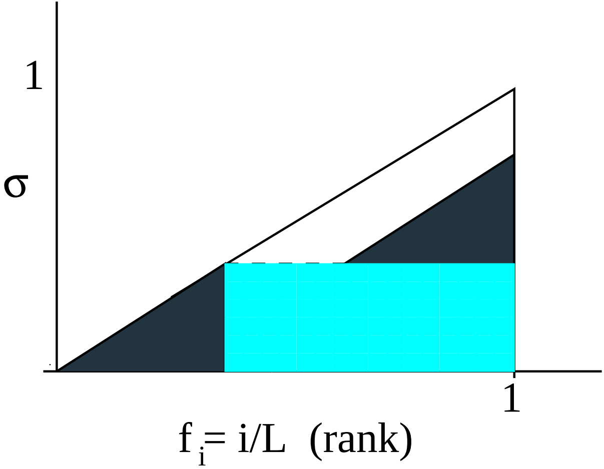
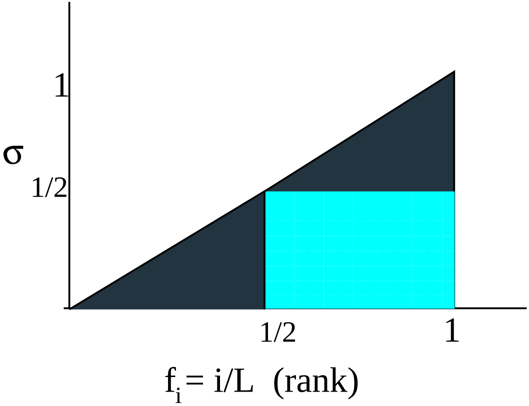


The system will no longer be able to support load when the minimum value of the threshold of the surviving fibers is 0.5



Now the total load $= \int_{1/2}^{1} f df = \sigma$

Or, $\sigma = 3/8$ which agrees very well with the numerical saturation value.

Ongoing work

So far considered the case when *all* fibers are loaded equally in the initial step.

What happens if the loading scheme is different? Let the fibers be loaded differently at the initial step (and in further steps as well).

There can be in principle, infinite number of paths. We need to consider some of the extreme cases.

Aim: To show that the results are highly dependent on the path and the existence of asymmetry.

As an extreme example, consider only a single fiber is loaded with load δ.
If $\delta >$threshold, breaks and redistribution takes place. Further breakings may take place.
After steady state is reached, another randomly selected fiber is loaded with δ.
The critical load σ_c is the total load placed in this manner when the system breaks

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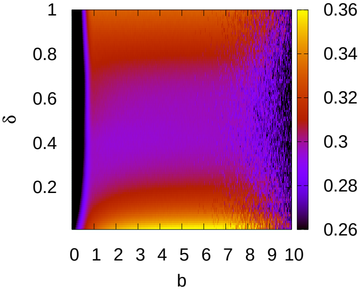
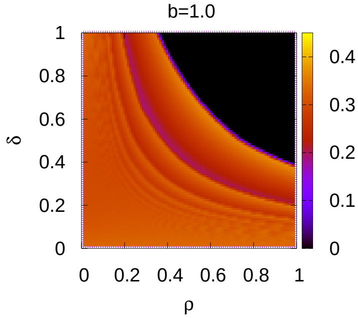
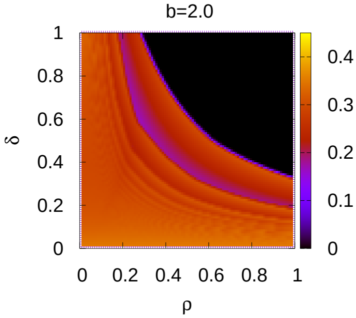
- Very small $b \implies$ homogeneous case; paths do not matter, $\sigma_c = 0.25$
- Very large b and small δ resembles quasi-static loading. σ_c close to 0.38
- Large δ - probability that the loaded fiber breaks is large - after redistribution small loads δ/N distributed to each - not far from the quasi-static case.



$\sigma_c \geq 0.25$ in general.

Consider now instead of one, a fraction ρ is loaded.





Usual effect of $b > 0$ observed here.

A third (and probably most interesting!) case:

A one step process (sudden loading) where a load q per fiber is put to p fraction of fibers ($b = 0$).

When $p = 1$, case of uniform (global) loading, the critical load at which the system breaks is $q_c = 1/4$.

Now, if $q = 1$ per fiber is placed initially on p fraction, all fibers where the load is first applied, are broken. The load ($= pN$) then gets redistributed to the $(1 - p)$ fraction of fibers uniformly.

Effectively, a load pN is applied to a system of $(1 - p)N$ fibers equally. Using the critical condition,

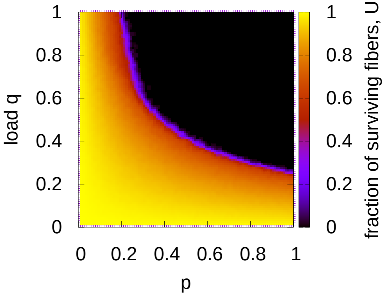
$$\frac{pN}{(1-p)N} = 0.25$$

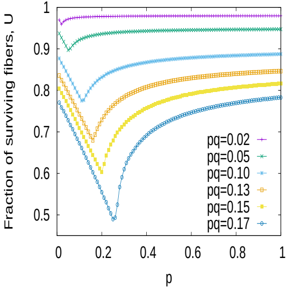
A critical value $p_c = 1/5$.

These extreme limits show that the total loads placed in the two cases are different for breakdown! ($0.25N$ in one case and $0.2N$ in the other).

Let the total load be constant, i.e., $pq =$ constant; things still depend on p (or q)

This is what we call the asymmetry in the system





Concluding remarks

- Loading paths are shown to strongly affect random fiber bundle models, with respect to redistribution scheme as well as loading rules.
- A modified load redistribution scheme leads to a larger value of the critical σ and can improve the strength of systems which can be controlled. Estimates of σ_c can be made analytically in certain cases.
- Exponents and relaxation behaviour grossly different from familiar models (not discussed)
- If in the initial step, all fibers are not loaded equally, results depend on the loading paths.
- Cases where both redistribution rule and non-uniform loading is allowed are discussed.
- An asymmetry exists between individual loads and fraction of fibers loaded. Total load does not govern the dynamics.

Thank you!