Geometric Calderón problem and its connection to several inverse problems

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Calderón problem - nonisotropic case

Consider a matrix $\sigma = (\sigma^{ij}(x))$ and consider the following BVP:

$$\partial_j \sigma^{jk}(x)\partial_k u(x) = 0 \text{ in } \Omega$$
$$u|_{\partial\Omega} = f$$

Define $\Lambda_\sigma : f \to \sigma^{jk}\partial_k u \nu_j|_{\partial\Omega}$.
Question: Does Λ_σ determine σ?
If $F : \Omega \to \Omega$ is a diffeomorphism fixing the boundary, then $\Lambda_{F_*\sigma} = \Lambda_\sigma$.
Here

$$(F_*\sigma)^{jk}(y) = \frac{1}{\det F_*(x)} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \sigma^{pq}(x)|_{x=F^{-1}(y)}.$$

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Geometric Calderón problem

Let (M, g) be a smooth compact Riemannian manifold with smooth boundary ∂M.

Consider

$$\Delta_g u = 0, \quad u|_{\partial M} = f \tag{1}$$

where

$$\Delta_g u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right), \quad \sqrt{g} = \sqrt{\det g}.$$

Consider the Dirichlet-to-Neumann map:

$$\Lambda_g : f \to \sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \nu_i, \text{ where } u \text{ is the solution to (1).}$$

Relation between these problems in $n \geq 3$.

$$g^{jk} = \det(\sigma)^{\frac{2}{n-2}} \sigma^{jk}.$$

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The inverse problem

The inverse problem: Does Λ_g determine g?

Answer is no. As mentioned already, $\psi : M \to M$ is a diffeomorphism fixing the boundary, then $\Lambda_{\psi^* g} = \Lambda_g$.

Conjecture

- *Let M be a smooth compact manifold with boundary with $n \geq 3$ and let g and $\widetilde{g}$ be smooth Riemannian metrics on M such that $\Lambda_g = \Lambda_{\widetilde{g}}$. Then there exists a diffeomorphism $\psi : M \to M$ identity on the boundary such that $g = \psi^* \widetilde{g}$.*
- *Let M be a smooth compact Riemannian manifold with boundary with $n = 2$ and let g and $\widetilde{g}$ be smooth Riemannian metrics on M such that $\Lambda_g = \Lambda_{\widetilde{g}}$. Then there exists a diffeomorphism $\psi : M \to M$ identity on the boundary such that $g = \varphi \psi^* \widetilde{g}$ for some positive function φ on M, $\varphi|_{\partial M} = Id$.*

In 2-dimensions, we also have conformal invariance of the Dircihlet problem. That is, in 2-dimensions, if we consider $\widetilde{g} = \varphi g$ for some smooth positive function φ on M, $\varphi|_{\partial M} = Id$, then $\Lambda_{\varphi g} = \Lambda_g$.

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Known results

Theorem (Lee-Uhlmann,1989)

*Let M be a compact simply connected real-analytic n-manifold with connected real-analytic boundary ∂M, $n \geq 3$. Let g and $\widetilde{g}$ be a real-analytic metrics on ∂M such that $\Lambda_g = \Lambda_{\widetilde{g}}$. Assume that **one** of the following conditions holds:*

- *M is strongly convex with respect to both g and $\widetilde{g}$*
- *either g or $\widetilde{g}$ extends to a complete real-analytic metric on a non-compact real-analytic manifold $\widetilde{M}$ without boundary containing M.*

Then there exists a real-analytic diffeomorphism $\psi : M \to M$ with $\psi_{\partial M} = Id$ and $g = \psi^* \widetilde{g}$.

(M, g) is said to be strongly convex if between any two points p and q in M, there is a unique length minimizing geodesic.

Known results

Theorem (Lassas-Uhlmann, 2001)

Assume that M is either a connected Riemann surface or if $n \geq 3$, (M, g) is a connected real-analytic Riemannian manifold with real-analytic boundary ∂M. Then

- *If $n = 2$, then Λ_g determines the conformal class of the metric g.*
- *If $n \geq 3$, then Λ_g determines the metric g up to the natural obstruction.*

Idea behind cloaking

Theorem (Greenleaf-Lassas-Uhlmann, 2003)

Let $\Omega \subset \mathbb{R}^n, n \geq 3$ *and* $g = g_{ij}$ *be a metric on* Ω. *Let* $D \subset \Omega$ *be such that there is a diffeomoprhism* $F : \Omega \setminus \{y\} \to \Omega \setminus \overline{D}$ *satisfying* $F|_{\partial\Omega} = Id$ *and such that*

$$\mathrm{d}F(x) \geq c_0 I, \quad det(\mathrm{d}F(x)) \geq c_1 \, dist_{\mathbb{R}^n}(x, y)^{-1}.$$

Let $\widetilde{g} = F_* g$ *and* $\widehat{g}$ *be an extension of* $\widetilde{g}$ *into* D *such that it is positive definite in the interior of* D. *Let* γ *and* $\widehat{\sigma}$ *be the corresponding conductivities of* g *and* $\widehat{g}$. *Then* $\Lambda_{\widehat{\sigma}} = \Lambda_\gamma$.

Remark

Note that $\widehat{\sigma}$ *can be changed arbitrarily inside* D *without changing boundary measurements.*

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Let $\Omega \subset \mathbb{R}^n, n \geq 3$ and $g = g_{ij}$ be a metric on Ω. Let $D \subset \Omega$ be such that there is a diffeomoprhism $F : \Omega \setminus \{y\} \to \Omega \setminus \overline{D}$ satisfying $F|_{\partial\Omega} = Id$ and such that

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Note that $\widehat{\sigma}$ can be changed arbitrarily inside D without changing boundary measurements.

Construction of the diffeomorphism

Let $\Omega = B(0,2) \subset \mathbb{R}^3$ be the ball with center 0 and radius 2. Consider $y = 0$ and consider the map

$$F(x) = \left(\frac{|x|}{2} + 1\right)\frac{x}{|x|}.$$

from $\Omega \setminus \{0\} \to \Omega \setminus \overline{D}$. Let $D + B(0,1)$ and consider the homogeneous conductivity $\gamma = 1$ and $\sigma = F_*\gamma$. Now σ can be extended continuously to a function $\widehat{\sigma}$ that is C^∞ smooth in D.

Two movies to illustrate cloaking.

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Boundary rigidity problem

Boundary rigidity problem in Riemannian geometry is the question whether the boundary distance function of a compact Riemannian manifold with boundary determines the metric g.

As before there is a natural obstruction here as well. That is, if $\widetilde{g} = \psi^* g$ with $\psi|_{\partial M} = Id$, then the boundary distance functions of both the metrics are the same.

Some additional hypothesis on the metric required.

One assumption is to assume that (M, g) is simple. (a) ∂M is strictly convex (b) for any point $x \in M$, $\exp_x^{-1} : M \to \exp_x^{-1}(M)$ is a diffeomorphism.

Conjecture

Let (M, g_i) be a compact simple Riemannian manifold with boundary and assume that $d_{g_1}(x, y) = d_{g_2}(x, y)$ for $(x, y) \in \partial M \times \partial M$. Then there exists a diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = Id$ such that $g_1 = \psi^* g_2$.

Positive answer in $n = 2$ by Pestov-Uhlmann, Annals of Math., 2005.

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Boundary rigidity problem in 2 dimensions

Theorem (Pestov-Uhlmann, 2005)

The same set up as in the conjecture with $n = 2$. Then if $d_{g_1} = d_{g_2}$, then $\Lambda_{g_1} = \Lambda_{g_2}$.

Assume this theorem is true, then Lassas-Uhlmann results shows that one can determine g up to the conformal factor and then using a result of Mukhemetov, one can show that the conformal factor is 1.

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Dynamic Dirichlet-to-Neumann map and boundary distance function

Consider the following boundary value problem:

$$(\partial_t^2 - \Delta_g)u = 0 \text{ in } (0,T) \times M$$
$$u|_{t=0} = \partial_t u|_{t=0} = 0 \text{ in } M$$
$$u|_{(0,T)\times\partial M} = f \text{ with } f \in H^2_{\text{loc}}, f = 0 \text{ for } t < 0.$$

$$\Lambda_g^h : f \to \sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \nu_i|_{(0,T)\times\partial M}$$

is the hyperbolic Dirichlet-to-Neumann map.

The boundary control method shows that if T is sufficiently large, then Λ_g^h uniquely determines g up to the natural obstruction.

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$$\Lambda_g^h : f \to \sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \nu_i|_{(0,T)\times\partial M}$$

is the hyperbolic Dirichlet-to-Neumann map.
The boundary control method shows that if T is sufficiently large, then Λ_g^h uniquely determines g up to the natural obstruction.

Theorem (Sylvester-Uhlmann)

Let (M, g_i) be compact simple Riemannian manifolds. If $\Lambda_{g_1}^h = \Lambda_{g_2}^h$, then $d_{g_1} = d_{g_2}$.

Linearized boundary rigidity problem

Linearizing the boundary rigidity problem near a known a simple metric g.

This leads to the following inverse problem: Determine the symmetric matrix f (symmetric 2-tensor field f) from tke knowledge of its integrals along all geodesics connecting boundary points.

$$\int_{-\infty}^{\infty} f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\mathrm{d}t.$$

Uniqueness question: If

$$I_g f(\gamma) = \int_{-\infty}^{\infty} f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\mathrm{d}t = 0$$

along all geodesics γ connecting boundary points, does it imply $f = 0$?

If $f = \mathrm{d}v$ with $v|_{\partial M} = 0$, then $I_g f(\mathrm{d}v)(\gamma) = 0$ for all γ connecting boundary points.

Open question: Is this the only obstruction?

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Geodesic ray transforms

We can study the following problem: Consider a simple metric g and consider $I_g f(\gamma)$ along geodesics connecting boundary points of a compact simple Riemannian manifold with boundary.

Conjecture

Let (M, g) be a compact simple Riemannian manifold with boundary. Let f be a function, 1-form or a higher rank symmetric tensor field. If $I_g f(\gamma) = 0$ along all geodesics connecting boundary points, then does it imply $f = 0$ (for functions) or $f = \mathrm{d}v$ with $v|_{\partial M} = 0$? Here d is the symmetrized covariant derivative.

For functions, the answer is yes by Mukhemetov
For 1-forms, the answer is yes by Anikonov-Romanov
For higher rank case, the conjecture is open. For analytic metrics, the answer is yes by Stefanov-Uhlmann.

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Helgason-type support theorems

The classical support theorem of Helgason is the following:

Theorem (Helgason)

Let f be a compactly supported distribution and suppose $Rf(H) = 0$ along all hyperplanes not intersecting a closed convex set K. Then $supp(f) \subset K$.

Theorem (Boman-Quinto, 1987)

Let W be an open, unbounded connected subset of $\mathbb{S}^{n-1} \times \mathbb{R}$ and let $\mu(x, \omega)$ be a strictly positive real-analytic function on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ that is even in ω. Let f be a compactly supported distribution such that

$$R_\mu f(\omega, p) = \int_{H(\omega,p)} f(x)\mu(x, \omega) \mathrm{d}x_H = 0$$

for $(\omega, p) \in W$. Then $f = 0$ on $\cup(H(\omega, p) | (\omega, p) \in W)$.

Proof of this theorem is based on analytic microlocal analysis.
This can be generalized to geodesic ray transforms.

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Theorem of Kawai-Kashiwara-Hörmander

Theorem

Let $u \in \mathcal{D}'(\Omega)$ and let f be a real-valued real analytic function and let $x_0 \in \Omega$ is a point in supp(u) such that

$$\mathrm{d}f(x_0) \neq 0, \quad f(x) \leq f(x_0) \text{ if } x \in supp(u).$$

Then $(x_0, \pm \mathrm{d}f(x_0)) \in WF_A(u)$.

Helgason-type support theorem for geodesic ray transforms

Theorem

- *(K., 2009) Let (M, g) be a compact simple Riemannian manifold with g real-analytic and with real-analytic boundary ∂M. Let $\mathcal{A}$ be an open subset of geodesics in M such that each geodesic $\gamma \in \mathcal{A}$ can be deformed to a point on ∂M by geodesics in $\mathcal{A}$. Let $M_{\mathcal{A}}$ be the set of points on these geodesics. If $If(\gamma) = 0$ for all $\gamma \in \mathcal{A}$, then $f = 0$ on $M_{\mathcal{A}}$.*
- *(K.-Stefanov, 2009) Let (M, g) be a compact simple Riemannian manifold with g real-analytic and with real-analytic boundary ∂M. Let K be a closed geodesically convex subset of M and let f be a symmetric 2-tensor field in M. Suppose $I_g f(\gamma) = 0$ for all γ not intersecting K, then $f = \mathrm{d}v$ on $M \setminus K$ and $v|_{\partial M} = 0$.*

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Support theorems

Problem

Prove a Helgason-type support theorem for the geodesic ray transforms of functions. Here (M, g) is a simple C^∞ Riemannian manifold.

Note: Uniqueness question has been settled by Mukhemetov.
This is a very subtle question. There is a famous counterexample by Boman involving weighted Radon transforms (with C^∞ weights).

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Connection between the support theorem and the Calderón problem

In the classical Calderón problem case, Helgason-type support theorem is used to prove uniqueness for Neumann measurements made on possibly small subsets of the boundary.

Reason: For the full data case: One ends up with the Fourier transform. For the partial data case, one ends up with the Radon transform. This was used by kenig-Sjöstrand-Uhlmann in their famous paper.

For a class of geometric Calderón problems, one ends up with the geodesic ray transform and hence the support theorem for the geodesic ray transform is useful. A recent preprint of Kenig and Salo deals with this case.

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