Exact phase diagram of a multispecies exclusion process

Arvind Ayyer

Department of Mathematics
Indian Institute of Science, Bangalore

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(joint with Dipankar Roy)

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The Model

- Two-species semipermeable ASEP with L sites.
- Particles of type $+, -$ as well as vacancies.
- Number n_0 of vacancies conserved.
- Bulk rules (action of an electric field)

$$+\,0 \underset{q}{\overset{1}{\rightleftharpoons}} 0\,+$$

$$0\,- \underset{q}{\overset{1}{\rightleftharpoons}} -\,0$$

$$+\,- \underset{q}{\overset{1}{\rightleftharpoons}} -\,+$$

- Left boundary

$$- \underset{\gamma}{\overset{\alpha}{\rightleftharpoons}} +$$

- Right boundary

$$+ \underset{\delta}{\overset{\beta}{\rightleftharpoons}} -$$

Matrix Ansatz

- Let $\tau = (\tau_1, \ldots, \tau_L)$ be a configuration.
- The steady state can be computed using

$$P(\tau) = \frac{1}{Z(L,n)} \langle W | X_{\tau_1} \ldots X_{\tau_L} | V \rangle.$$

- The algebra of the matrices and boundary vectors is given by

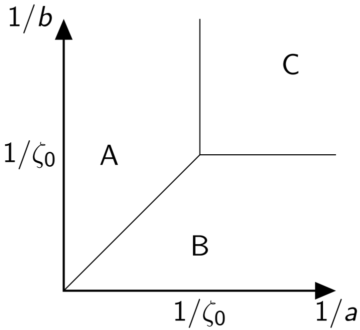
$$\begin{aligned} X_+X_- - qX_-X_+ &= X_+ + X_- \\ X_+X_0 - qX_0X_+ &= X_0 \\ X_0X_- - qX_-X_0 &= X_0 \\ (\beta X_+ - \delta X_-)|V\rangle &= |V\rangle \\ \langle W|(\alpha X_- - \gamma X_+) &= \langle W|. \end{aligned}$$

- Solved by M. Uchiyama (*Chaos, Solitons and Fractals*, 2007)

The Phase Diagram

Take $L \to \infty$ such that $n_0/L \to \rho_0$.

$$\begin{aligned} a &= \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \\ b &= \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}, \\ \zeta_0 &= \frac{1+\rho_0}{1-\rho_0}. \end{aligned}$$



Densities and Currents

Region	Density of $+$	Current of $+$
A	$\frac{1}{1+a}$	$(1-q)\frac{a}{(1+a)^2}$
B	Piecewise constant	$(1-q)\frac{b}{(1+b)^2}$
C	$\frac{1-\rho_0}{2}$	$(1-q)\frac{1-\rho_0^2}{4}$

Simulations

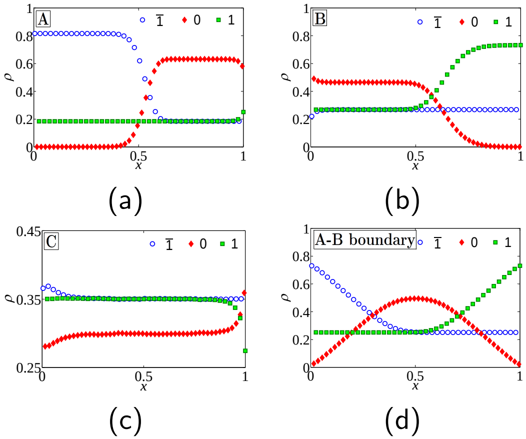
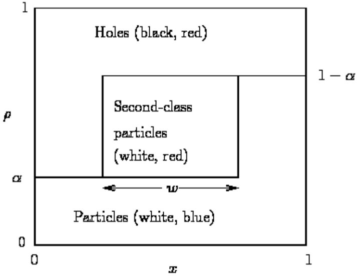


Figure: Plots of the densities of $+$ (green squares), 0 (red diamonds), and $-$ (blue circles), versus the scaled position for the semipermeable ASEP with $n = 200$, $\theta_0 = 0.3$ and $\theta_1 = 0.7$.

The Fat Shock

0's form a clump, a region which we call the **fat shock**.
All qualitative features are explained by this construction.
Proved rigorously for the semipermeable TASEP [A., Lebowitz, Speer, *J. Stat. Phys* 2009]



The Model

- Introduced recently by Cantini, Garbali, de Gier and Wheeler (*J Phys. A, 2016*).
- One-dimensional lattice of size L
- r species of charges, denoted j and $\bar{j} \equiv -j$, and 0's.
- Total number of charges of species j is n_j.
- Number n_0 of vacancies conserved.
- Bulk rules (action of an electric field)

$$j\,k \underset{q}{\overset{1}{\rightleftharpoons}} k\,j \quad \text{if } j > k$$

- Left boundary

$$\bar{j} \underset{\gamma}{\overset{\alpha}{\rightleftharpoons}} j$$

- Right boundary

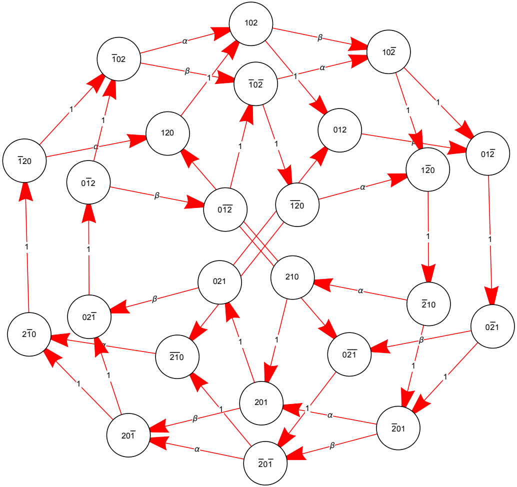
$$j \underset{\delta}{\overset{\beta}{\rightleftharpoons}} \bar{j}$$

Basic properties

- *Ergodic*.
- Even the mTASEP ($q = \gamma = \delta = 0$) is ergodic.
- Charge-conjugation symmetry

$$
\begin{array}{ccc}
j & \longleftrightarrow & \bar{j}, \\
1 & \longleftrightarrow & q, \\
\alpha & \longleftrightarrow & \beta, \\
\gamma & \longleftrightarrow & \delta \\
\text{change} & \text{direction of motion} &
\end{array}
$$

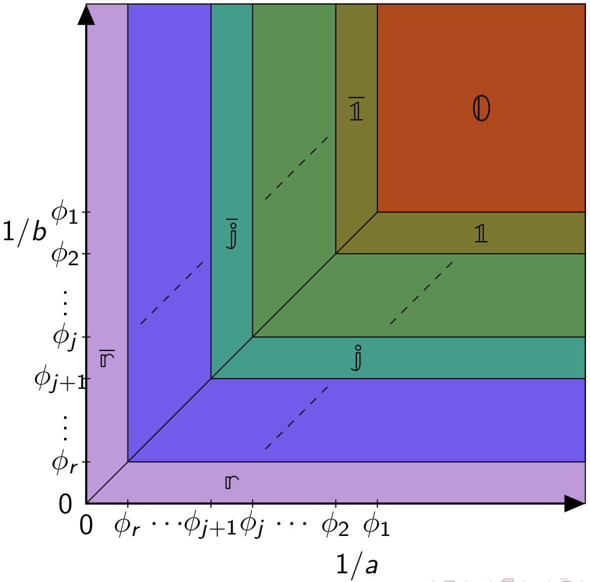
Example: $r = 2$ mTASEP, $n_0 = n_1 = n_2 = 1$



Thermodynamic Limit

- Take $L \to \infty$ and $n_j \to \infty$ for each j such that $n_j/L \to \theta_j > 0$.
- $\Theta_k = (\theta_k + \cdots + \theta_r)/2$
- $\phi_k = \Theta_k/(1 - \Theta_k)$ for $1 \leq k \leq r$
- Let $f(x) = 1/(1 + x)$.

Phase diagram



Densities

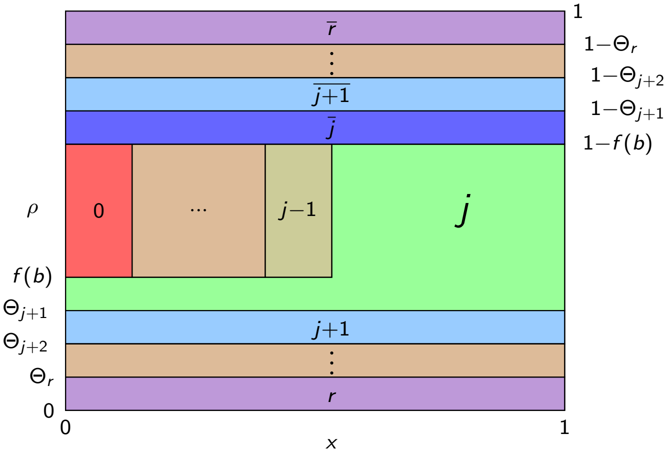
Phase $\downarrow$	Densities in the bulk: Species k	Densities in the bulk: Values of $\rho_k, \rho_{\bar{k}}$
$\mathbb{0}$	$k = 0$	$\rho_0 = \theta_0$
	$1 \leq k \leq r$	$\rho_k = \rho_{\bar{k}} = \theta_k/2$
$\mathbb{j}$	$k = 0$	ρ_0 piecewise constant
	$1 \leq k \leq j-1$	$\rho_{\bar{k}} = 0$ ρ_k piecewise constant
	$k = j$	$\rho_{\bar{j}} = f(b) - \Theta_{j+1}$ ρ_j piecewise constant
	$j+1 \leq k \leq r$	$\rho_k = \rho_{\bar{k}} = \theta_k/2$
$\bar{\mathbb{j}}$	$k = 0$	piecewise constant
	$1 \leq k \leq j-1$	$\rho_k = 0$ $\rho_{\bar{k}}$ piecewise constant
	$k = j$	$\rho_j = f(a) - \Theta_{j+1}$ $\rho_{\bar{j}}$ piecewise constant
	$j+1 \leq k \leq r$	$\rho_k = \rho_{\bar{k}} = \theta_k/2$

Currents

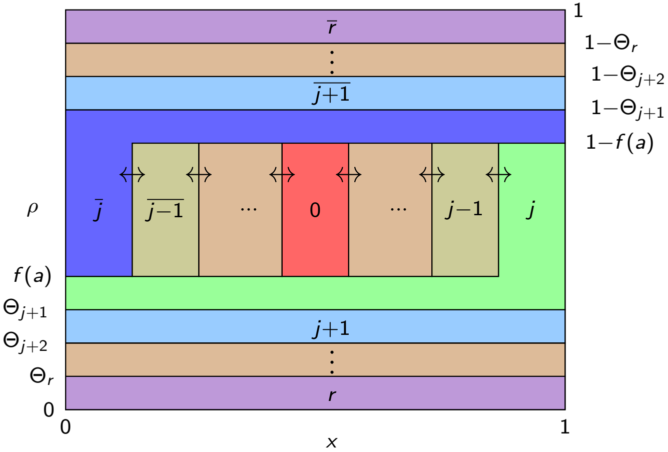
Phase	Currents	
$\downarrow$	Species k	Value of J_k
$\mathbb{0}$	$1 \leq k < r$	$(1-q)(\Theta_k - \Theta_{k+1})(1 - \Theta_k - \Theta_{k+1})$
	$k = r$	$(1-q)\Theta_r(1-\Theta_r)$
$\mathbb{j}$	$1 \leq k \leq j-1$	0
	$k = j$	$(1-q)\left(f(b)(1-f(b)) - \Theta_{j+1}(1-\Theta_{j+1})\right)$
	$j+1 \leq k < r$	$(1-q)(\Theta_k - \Theta_{k+1})(1 - \Theta_k - \Theta_{k+1})$
	$k = r$	$(1-q)\Theta_r(1-\Theta_r)$
$\bar{\mathbb{j}}$	$1 \leq k \leq j-1$	0
	$k = j$	$(1-q)\left(f(a)(1-f(a)) - \Theta_{j+1}(1-\Theta_{j+1})\right)$
	$j+1 \leq k < r$	$(1-q)(\Theta_k - \Theta_{k+1})(1 - \Theta_k - \Theta_{k+1})$
	$k = r$	$(1-q)\Theta_r(1-\Theta_r)$

Nested Fat Shock: Phase $\mathring{j}$

- Dynamical Localisation
- Dynamical Expulsion



Nested Fat Shock: $\mathring{\jmath} - \bar{\mathring{\jmath}}$ boundary



Simulations of the mTASEP with $r = 2$

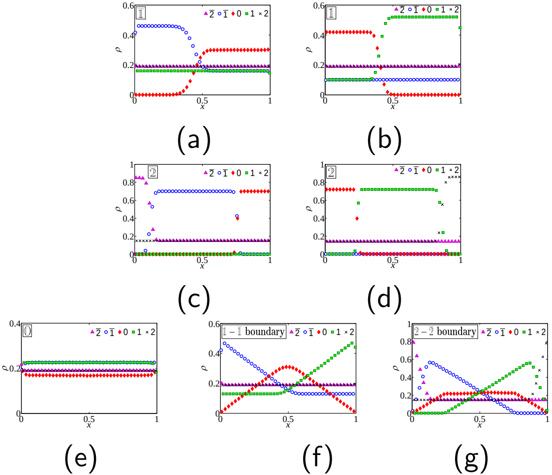


Figure: Plots of the densities of particles 2 (black crosses), 1 (green squares), 0 (red diamonds), $\overline{1}$ (blue circles), and $\overline{2}$ (pink triangles) with $n = 1000$, $\theta_0 = 0.17, \theta_1 = 0.45$ and $\theta_2 = 0.38$

Movie

Motion of the nested fat shock on the $2 - \overline{2}$ shock line.

Thank you for your attention!