Numerical Hydrodynamics: Part 2

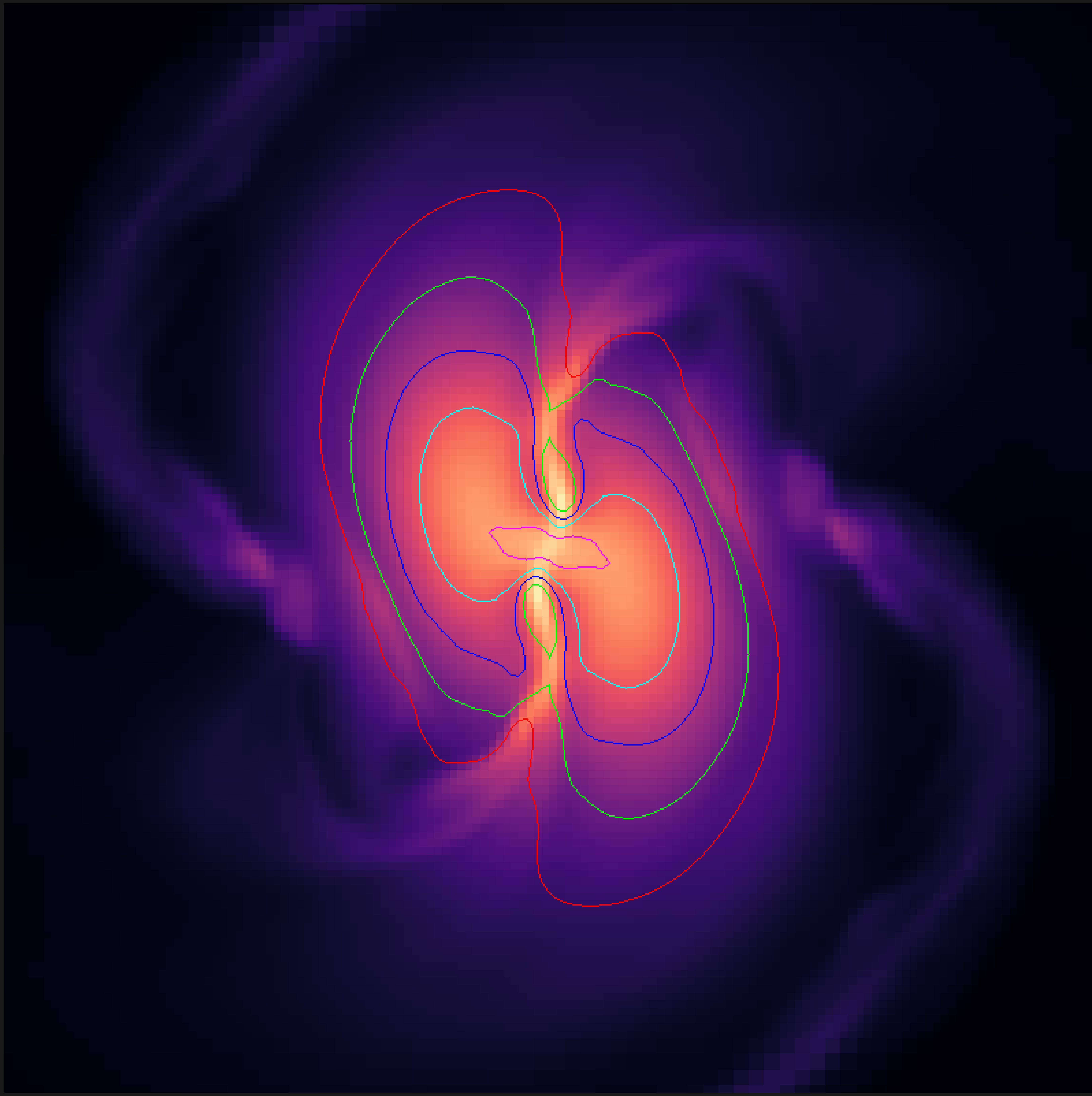
Ian Hawke

@IanHawke
github.com/IanHawke
orcid.org/0000-0003-4805-0309
STAG, University of Southampton

ianhawke.github.io/slides/icts-2020
Additional material at github.com/IanHawke/icts-2020

What we covered

- Balance laws are generic;
- lead to shocks;
- shocks appear in mergers;
- start from Newtonian CFD;
- GR gives
  - increased cost;
  - increased complexity;
- Merger problem gives
  - surface/atmosphere;
  - lots more physics.



Balance laws

All the terms:

$$\partial_t \mathbf{q} + \nabla \cdot \mathbf{f} + A \cdot \nabla \mathbf{q} = \nabla \left(D \cdot \nabla \mathbf{q} \right) + \mathbf{s}.$$

In mergers

- ignore diffusive term;
- simple models don't have A term;
- source terms often local, well-behaved.

Characteristics

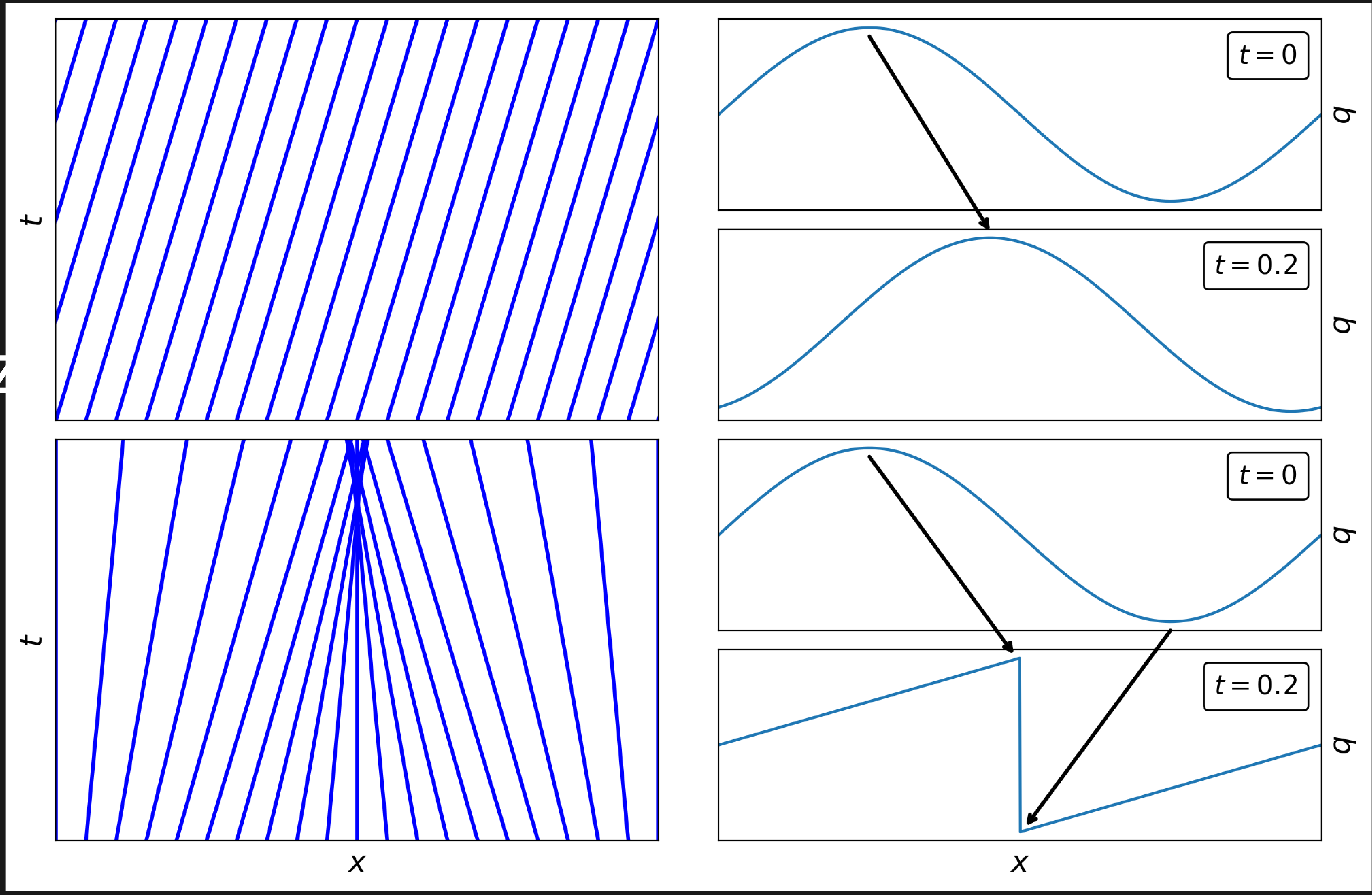
For $\partial_t q + \partial_x f = 0$ get local speed

$$\partial_t q + {\color{red}\partial_q f}\, \partial_x q = 0.$$

For system $\partial_t \mathbf{q} + A \partial_x \mathbf{q} = 0$ diagonaliz
to get

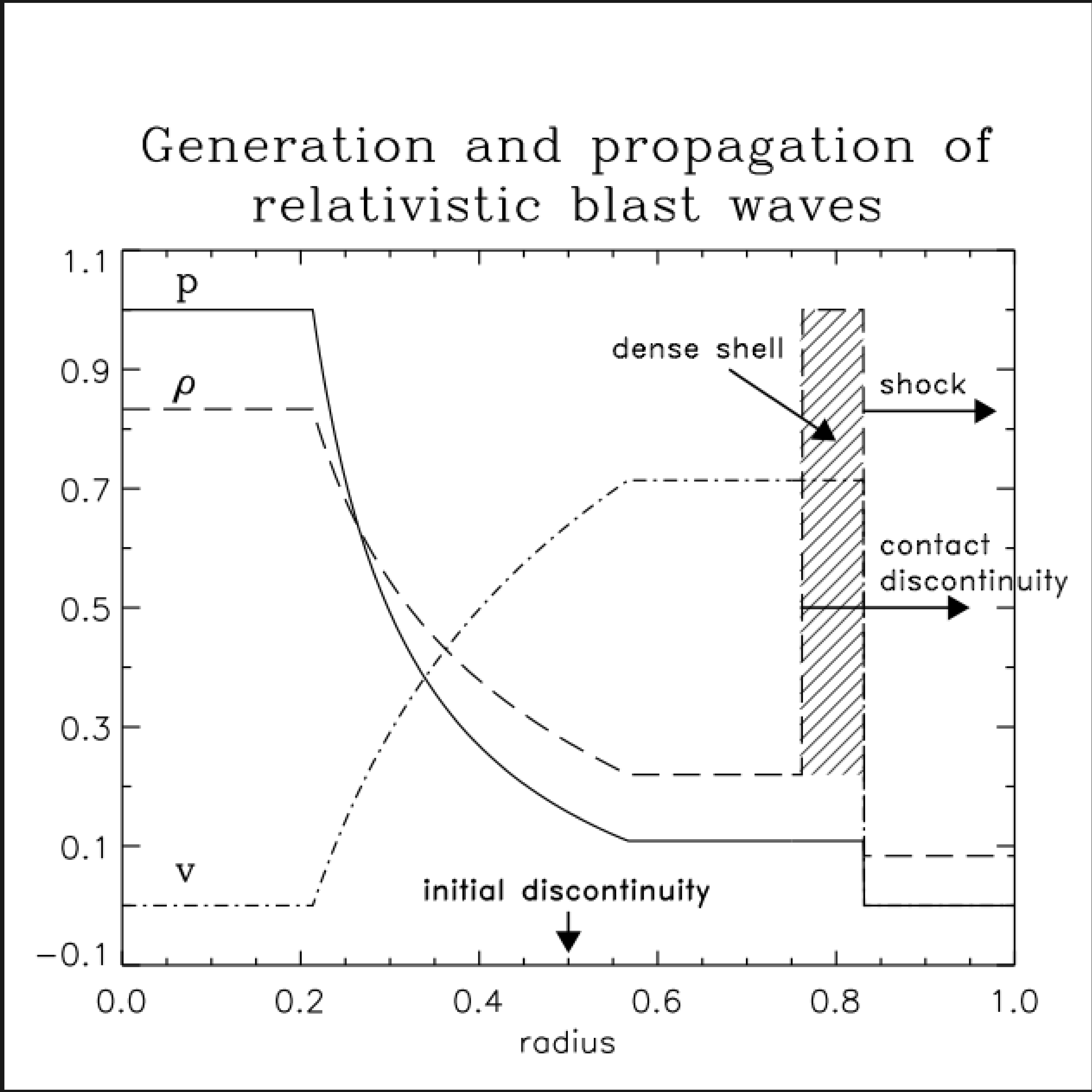
$$\mathbf{w} = L\mathbf{q}, \qquad \partial_t \mathbf{w} + \Lambda \partial_x \mathbf{w} = 0.$$

Extend to nonlinear case with care.



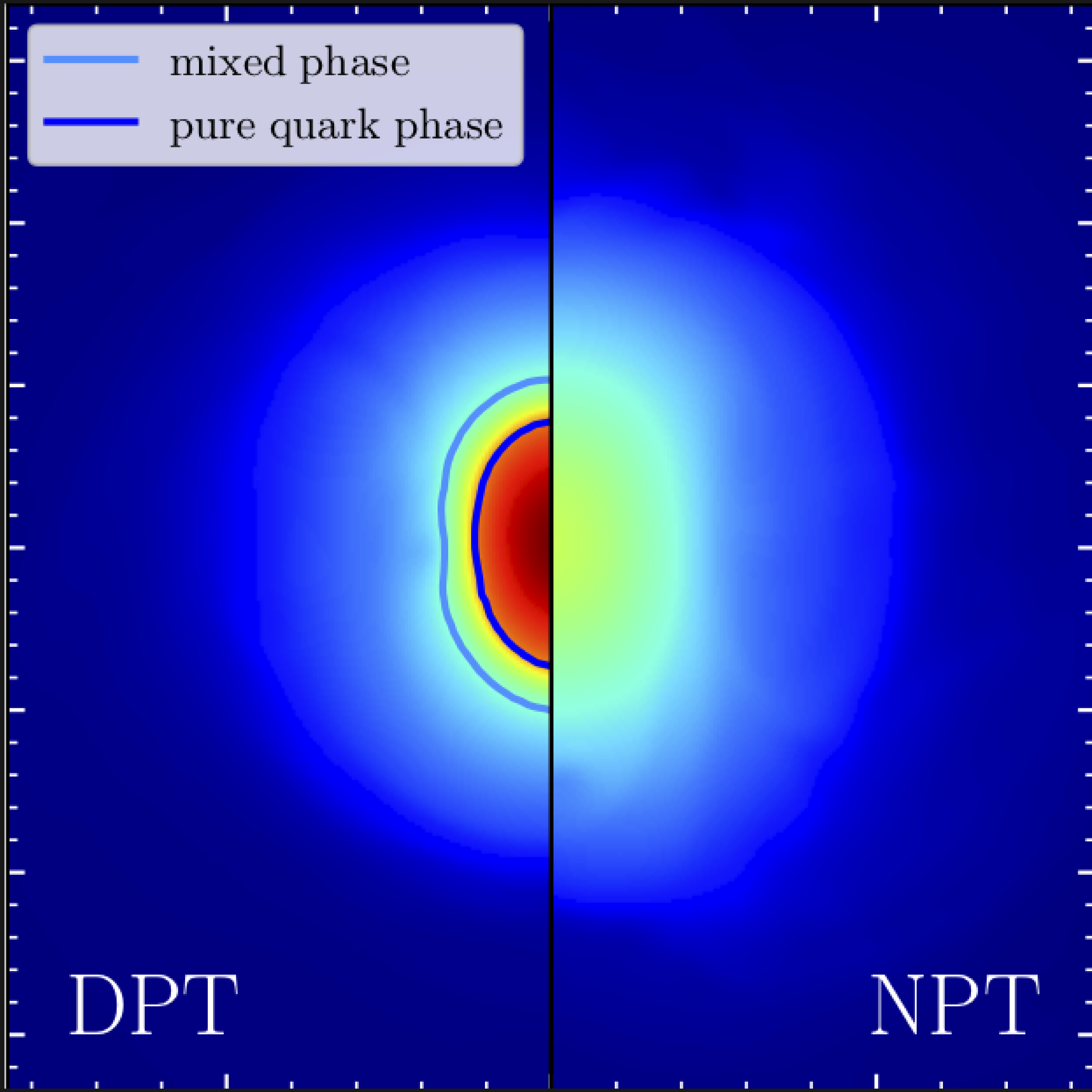
Wave types

- Rarefaction:
  - nonlinear
  - continuous;
- Shocks:
  - nonlinear
  - discontinuous;
- Contacts:
  - linear
  - discontinuous.
- Compound waves:
  - nonlinear, a mess;
  - MHD, phase transitions.



Questions...

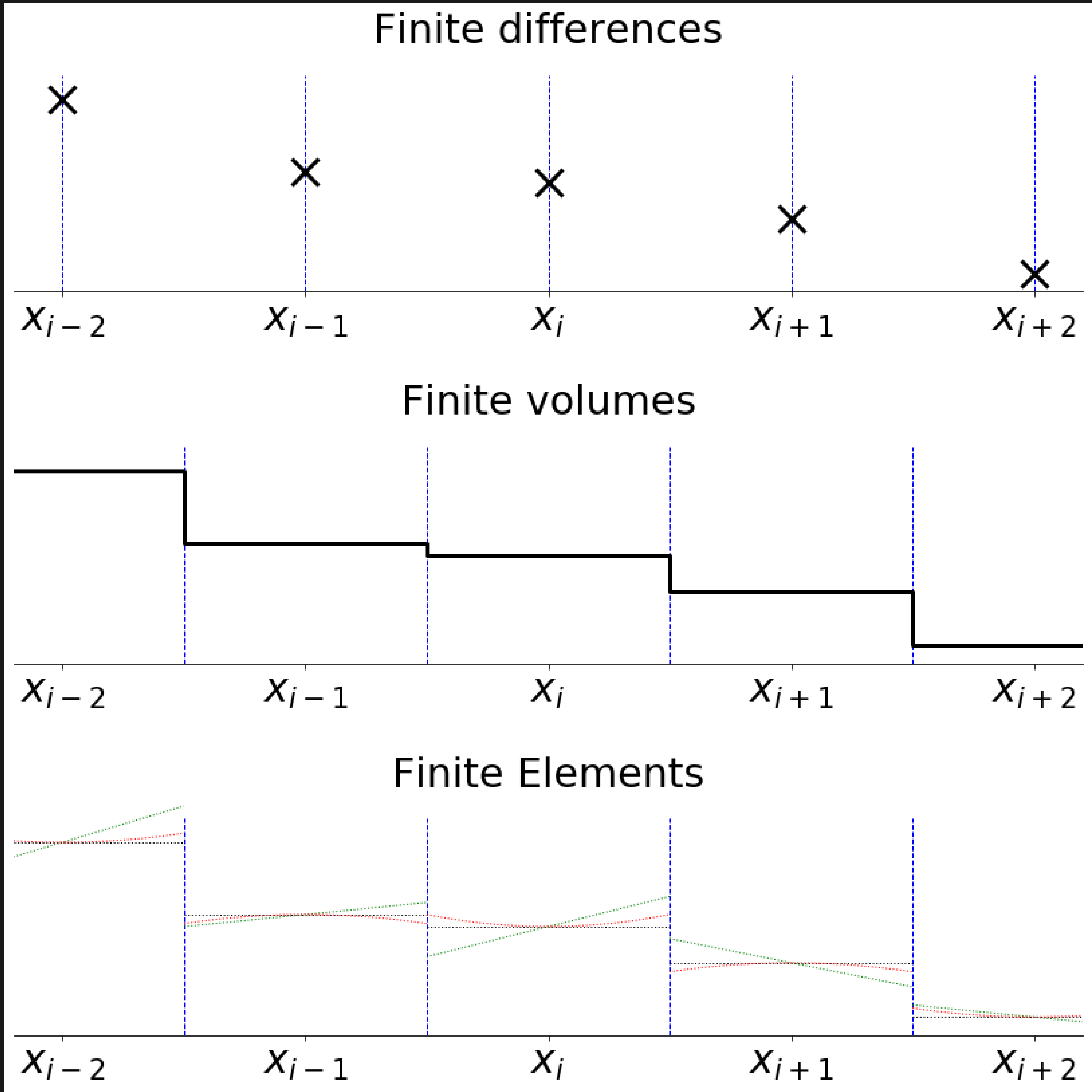
- Stretch out;
- have a break;
- add questions to the chat.



Weih et al, 1912.09340.

Grids and approximations

- Finite differences:
  store point values q_i.
- Finite volumes:
  store cell averages $\hat{q}_i$.
- Finite elements (DG):
  store modal coefficients $q_i^{(m)}$.



Fluxes and telescoping

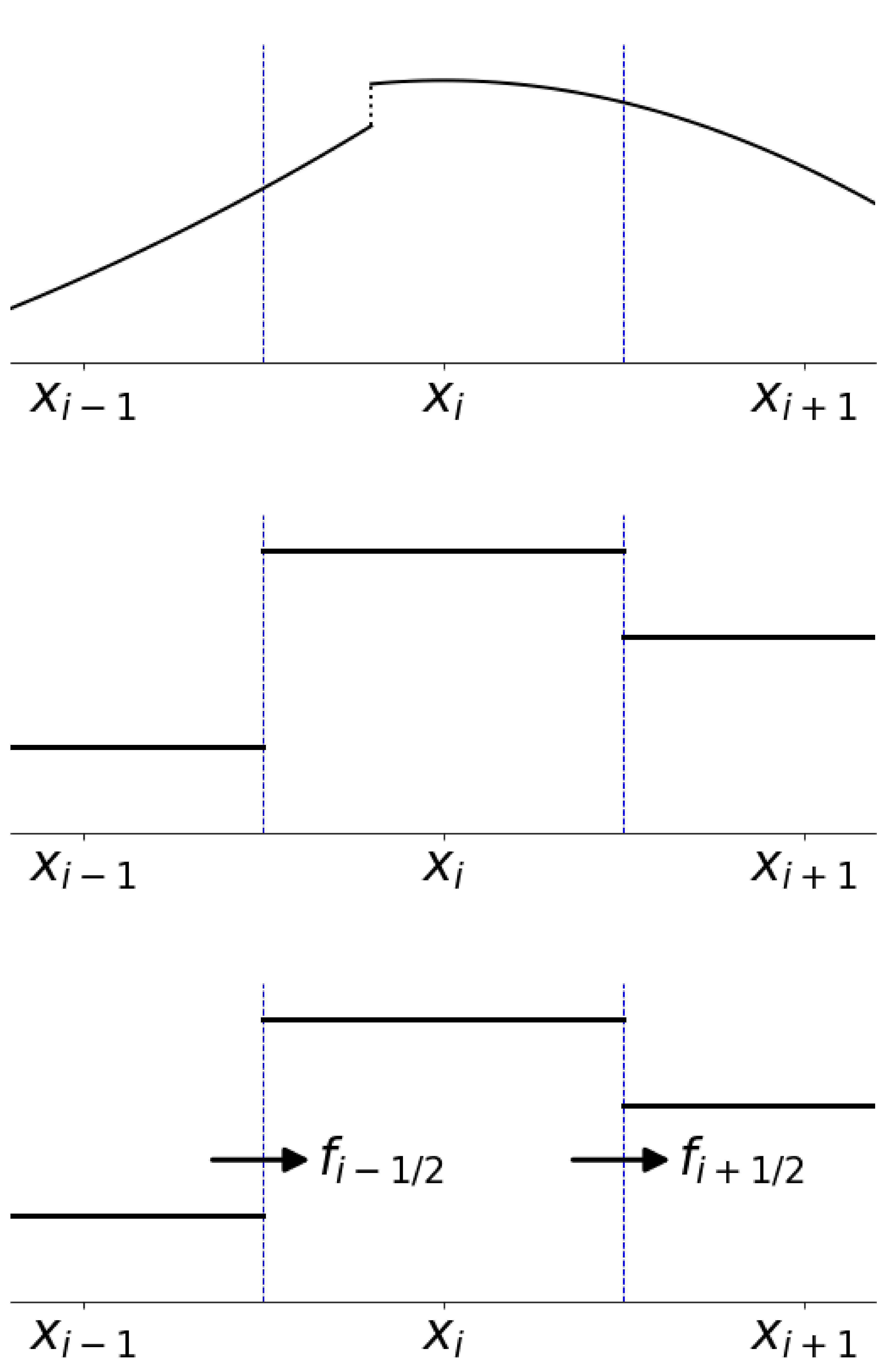
Integrate over cell $i \to \hat{q}_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_i + \frac{1}{|V_i|}\oint_{\partial V_i} f(q) = 0.$$

Restrict to one dimension:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_i = \frac{1}{\Delta x}\left[f_{i-1/2} - f_{i+1/2}\right].$$

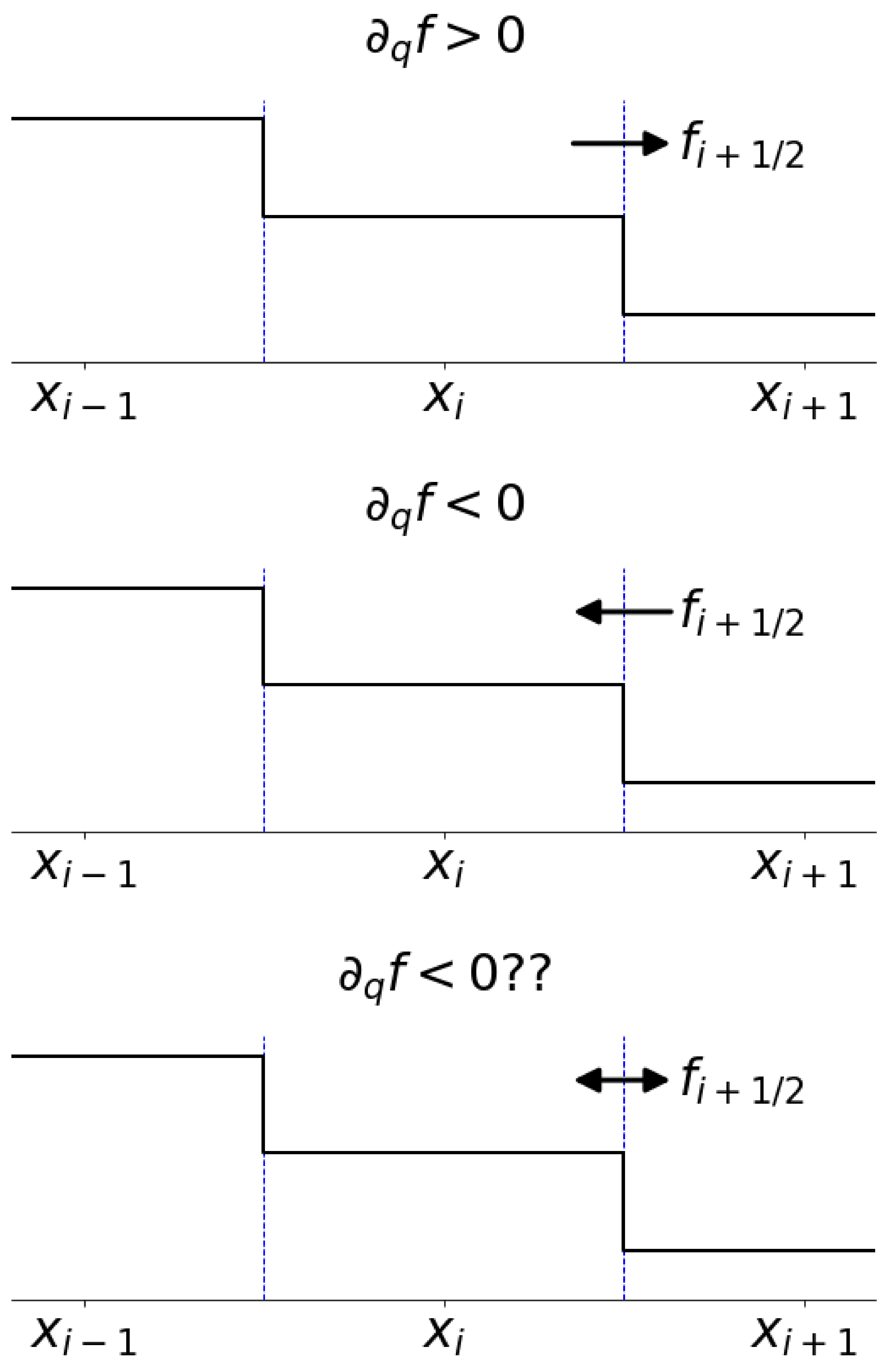
Gives *discrete* conservation.



Computing the intercell flux

Godunov:

- $q(x) = \hat{q}$ in cell i;
- $$f_{i+1/2} = F(\hat{q}_i, \hat{q}_{i+1});$$
- $$\partial_q f > 0 \implies f_{i+1/2} = f(\hat{q}_i);$$
- Systems: use characteristic variables;
- Nonlinear: solve *Riemann Problem*, usually approximately!

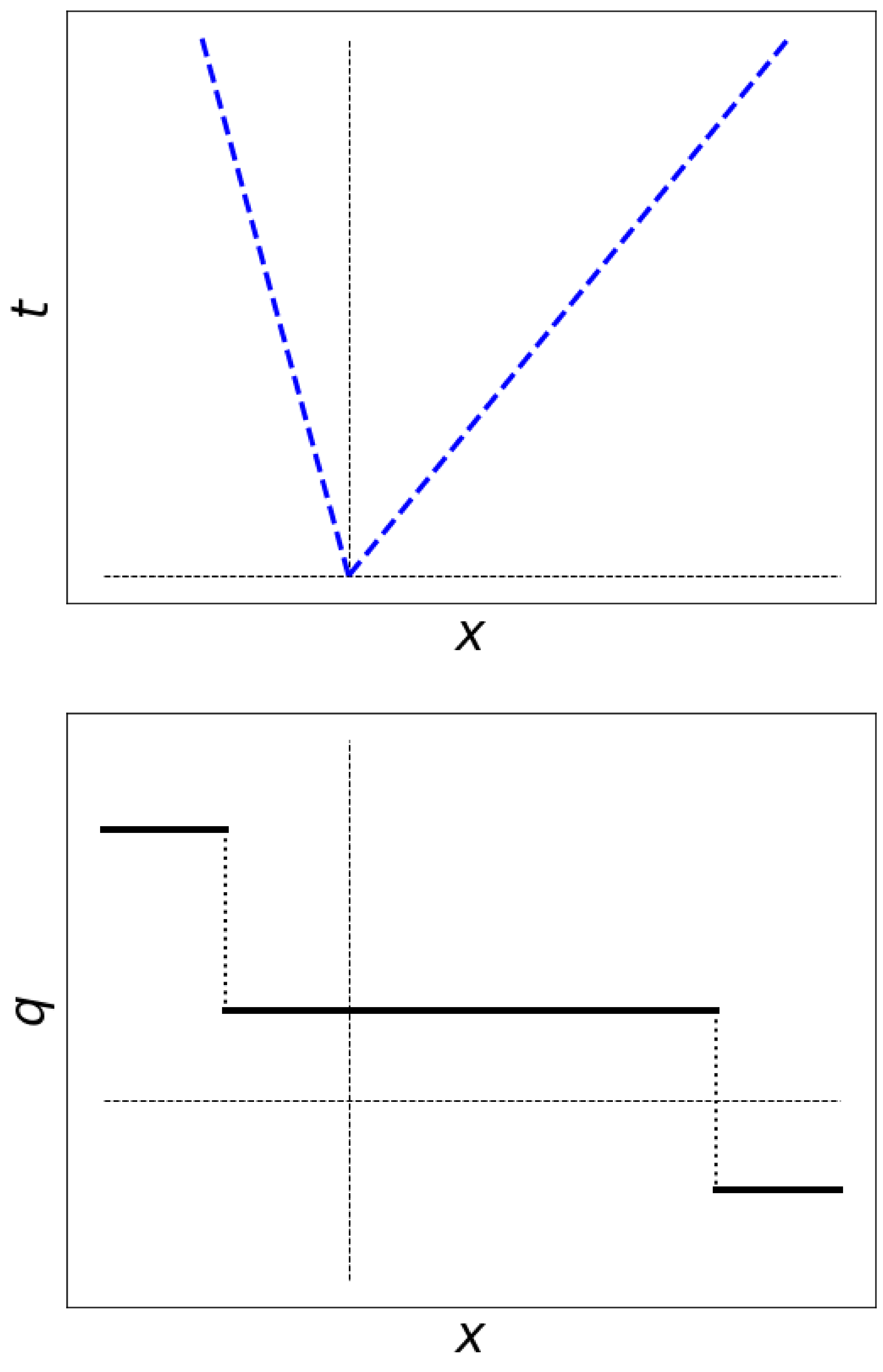


Approximate Riemann Solvers

HLLE: to find

$$f_{i+1/2} = F(\hat{q}_i, \hat{q}_{i+1}) = F(q_L, q_R) :$$

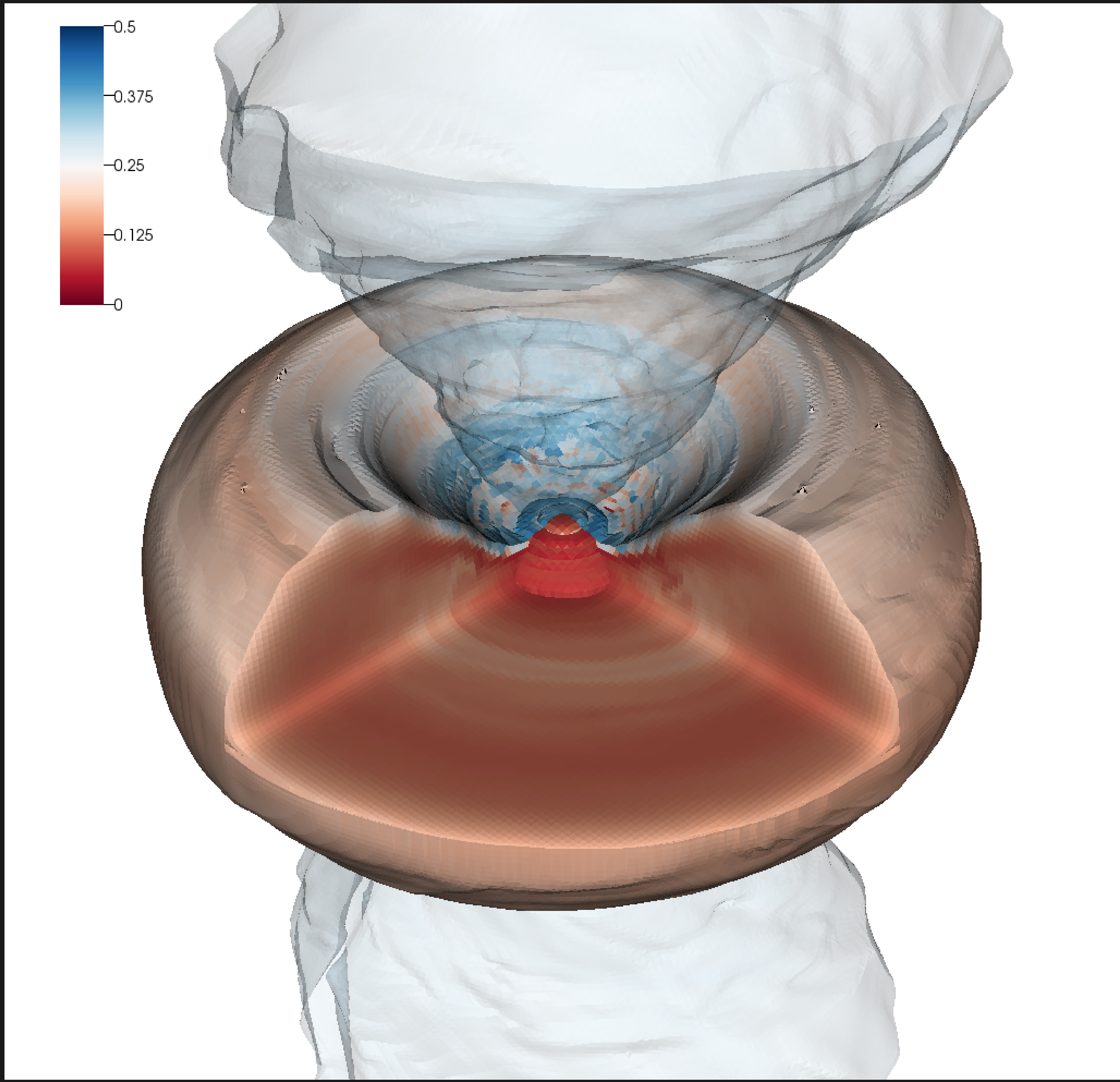
- assume fastest speeds are $\xi_\pm$;
- impose conservation;
- $$q_* = \frac{\xi_+ q_R - \xi_- q_L - f(q_R) + f(q_R)}{\xi_+ - \xi_-};$$
- get flux from appropriate state.



Questions...

- Stretch out;
- take a chance to refocus;
- ensure you're ready for more detail;
- add questions to the chat.

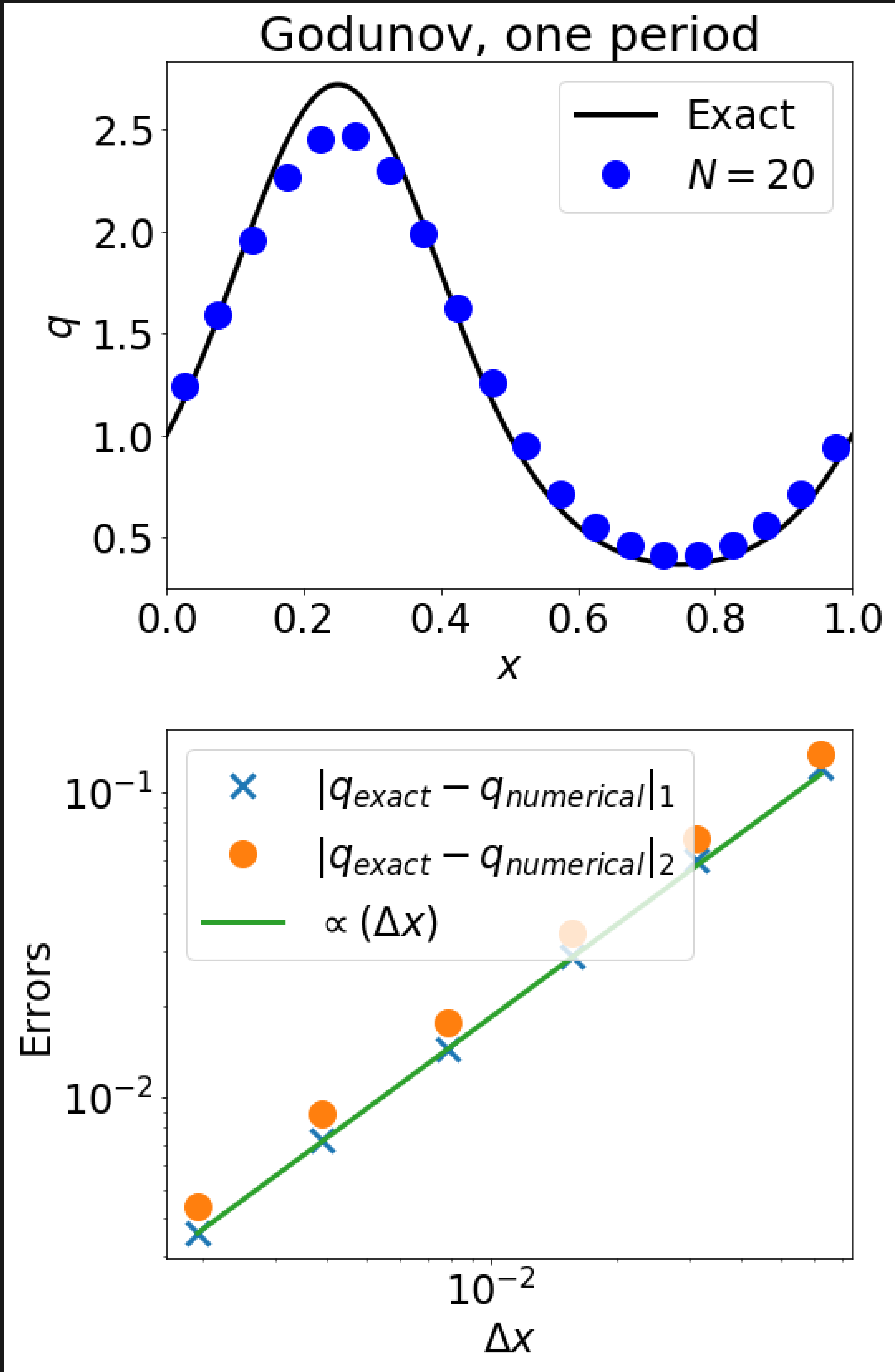
Bernuzzi, 2004.06419.



Dimensions, costs, and accuracy

Godunov isn't good enough:

- Error $\propto (\Delta x)^1$.
- Computational cost $\propto (\Delta x)^{-4}$.
- Extrema are *clipped*.
- GWs need better phase accuracy.
- Need higher order methods. But...
- ...leads to problems with shocks.



Reconstruct-Evolve-Average

Rethink Godunov as three steps:

1. *Reconstruct*:

$$\hat{q} \to q(x);$$

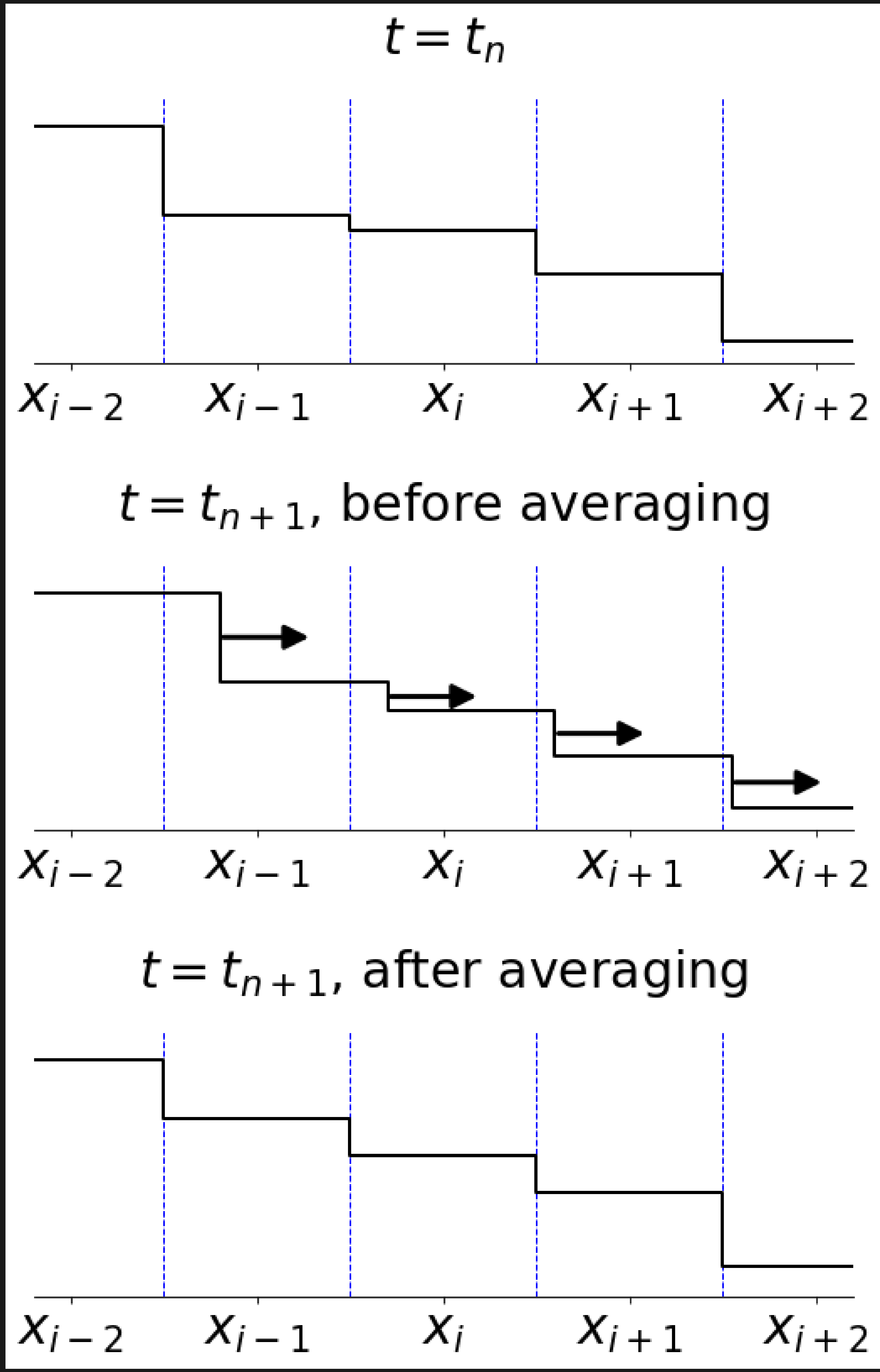
2. *Evolve*:

$$q(x);$$

3. *Average*:

$$q(x) \to \hat{q}.$$

Reconstruction loses accuracy.



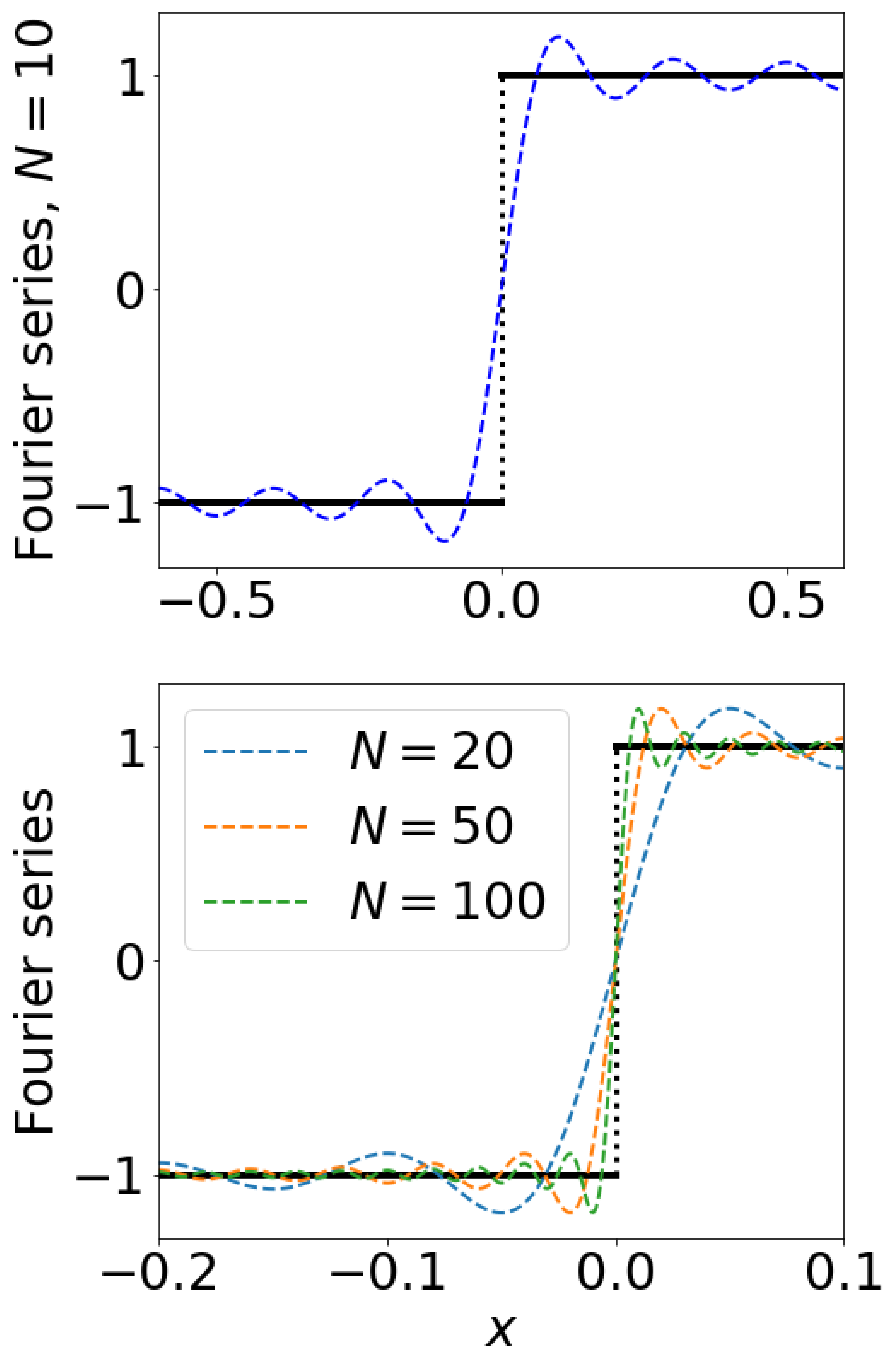
Monotonicity, Gibbs oscillations, and Godunov's theorem

Fourier Series:
discontinuities $\implies$ oscillations.

- Don't converge with Δx;
- Don't converge with more modes;
- Don't go away with different function basis.

Monotonicity: scheme doesn't introduce oscillations.

Godunov's theorem: linear monotonic schemes are first order accurate.



Slope limiting

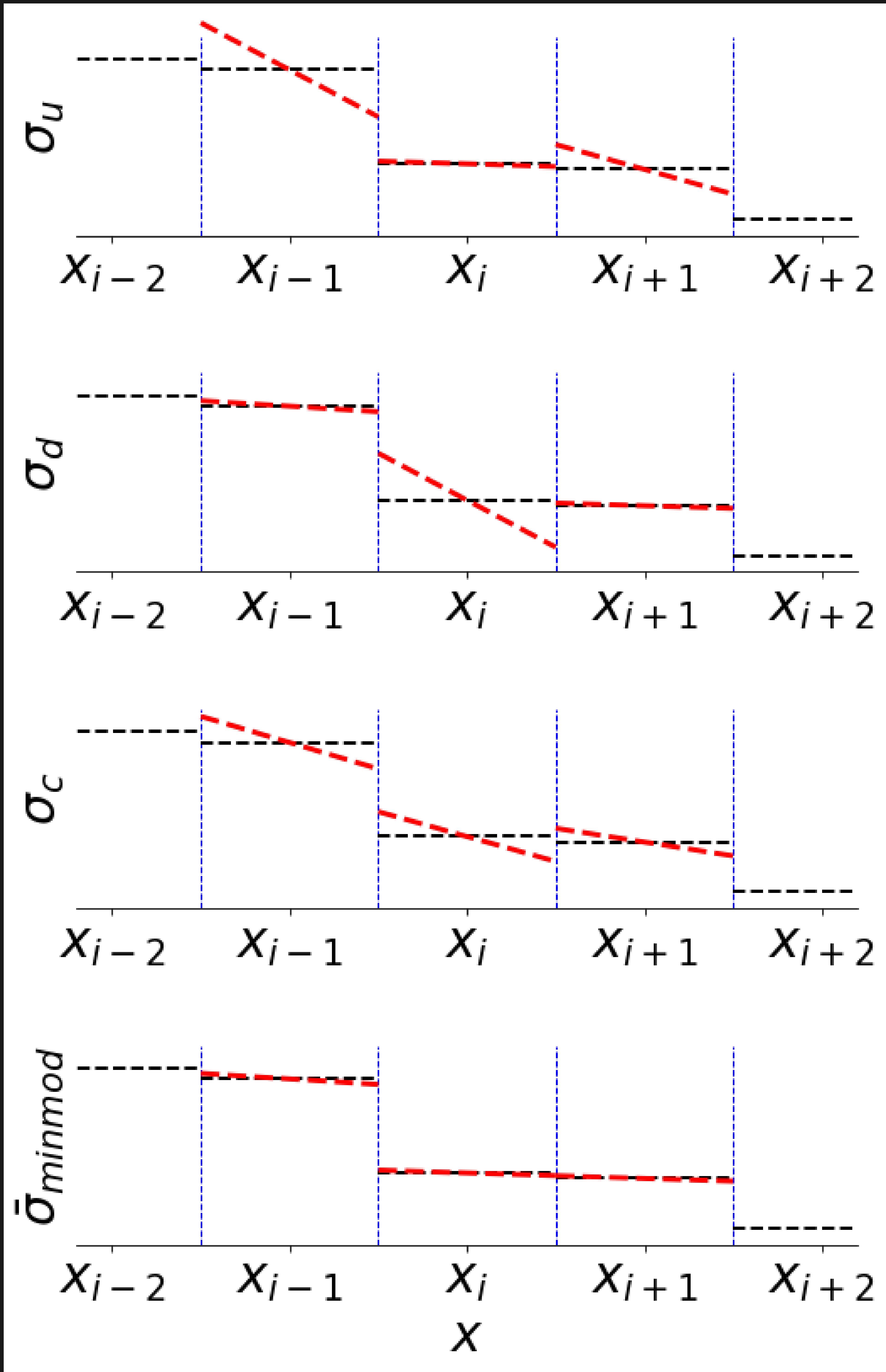
Assume $q(x) = \hat{q}_i + \frac{x - x_i}{2}\sigma$.

Slope σ could be

- Upwind: $\sigma_\text{u} = \hat{q}_{i+1} - \hat{q}_i$;
- Downwind: $\sigma_\text{d} = \hat{q}_i - \hat{q}_{i-1}$;
- Centered: $\sigma_\text{c} = \frac{1}{2}(\hat{q}_{i+1} - \hat{q}_{i-1})$.

All would give oscillations. *Limit* slope:

$$\bar{\sigma} \equiv \bar{\sigma}(\sigma_\text{u}, \sigma_\text{d}) \overset{\text{(eg)}}{=} \begin{cases} 0 & \text{if } \sigma_\text{u} \cdot \sigma_\text{d} \le 0 \\ \sigma_\text{u} & \text{if } |\sigma_\text{u}| < |\sigma_\text{d}| \\ \sigma_\text{d} & \text{if } |\sigma_\text{u}| > |\sigma_\text{d}| \end{cases}.$$



Finite difference methods

In N-d, finite volume needs a surface integral: expensive.

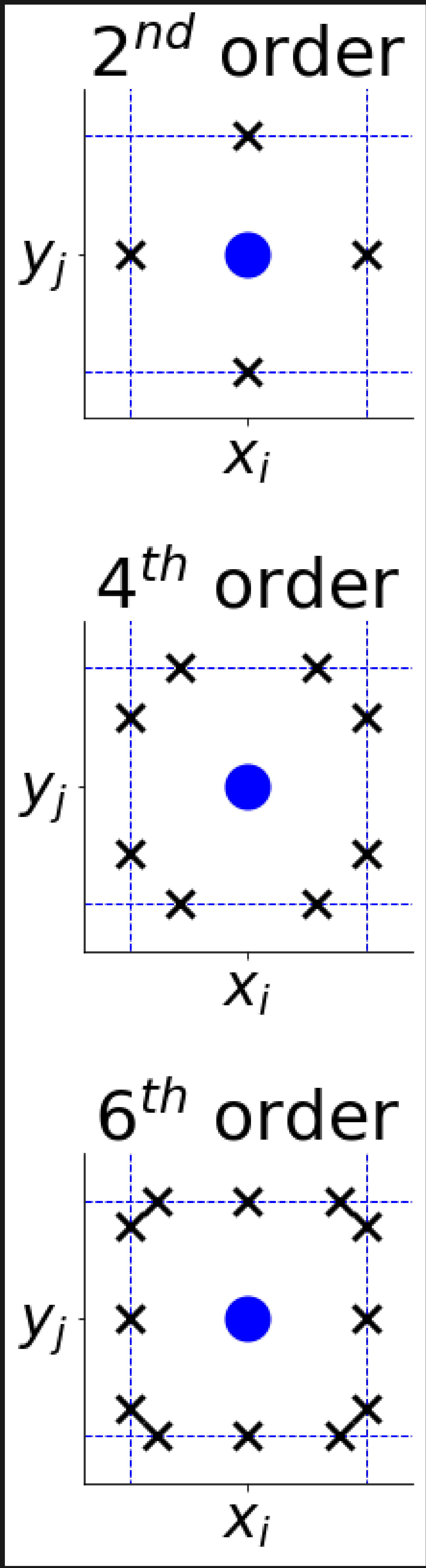
Instead, write a finite difference method as

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = \frac{1}{\Delta x} \left[f_{i-1/2} - f_{i+1/2} \right] .$$

Now $f_{i\pm 1/2}$ *not* intercell fluxes. Directly reconstruct flux.

For stability must *split flux*:

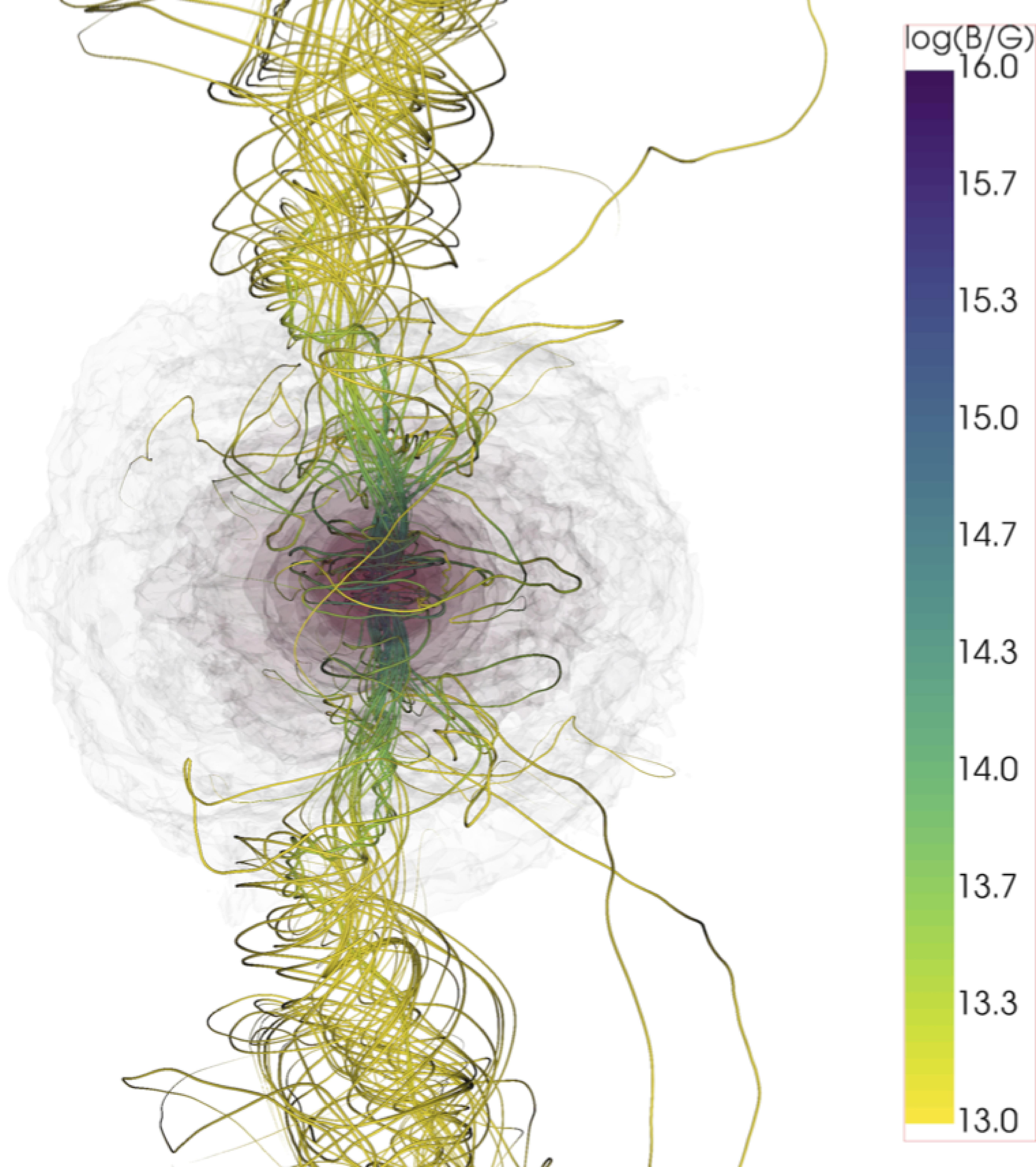
$$f = f^{(+)} + f^{(-)}, \quad f^{(\pm)} = \tfrac{1}{2}(f \pm \max|\lambda| q) .$$



Questions...

- One topic left, so
- deep breath;
- stretch out;
- drink something;
- screen break;
- add questions to the chat.

Ciolfi, 2001.10241.



MHD

Constraint $C = \nabla \cdot \mathbf{B} = 0$ needed.

Either

- Modify equations of motion so

$$\partial_t C \sim -\alpha C \qquad \Longrightarrow \qquad C \sim e^{-\alpha t};$$

- Find discrete scheme so, when $\nabla \to D$,

$$D \cdot \mathbf{B} = 0.$$

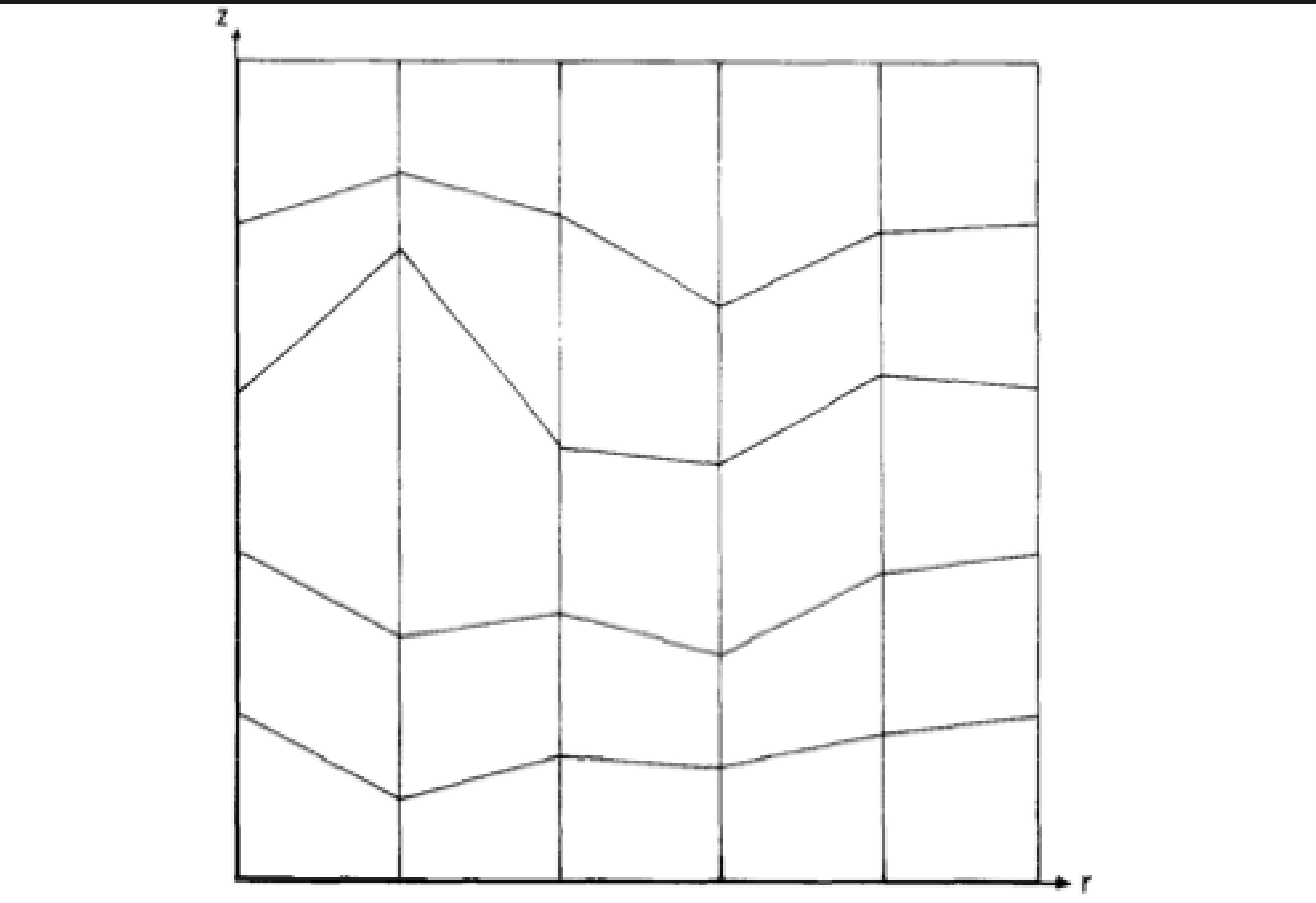


FIG. 1. The computation mesh for a Lagrangian calculation without correction for $\nabla \cdot \mathbf{B} \neq 0$ is shown after 209 time steps, corresponding to 160 signal transit times across the mesh. The axis of symmetry coincides with the left boundary; the velocity, **u**, is zero on the top, right, and bottom boundaries. The initial field is parallel to the axis and uniform. The plasma beta is 1.3×10^{-3}. The displacements of the vertices of the mesh are parallel to the magnetic field, and result from velocities approximately equal to 2×10^{-3} times the Alfven speed.

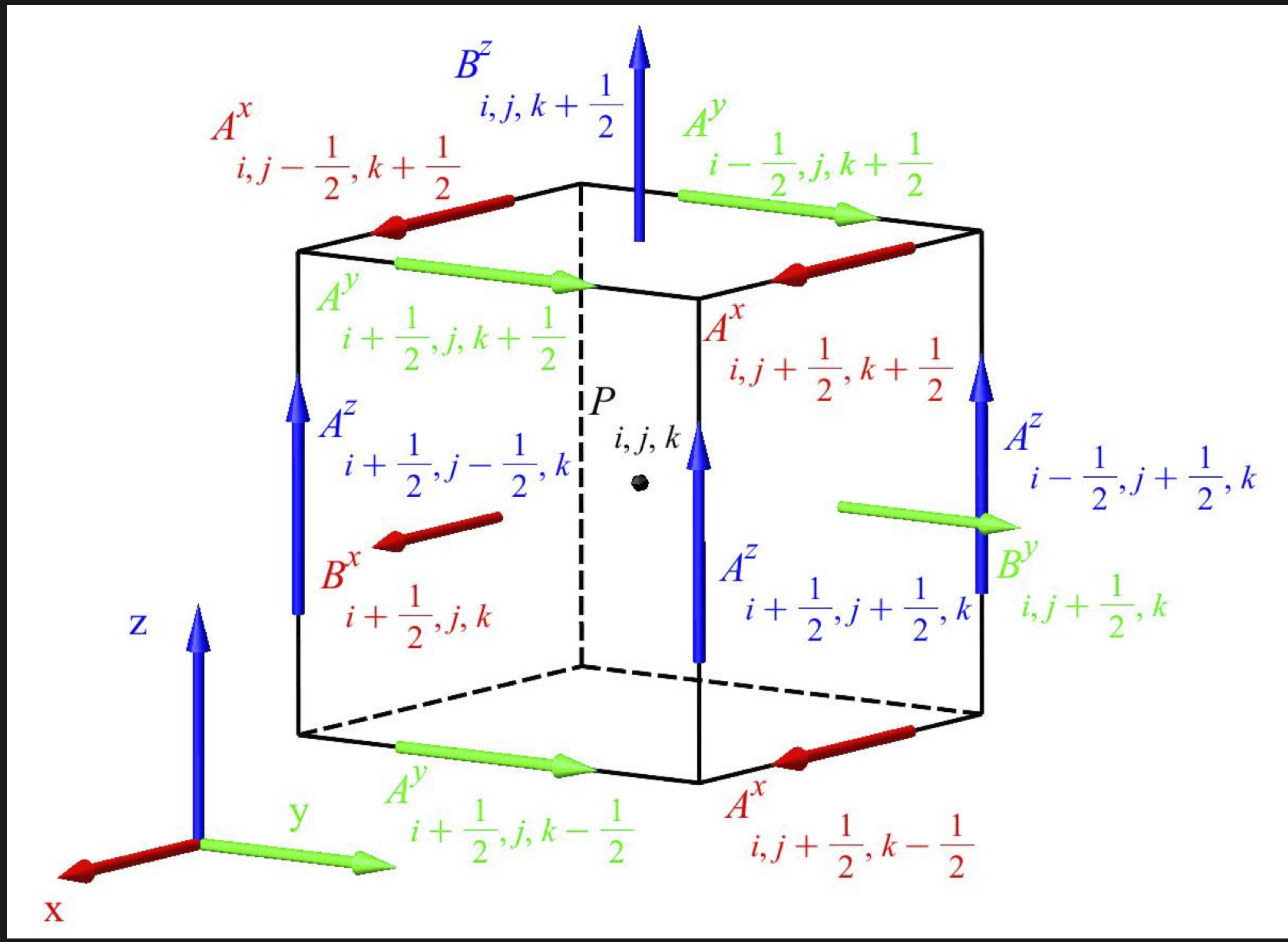
Vector Potentials

Use vector potential $\mathbf{A}$, so $\mathbf{B} = \nabla \times \mathbf{A}$.

- Automatically preserves constraint.
- *EM* gauge freedom. Prefer *Lorenz* gauge,

$$\partial_t \mathbf{A} + \nabla \Phi \sim \mathbf{E},$$
$$\partial_t \Phi + \nabla \cdot \mathbf{A} \sim 0.$$

- Grid structure is conceptually more complex.

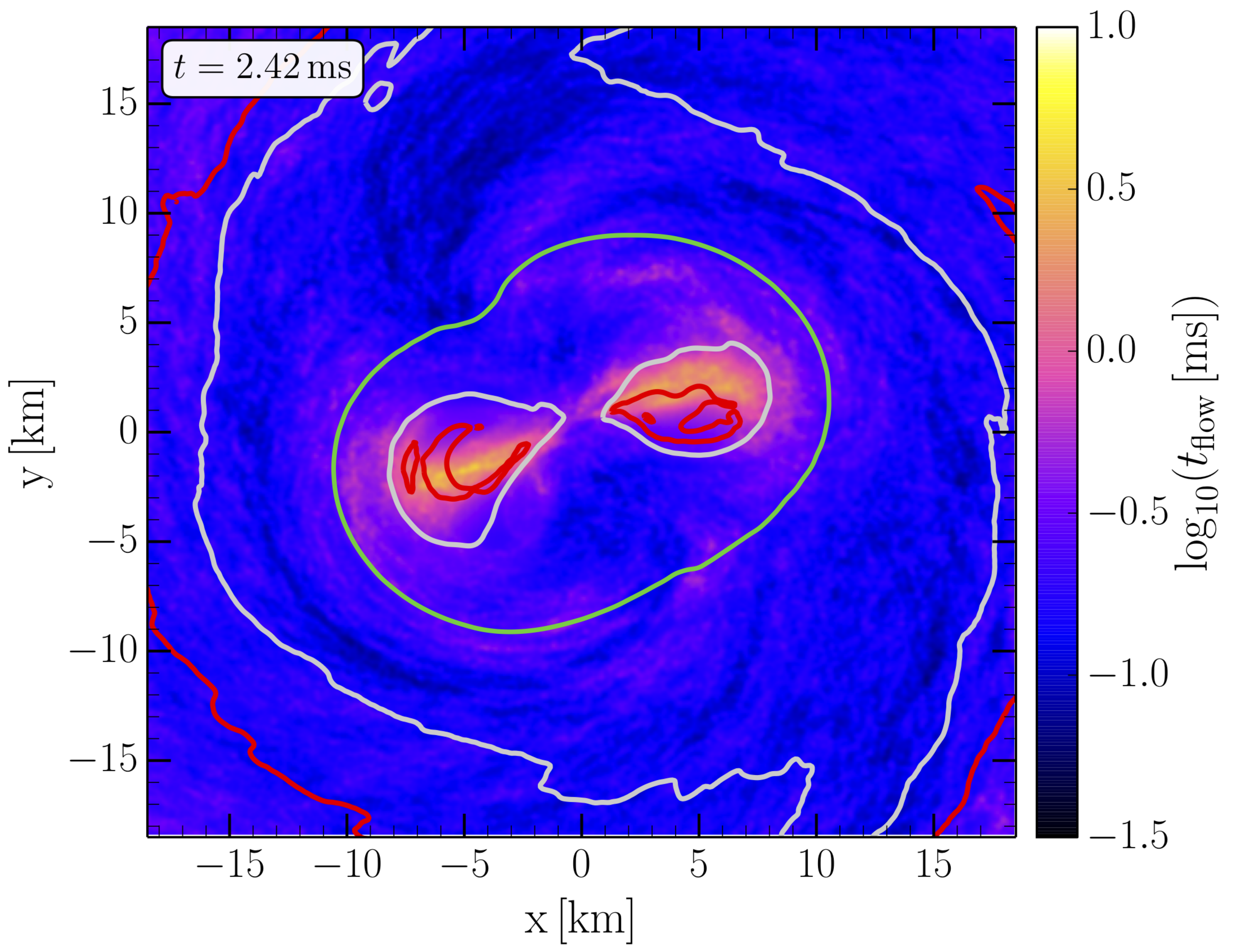


Summary

We have discussed

- the three key approaches,
  - finite volumes;
  - finite differences;
  - finite elements,
- discrete flux conservation;
- reconstruction, monotonicity and Gibbs oscillations;
- MHD and constraints.

Next lecture: some aspects of the future.



Alford et al, 1707.09475.