§ Introduction (self-introduction)

I think no one here knows me
(except for Japanese participants, I hope),
because I am not a knot theorist.

My name is Hideki Miyachi.
I am working in Teichmüller theory.
I mainly use quasiconformal mappings
for studying, and complex analysis

My personal belief :

Topological invariants coming from surface
appear in the "bdy at ∞" of
Teichmüller space (i.e. degeneration of cpx structure)

In the series of lectures, I would like to give a (kind of) grimpse.

Unfortunately, I do not (can not) give any "Knot invariant" in this lecture.

But, I want to get a good connection

Teichmüller theory (complex analysis) $\Longleftrightarrow$ Lowdimensional top Knot theory

to contribute Knot theory (low dim top) in (near) future.

1. Introduction of Teichmüller space

2. Bers - Thurston classification of the mapping class group

3. Isometry of Teichmüller space

(Introduction to the extremal length geometry of Teichmüller space.)

§ 1 Teichmüller theory

§1.1. S : compact orientable surface (2-dim mfd)

S is of the type $(g.n)$ if
genus$(S) = g$
$\#$(components of ∂S) $= n$

$\#$ handle $=$ genus

A Riemann surface is a 1-dimensional complex mfd..

A Riemann surface X is said to be of (analytically) finite type $(g.n)$ if

$$X = (\text{closed R.S of genus } g) - (n \text{ pts})$$

$$\text{Type } (1.0) \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

$$\text{Type } (1,1) \cong \mathbb{C}$$

Each of these Riemann surfaces is simply connected.

Thm (Koebe) Any simply connected Riemann surface is biholomorphically equivalent to one of the following:

① $\hat{\mathbb{C}}$ ② $\mathbb{C}$ ③ $\mathbb{D}$

Furthermore, two of those are not biholomophically equivalent.

§1.2 Classification via universal covering space

Let X be a Riemann surface.
Let $\tilde{X}$ be the universal covering space

① If $\tilde{X} \cong \hat{\mathbb{C}}$, $X \cong \hat{\mathbb{C}}$

$\because$ Indeed, the deck group is trivial, because any element in $\mathrm{Aut}(\hat{\mathbb{C}})$ has fixed pts on $\hat{\mathbb{C}}$.

② If $\tilde{X} \cong \mathbb{C}$, $X \cong \mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
or T : torus

$\because$ $\mathrm{Aut}(\mathbb{C}) = \{az+b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$
$az+b \in \mathrm{Aut}(\mathbb{C})$ does not have a fixed pt if and only if $a = 0$ and $b \neq 0$.
Hence, we can see that the Deck group of X contained in the abelian subgroup

$$\{z+b \mid b \in \mathbb{C}\}.$$

We can see that any discrete group is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. //

A Riemann surface X is said to be hyperbolic if

(universal covering of X) $\cong \mathbb{D}$ (bihol).

Then, the deck group $\subset \mathrm{Aut}(\mathbb{D})$.

$\rightsquigarrow$ Any hyperbolic Riemann surface admits a unique metric ds_X

s.t. $\pi^* ds_X = ds_{\mathbb{D}}$

where $ds_{\mathbb{D}} = \dfrac{2|dz|}{1-|z|^2}$ the hyperbolic metric on $\mathbb{D}$.

We call ds_X the hyperbolic metric on X.

§2 Teichmüller space

S : compact orientable surface with $\overset{-n+2-2g}{\overset{\|}{\chi(S)}} < 0$.

Namely, S is not one of :

(0,0) (0,1) (0,2) (1,0)

A marked Riemann surface of type (g, n) is a pair (X, f)

X : a R.S. of analytically finite type (g, n).

$f : \mathrm{Int}(S) \to X$: orientation preserving homeomorphism.

Two marked R.S (X_1, f_1) and (X_2, f_2) are said to be Teichmüller equivalent if

$\exists\, h : X_1 \to X_2$: biholomorphism

s.t.

$$\begin{tikzcd} & X_1 \arrow[dd, "h"] \\ \mathrm{Int}(S_1) \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & X_2 \end{tikzcd}$$

$h \circ f_1 \sim f_2$
(homotopic)

$$T(S) = \{(X, f) \mid \text{marked R.S of type } (g,n)\} \big/ \underset{\text{Teichmüller eq}}{\sim}$$

The Teichmüller space of S
The Teichmüller space of type (g,n).

§3 Teichmüller space as a quasiconformal deformation space

§3.1 Quasiconformal mapping (Quick review)

D_1, D_2 : domains in $\mathbb{C}$
$f : D_1 \to D_2$: an orientation preserving homeomorphism
f is said to be a K-quasiconformal (K-qc) if

① f has distributional derivatives f_z and $f_{\bar{z}}$ in $L^1_{loc}(D_1)$.

② $|f_{\bar{z}}| \leq k |f_z|$ a.e. on D_1, $k = \dfrac{K-1}{K+1}$

◎ Propaties

① f is 1-qc $\Leftrightarrow$ f is conformal.
(Weyl's lemma)

② f is K_1-qc, g is K_2-qc
$\Rightarrow$ $f \circ g$ is K_1K_2-qc.

③ f is K-qc $\Leftrightarrow$ f^{-1} is K-qc

④ (Gehring-Lehto) f is totally differentiable a.e on D_1

⑤ $f_z \neq 0$ a.e. on D_1

$\mu_f = \dfrac{f_{\bar{z}}}{f_z} \in L^\infty(D_1)$: the Bertrami differential of f

①. ③ $\leadsto$ We can consider K-qc between Riemann surfaces

In this case, the Bertrami differential is $(-1,1)$-L^∞-form $\mu = \mu(z)\dfrac{d\bar{z}}{dz}$.

- Geometric meaning of qc.

Suppose $0 \in D_1$ and $f : D_1 \to D_2$ is totally differentiable at $z=0$.
We also assume that $f(0)=0$.

$A = f_z(0)$, $B = f_{\bar{z}}(0)$. Assume $A \neq 0$.

$$f(z) = Az + B\bar{z} + o(|z|) \qquad (|z| \to 0)$$

Note

$$Az + B\bar{z} = A\left(z + \frac{B}{A}\bar{z}\right)$$

Set

$$\mu = \frac{B}{A} = |\mu| e^{2i\theta}, \quad |\mu| \leq k$$

$$f(e^{i\theta}z) = A\left(e^{i\theta}z + \mu e^{2i\theta} \cdot e^{-i\theta}\bar{z}\right) + o(|z|)$$

$$= Ae^{i\theta}\left(z + |\mu|\bar{z}\right) + o(|z|)$$

Set

$$\mu = \frac{B}{A} = |\mu| e^{2i\theta}, \quad |\mu| \le k$$

$$f(e^{i\theta}z) = A\left(e^{i\theta}z + \mu e^{2i\theta}\cdot e^{-i\theta}\bar{z}\right) + o(|z|)$$

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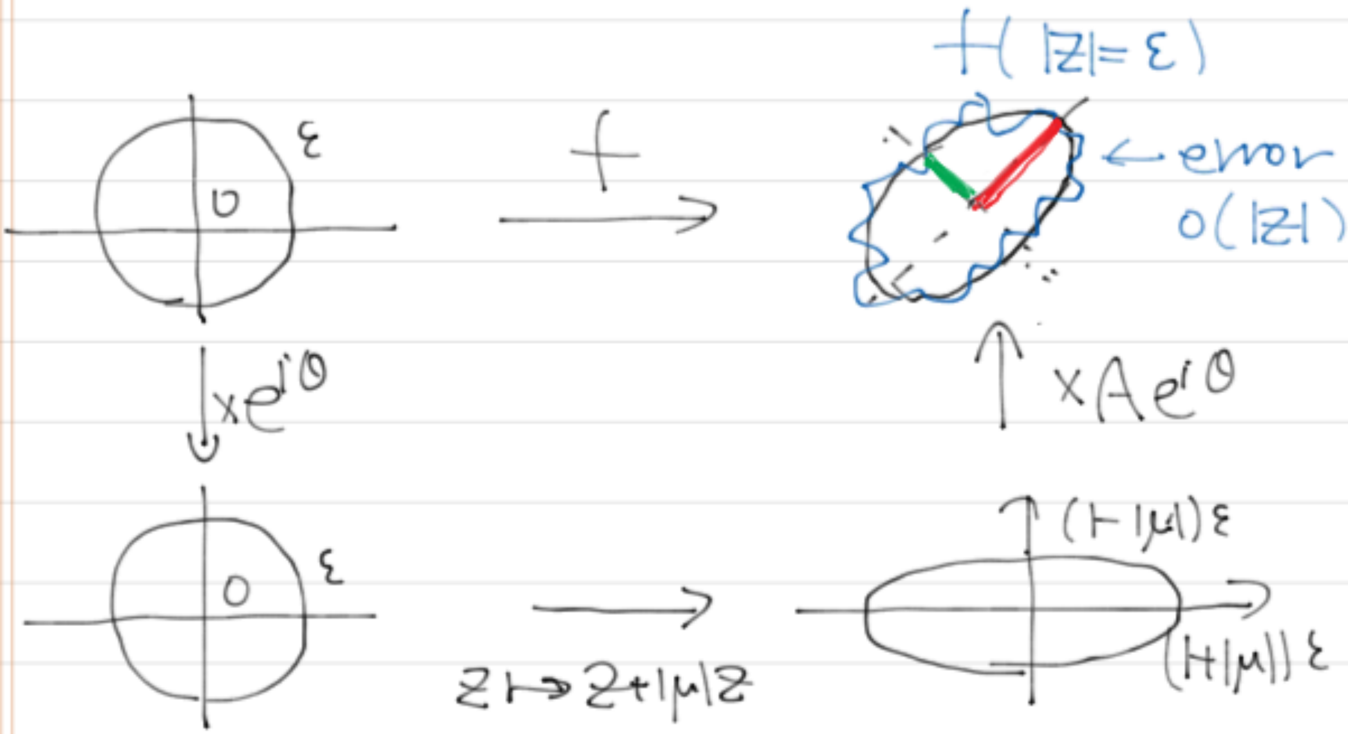


$$\frac{\text{major axis}}{\text{minor axis}} = \frac{1+|\mu|}{1-|\mu|} \le \frac{1+k}{1-k} = K$$

Ahlfors–Bers' measurable Riemann mapping Thm.

$$\{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : qc\} \longrightarrow M(\hat{\mathbb{C}}) \quad \cdots \circledast$$

$$f \xrightarrow{\text{map}} \mu_f$$

We call f a μ_f-qc

$$M(\hat{\mathbb{C}}) = \{\mu \in L^\infty(\hat{\mathbb{C}}) \mid \|\mu\|_\infty < 1\}$$

unit ball in $L^\infty(\hat{\mathbb{C}})$.

$\boxed{\text{Thm}}$ (Ahlfors–Bers)

The map $\circledast$ is surjective.

- If $\mu_f = \mu_g \Rightarrow g = A \circ f$ $(\exists A \in \mathrm{Aut}(\hat{\mathbb{C}}))$

Namely for $\mu \in L^\infty(\hat{\mathbb{C}})$, $\exists!\ f_\mu : \mu$-qc on $\hat{\mathbb{C}}$

we call normalized μ-qc

s.t.

- $\mu_{f_\mu} = \mu$
- f_μ fixes $0, 1, \infty$.

Moreover, $z \in \hat{\mathbb{C}}$,

$$M(\hat{\mathbb{C}}) \ni \mu \longmapsto f_\mu(z) \in \hat{\mathbb{C}} \quad \text{is holomorphic.}$$

fix

§3.2 Topology of Teichmüller space

Ahlfors-Bers' thm implies that (X: R.S.)

$$\forall \mu \in M(X) = \left\{ \mu = \mu(z)\frac{d\bar{z}}{dz} \;\middle|\; \|\mu\|_\infty < 1 \right\}$$

$$\exists f_\mu : X \longrightarrow {}^{\exists} X_\mu \;:\; \mu\text{-qc}$$

X_μ ← Riemann surface

◎ $y_1 = (X_1, f_1),\ y_2 = (X_2, f_2) \in T(S)$ ← type (g, n)

$$K^*(y_1, y_2) = \inf\{ K \mid {}^{\exists} h : X_1 \to X_2 : K\text{-qc} \text{ s.t. } h \circ f_1 \sim f_2 \}$$

$\sim$: homotopy

$$d_T(y_1, y_2) = \frac{1}{2} \log K^*(y_1, y_2)$$

The Teichmüller distance

$$\begin{tikzcd} & X_1 \arrow[dd, "h"] \\ S' \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & X_2 \end{tikzcd}$$

homotopically commutative

$\boxed{\text{Thm}}$ (Teichmüller) $(T(S), d_T)$ is a complete metric space s.t.

$$T(S) \cong \mathbb{R}^{6g-6+2n}$$

But, not isometric. In fact, we have

$\boxed{\text{Thm}}$ (M)

There is no map $\phi : T(S) \to T(S)$ s.t.

① ϕ admits a quasi inverse ψ, that is

$$\sup_{x \in T(S)} \{ d_T(\phi \circ \psi(x), x), d_T(\psi \circ \phi(x), x) \} < \infty$$

② ϕ satisfies

$$| d_T(\phi(x), \phi(y)) - K d_T(x, y) | \leq D$$

for some $K \neq 1$ and $D \geq 0$.

$\boxed{\text{Note}}$: (Royden, Earle-Kra, Markovic, Ivanov, M)

$$\mathrm{Isom}(T(S), d_T) \cong \mathrm{Mod}^*(S)$$

$$\mathrm{Mod}^*(S) = \mathrm{Homeo}(S) / \text{isotopy}$$

The extended mapping group of S

Unless $(g, n) = (0, 4), (1, 1), (1, 2)$.

⇒ This is a connection

$\boxed{\text{Complex analysis}} \Leftrightarrow \boxed{\text{Topology}}$

in Teichmüller geometry

ダンニャワード

§4. Bers–Thurston classification

§4.1. Thurston theory: Measured foliations.

Let

$$\mathcal{S} = \{\text{non trivial, non peripheral s.c.c on } S\} / \text{free homotopy}$$

$$\mathcal{WS} = \{t\alpha \mid t \geq 0,\ \alpha \in \mathcal{S}\}$$ weighted s.c.cs on S

formal product

$$\mathcal{S} \ni \alpha \hookrightarrow 1\cdot\alpha \in \mathcal{WS}$$

For $t\alpha, s\beta \in \mathcal{WS}$, we define the intersection number by

$$i(t\alpha, s\beta) = ts \min\{|\alpha' \cap \beta'| \mid \alpha' \in \alpha,\ \beta' \in \beta\}$$

usual geometric # number

$\mathcal{R} = [0,\infty)^{\mathcal{S}} = \{$ non-negative functions on $\mathcal{S}\}$

with pointwise convergent topology.

We embed $\mathcal{WS}$ into $\mathcal{R}$ by

$$\mathcal{WS} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto i(t\alpha, \beta)] \in \mathcal{R}$$

The closure of the image is called the space of measured foliations on S.

Thm (Thurston, Bonahon, Rees)
The intersection number

$$i : \mathcal{WS} \times \mathcal{WS} \longrightarrow [0,\infty)$$

extends continuously to

$$i : \mathcal{MF} \times \mathcal{MF} \longrightarrow [0,\infty)$$

Let

$$PR = (R - \{0\}) / \mathbb{R}_{>0}$$

The action : $\mathbb{R}_{>0} \times R \to R$

$(t, f) \mapsto tf$

proj : $R - \{0\} \to PR$: projection

$$PMF = \mathrm{proj}(MF - \{0\}) \subset PR$$

the space of projective measured foliations

Note The extended mapping class group

$$\mathrm{Mod}^*(S) = \mathrm{Homeo}(S) / \text{isotopy}$$

acts on $\mathcal{R}$ as follows:

$$[\omega] \in \mathrm{Mod}^*(S), \ f \in \mathcal{R}$$

$$[\omega]_*(f)(\alpha) = f(\omega^{-1}(\alpha)) \quad (\alpha \in \mathcal{S})$$

This action induces actions on $\mathcal{MF}$ and $P\mathcal{MF}$ of $\mathrm{Mod}^*(S)$.

⊙ $\mathrm{Mod}(S) = \mathrm{Homeo}^+(S) / \text{Isotopy}$

The mapping class group

§4.2. Thurston compactification

$y \in (Y, f) \in T(S)$, $\alpha \in \mathcal{S}$

$$l_y(\alpha) = l_Y(f(\alpha))$$

$$= \left(\begin{array}{c} \text{the length of the closed geodesic} \\ \text{homotopic to } f(\alpha) \end{array} \right)$$

the hyperbolic length of 'α' on y

$$\begin{array}{cccc} \widetilde{\Phi}_{Th} : & T(S) & \longrightarrow & \mathbb{R}^{\mathcal{S}} \\ & \cup\!\!\!\!\!\!\;\text{ } & & \cup \\ & y & \longmapsto & [\mathcal{S} \ni \alpha \mapsto l_y(\alpha)] \end{array}$$

Set $\Phi_{Th} := \mathrm{proj} \circ \widetilde{\Phi}_{Th} : T(S) \to P\mathbb{R}^{\mathcal{S}}$

$\boxed{\text{Thm}}$ (Thurston) $\Phi_{Th}(T(S))$ is relatively cpt.

$$\frac{l}{\max(l_i)} \leftarrow \text{filling fam}$$

$$\overline{T(S)}^{Th} = \Phi_{Th}(T(S)) \sqcup PMF$$

disjoint union

$$\cong \mathbb{B}^{6g-6+2n}$$

Thurston compactification

§4.3 Thurston classification

The extended mapping class group

$$\mathrm{Mod}^*(S) = \mathrm{Homeo}(S) / \mathrm{Isotopy}$$

acts on $T(S)$ as follows:

$$[\omega] \in \mathrm{Mod}^*(S), \quad y = (Y, f) \in T(S)$$

$$[\omega]_*(y) = (Y, f \circ \omega^{-1})$$

<u>Note</u>

$$\ell_{[\omega]_*(y)}(\alpha) = \ell_Y(f \circ \omega^{-1}(\alpha))$$

$$= \ell_y(\omega^{-1}(\alpha))$$

$\Rightarrow$ $[\omega]_* \circ \widetilde{\Phi}_{Th} = \widetilde{\Phi}_{Th} \circ [\omega]_*$ on $T(S)$

$\Rightarrow$ Any element in $\mathrm{Mod}^*(S)$ extends to $\overline{T(S)}^{Th}$.

Brower fixed pts thm

$\Rightarrow$ Any element $[\omega] \in \mathrm{Mod}^*(S) - \{id\}$
has a fixed pt in $\overline{T(S)}^{Th}$.

orientable

$\boxed{\text{Thm}}$ (Thurston) $[\omega] \in \mathrm{Mod}(S)$

① $[\omega]$ is of finite order
$\Leftrightarrow$ $[\omega]_*$ has a fixed pt in $T(S)$
$\Leftrightarrow$ $[\omega]$ is represented by a conformal automorphism on some R.S. X.

② $[\omega]$ is reducible. i.e.
$\exists\ \mathcal{A} = \{\alpha_1, \cdots, \alpha_n\} \subset \mathcal{S}$ disjoint
s.t. $\omega(\mathcal{A}) = \mathcal{A}$ (in the homotopy sense)

③ $[\omega]$ is irreducible i.e.
ω does not have fixpts in $\mathcal{S}$
$\Longleftrightarrow$ $\exists [\mu_s], [\mu_u] \in \mathcal{PMF}$ s.t
Thurston
$[\omega]_*([\mu_s]) = [\mu_s]$, $[\omega]_*([\mu_u]) = [\mu_u]$

In the third case, we call $[\omega]$ (a class of) pseudo-Anosov.

Note

① Two $[\mu_s]$, $[\mu_u]$ bind on S, i.e

$$i(\mu_s, F) + i(\mu_u, F) > 0$$
$$\forall F \in \mathcal{MF} - \{0\}$$

② $\forall y \in T(S)$

$$[\omega]_*^n(y) \to [\mu_s]$$
$$[\omega]_*^{-n}(y) \to [\mu_u] \quad (n \to \infty)$$
$$\text{in } \overline{T(S)}^{Th}$$

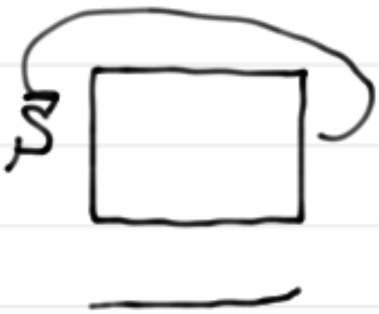
$(\exists \lambda > 1)$ $[\omega]_*(\mu_s) = \frac{1}{\lambda}\mu_s \; (= \omega(\mu_s))$

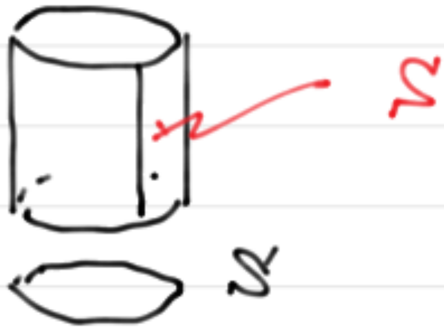
$[\omega]_*(\mu_u) = \lambda \mu_u \; (= \omega(\mu_u))$ in $\mathcal{MF}$

- Mapping torus

$\phi = [\omega] \in \mathrm{Mod}(S)$

$M_\phi = S \times [0,1] / (\omega(x), 0) \sim (x,1)$





M^3 is a compact 3-mfd which fibers over S^1

· $\chi(S) < 0$

① ϕ is of finite order $\Leftrightarrow$ $\mathbb{H}^2 \times \mathbb{R}$-structure

② ϕ is reducible $\Leftrightarrow$ M^3 contains non peripheral embedded incompressible torus

splits M^3 →

③ ϕ is p.A. $\Leftrightarrow$ M^3 admits a hyperbolic structure

§4.4. Bers classification

$[\omega] \in \mathrm{Mod}(S)$

$$a([\omega]) = \inf\{ d_T(y, [\omega]_*(y)) \mid y \in T(S) \}$$

Translation length of $[\omega]$ on $T(S)$

	$a([\omega]) = 0$	$a([\omega]) > 0$
"inf" attains.	elliptic	hyperbolic
"inf" not attain	parabolic	pseudo-hyperbolic

(compare : classification of $\mathrm{M\ddot{o}b}(\mathbb{D})$)

$\boxed{\text{Thm}}$ (Bers) $[\omega] \in \mathrm{Mod}(S)$

① $[\omega]$: elliptic $\Leftrightarrow$ $[\omega]$ of finite order
② $[\omega]$: hyperbolic $\Leftrightarrow$ $[\omega]$: pA
③ $[\omega]$: pseudo hyp / parabolic $\Leftrightarrow$ $[\omega]$: reducible

$\circ$ $[\omega]$: parabolic or pseudo-hyperbolic

$\Leftrightarrow$ $[\omega]$ reducible

$\exists\, \mathcal{A}$: a system of disjoint curves in S

s.t. $\omega(\mathcal{A}) = \mathcal{A}$ (in the homotopy sense)

$\exists\, n \in \mathbb{N}$ s.t. ω^n fixes (setwise)

any components of

$S - \bigcup_{\alpha \in \mathcal{A}} \alpha$

① ω is parabolic $\Leftrightarrow$ $\forall S'$: a component of

$S - \bigcup_{\alpha \in \mathcal{A}} \alpha$

$\omega|_{S'}$: finite order

② ω is pseudo-hyp $\Leftrightarrow$ $\exists S'$: component

s.t.

$\omega|_{S'}$: pA

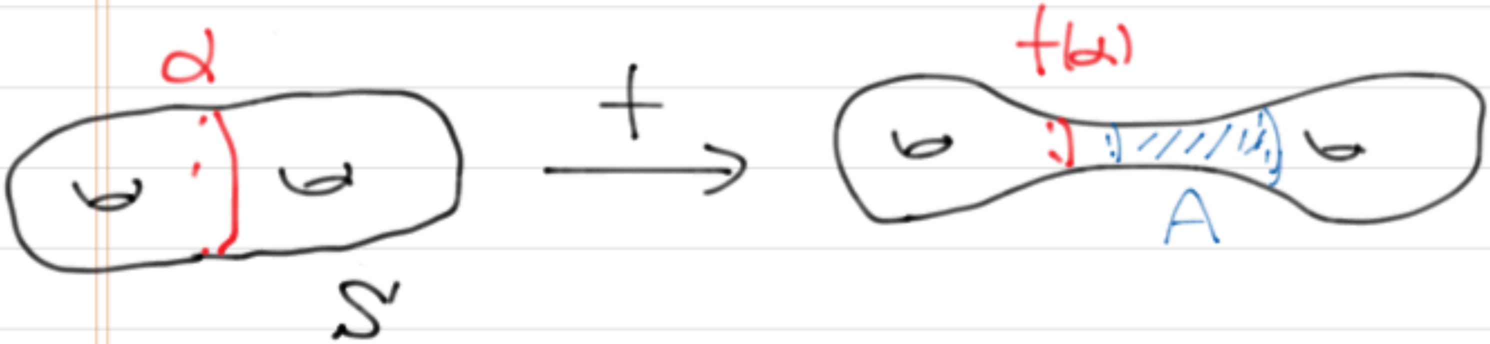
§5. Isometry of Teichmüller space
− Introduction to the extremal length geometry −

§5.1. Extremal length

$y = (Y, f) \in T(S)$, $\alpha \in \mathcal{S}$

Extremal length

$$\mathrm{Ext}_y(\alpha) = \inf_A \left\{ 1/\mathrm{mod}(A) \mid \mathrm{core}(A) \sim f(\alpha) \right\}$$

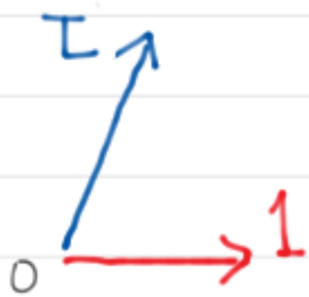


where mod(A) is the modulus of A :

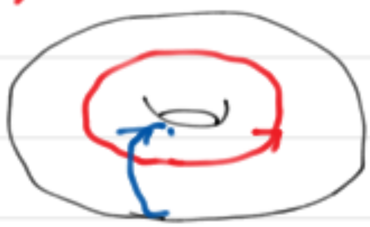
If $A \cong \{1 < |z| < r\}$ (biholomorphic)

$$\mathrm{mod}(A) = \frac{1}{2\pi} \log r$$

$\boxed{\text{Example}}$ When $Y = \mathbb{C}/\Gamma_\tau$: torus



$\Gamma_\tau = \langle z+1,\ z+\tau \rangle$
lattice



Then

$$\mathrm{Ext}_Y(\bigcirc) = \frac{1}{\mathrm{Im}\,\tau}$$

In general,

$\mathrm{Ext}_Y(\alpha)$ small $\Longleftrightarrow$ Y contains wide annulus

$\rightsquigarrow$ $f(\alpha)$ is short in Y

$\boxed{\text{Thm}}$ (Kerckhoff) If we set

$$\mathrm{Ext}_y(t\alpha) = t^2\, \mathrm{Ext}_y(\alpha),$$

Ext_y extends continuously to $\mathcal{MF}$.

$\boxed{\text{Note}}$ $\mathcal{MF}$ has a canonical PL-structure (Thurston)

Hence, "**directional derivative**" is defined.
Namely, we have the "**tangent cone**" at each point of $\mathcal{MF}$.

We can see that Ext_y is differentiable (M).

$\boxed{\text{Thm}}$ (Kerckhoff)

$$d_T(y_1, y_2) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)}$$

From the Kerckhoff's formula, we often call the geometry of the Teichmülle distance the extremal length geometry of Teichmülle space.

§ 5.2. Compactification in the extremal length geometry

We define

$$\tilde{\Phi}_{GM} : \begin{array}{ccc} T(S) & \longrightarrow & \mathbb{R} \\ \Psi & & \Psi \\ y & \longmapsto & \left[\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{\frac{1}{2}} \right] \end{array}$$

and

$$\Phi_{GM} = \mathrm{proj} \circ \tilde{\Phi}_{GM} : T(S) \to P\mathbb{R}$$

$\boxed{\text{Thm}}$ (Gardiner - Masur)

- $\Phi_{GM}(T(S))$ is relatively compact.
- $\underline{\overline{T(S)}^{GM}} = \Phi_{GM}(T(S)) \cup \partial_{GM} T(S)$

$\Longrightarrow$ $PMF \subsetneq \partial_{GM} T(S)$.

closure of the image of Φ_{GM}.

Set $|xy| = d_T(x,y)$ $(x, y \in T(S))$

$\boxed{\text{Thm}}$ (M) Fix $x_0 \in T(S)$. For $y_1, y_2 \in T(S)$,

set $i_{x_0}(y_1, y_2) = \exp(-2\langle y_1 | y_2 \rangle_{x_0})$,

where $\langle y_1 | y_2 \rangle_{x_0} = \frac{1}{2}(|x_0 y_1| + |x_0 y_2| - |y_1 y_2|)$.

Gromov product

Then, i_{x_0} extends continuously to $(\overline{T(S)}^{GM})^2$.

Furthermore, $i_{x_0}([F],[G])^2 = \dfrac{i(F,G)^2}{\mathrm{Ext}_y(F)\,\mathrm{Ext}_y(G)}$

$(F, G \in \mathcal{MF} - \{0\})$

$\rightsquigarrow$ This observation connects

$\boxed{\text{Complex analysis}} \longleftrightarrow \boxed{\text{topology}}$

at infinity of $T(S)$

In fact, when

$$x_n \to [F] \\ y_n \to [G] \quad \in \mathcal{PMF} \qquad \text{in } \overline{T(S)}^{GM}$$

$$\boxed{\langle x_n | y_n \rangle \to \infty} \iff \boxed{i(F,G)=0}$$

Teichmüller geometry — Topology

We can detect a topological condition

" $i(F,G)=0$ "

from the inside of $T(S)$

④ Gromov hyperbolic space

A metric space (X, d) is Gromov δ-hyperbolic if $\cdot\, x_0 \in X$: base pt

$\forall a, b, c \in X$

$$\langle a | b \rangle_{x_0} \geq \min\{\langle a|c\rangle_{x_0}, \langle b|c\rangle_{x_0}\} - \delta$$

$\boxed{\text{Known}}$ Set

$$\mathrm{Seq}(X) = \left\{ \{x_n\} \subset X \;\middle|\; \langle x_n | x_m \rangle \to \infty \atop n, m \to \infty \right\}$$

Then a relation "$\langle x_n | y_m \rangle \to \infty$" is an equivalence relation if (X, d) is Gromov hyperbolic

Prop (Masur-Wolf, McCarthy-Papadopoulos, Ivanov, Brock-Farb)

$(\mathcal{T}(S), d_T)$ is not Gromov hyperbolic

$\because$

Consider a seq $\{x_n\}, \{y_n\}, \{z_n\}$

$x_n \to [\alpha]$

$y_n \to [\beta]$ in $\overline{\mathcal{T}(S)}^{GM}$

$z_n \to [\gamma]$

Then $\langle x_n | z_n \rangle \to \infty$

$\langle y_n | z_n \rangle \to \infty$

but

$\langle x_n | y_n \rangle \not\to \infty$

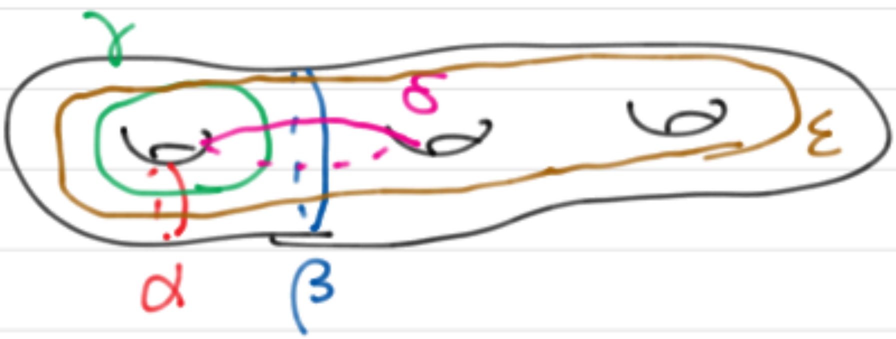
§5.3. Curve graph

A curve graph $C(S)$ is a graph with

Vertices $= \mathcal{S}$

Edges : $\alpha, \beta \in \mathcal{S}$. α and β are connected by an edge if

$$i(\alpha, \beta) = 0$$



For $[\omega] \in \mathrm{Mod}^*(S)$,

$[\omega]_*(\alpha) = \omega(\alpha)$ gives a simplicial automorphism and hence

$\exists$ $\mathrm{Mod}^*(S) \longrightarrow \mathrm{Aut}(C(S))$ homo.

$\boxed{\text{Thm}}$ (Ivanov - Korkmaz - Luo)

If S is not of type $(0,4), (1,1), (1,2)$,

$$\mathrm{Aut}(C(S)) \cong \mathrm{Mod}^*(S)$$

§5.4. Rough sketch of the proof

"$\mathrm{Isom}(T(S)) \cong \mathrm{Mod}^*(S)$"

① $\phi \in \mathrm{Isom}(T(S))$

(Liu-Su) ϕ extends homeomorphically to $\overline{T(S)}^{GM}$

Notice that for any $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$

$$\langle x_n | y_n \rangle \to \infty \iff \langle \phi(x_n), \phi(y_n) \rangle \to \infty$$

Hence, for $x_n \to p$, $y_n \to q$ $\in \partial_{GM} T(S)$

$$i_{x_0}(p, q) = 0 \iff i_{x_0}(\phi(p), \phi(q)) = 0$$

(2) We can see that

$$\phi(\mathcal{PMF}) = \mathcal{PMF}$$

(3) $[F] \in \mathcal{PMF}$

$$N([F]) = \{ G \in \mathcal{MF} \mid i(G, F) = 0 \}$$

(1) and (2) $\Rightarrow$ $\forall \alpha \in \mathcal{S}$

$$\phi(N([\alpha])) = N(\phi([\alpha]))$$

(4) (Ivanov) $F \in \mathcal{MF} - \{0\}$

$N([F])$ is of codimension one in $\mathcal{MF}$

$\Longleftrightarrow$ $[F] = [\alpha]$

⑤ Hence Φ induces

$$\Phi_* : \mathcal{S} \to \mathcal{S} \quad : \text{bijection}$$

⑥ By ①, $\alpha, \beta \in \mathcal{S}$

If $i(\alpha, \beta) = 0 \Rightarrow i(\Phi_*(\alpha), \Phi(\beta)) = 0$

$\rightsquigarrow \Phi_* \in \mathrm{Aut}(C(S))$

$\rightsquigarrow$ We have a homomorphism

$$\begin{tikzcd}
\mathrm{Isom}(T(S), d_T) \arrow[r] & \mathrm{Aut}(C(S)) \arrow[d, "\cong"] \\
& \mathrm{Mod}^*(S)
\end{tikzcd}$$

The action of $\mathrm{Mod}^*(S)$ on $T(S)$

If $(g,n) \neq (1,1)$, $(0,4)$, $(1,2)$

$\omega : T(S) \to T(S)$

Asymptotically conservative

$\Leftrightarrow \forall \{x_n\}, \{y_n\} \subset T(S)$

$\langle x_n | y_n \rangle \to \infty$

$\Leftrightarrow \quad (\omega(x_n) | \omega(y_n)) \to \infty$

$x = \{x_n\}$: seq. in $T(S)$

$\mathrm{Vis}(x) = \{\{y_n\} \mid \langle x_n | y_n \rangle \to \infty\}$

- ω_1, ω_2 close at ∞

$\Leftrightarrow \quad \forall \{x_n\}, \{y_n\}$ in $T(S)$

$\mathrm{Vis}(\{x_n\}) = \mathrm{Vis}(\{y_n\})$

$\Rightarrow \quad \mathrm{Vis}(\{\omega(x_n)\}) = \mathrm{Vis}(\{\omega(y_n)\})$

- ω' is an asymptotic inverse of ω

$\Leftrightarrow \quad \omega' \circ \omega$ and $\omega \circ \omega'$ are close to the identity at ∞

If AC ω admits asymptotic quasiinverse, we call invertible.

$$AC_{Inv}(T(S)) = \{\omega \mid AC \ \& \text{ invertible}\}$$

<u>Thm</u> ([M]) $\exists$ homo

$$[\cdot] : AC_{Inv}(T(S)) \longrightarrow Aut(C(S))$$

s.t.

$$\begin{tikzcd}
Mod^*(S) \arrow[r, leftrightarrow] & Isom(T(S)) \arrow[r, leftrightarrow] \arrow[dr, "\cong"'] & AC_{Inv}(T(S)) \arrow[d, "quot"] \arrow[dr, "{[\cdot]}"] & \\
& & \mathcal{AC} \arrow[r, "\cong"'] & Aut(C(S))
\end{tikzcd}$$

"close at ∞" is semigroup congruence

quotient semigp by the relation "close at ∞"