Flowing granular materials.

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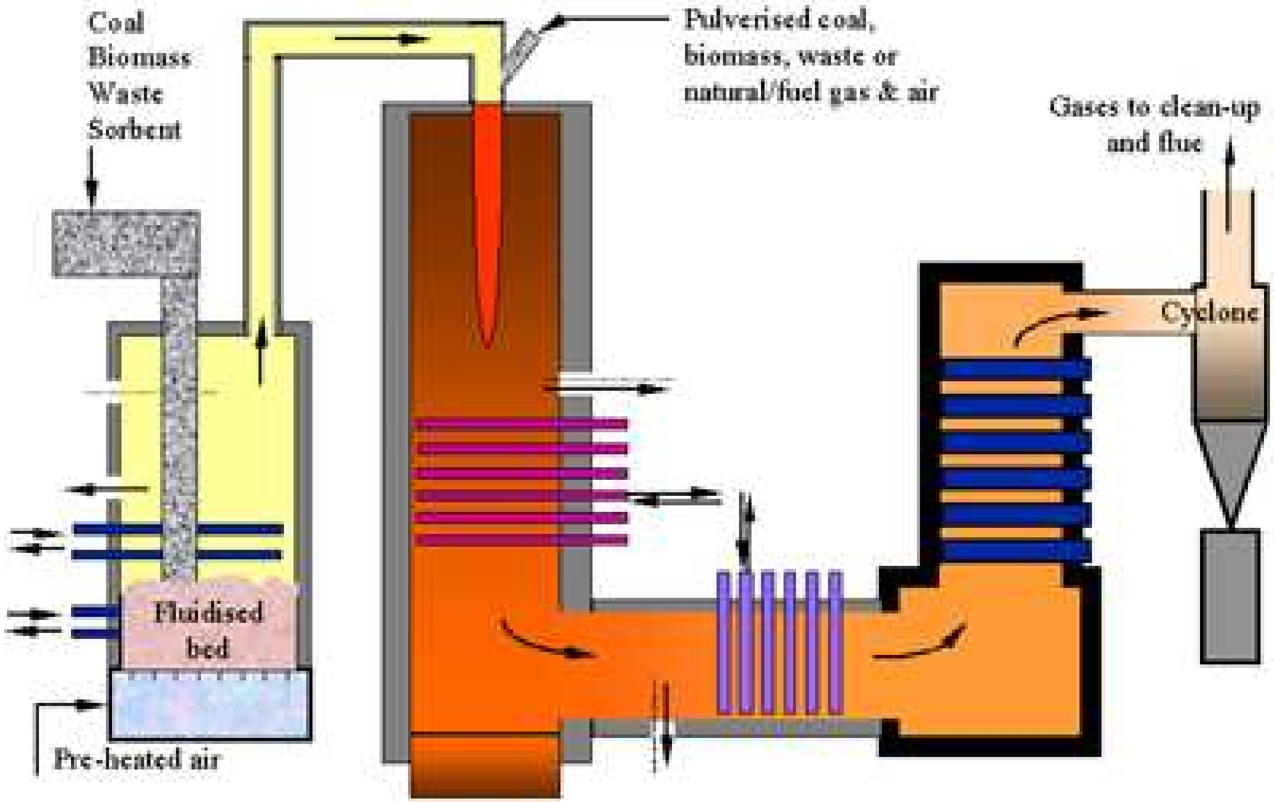
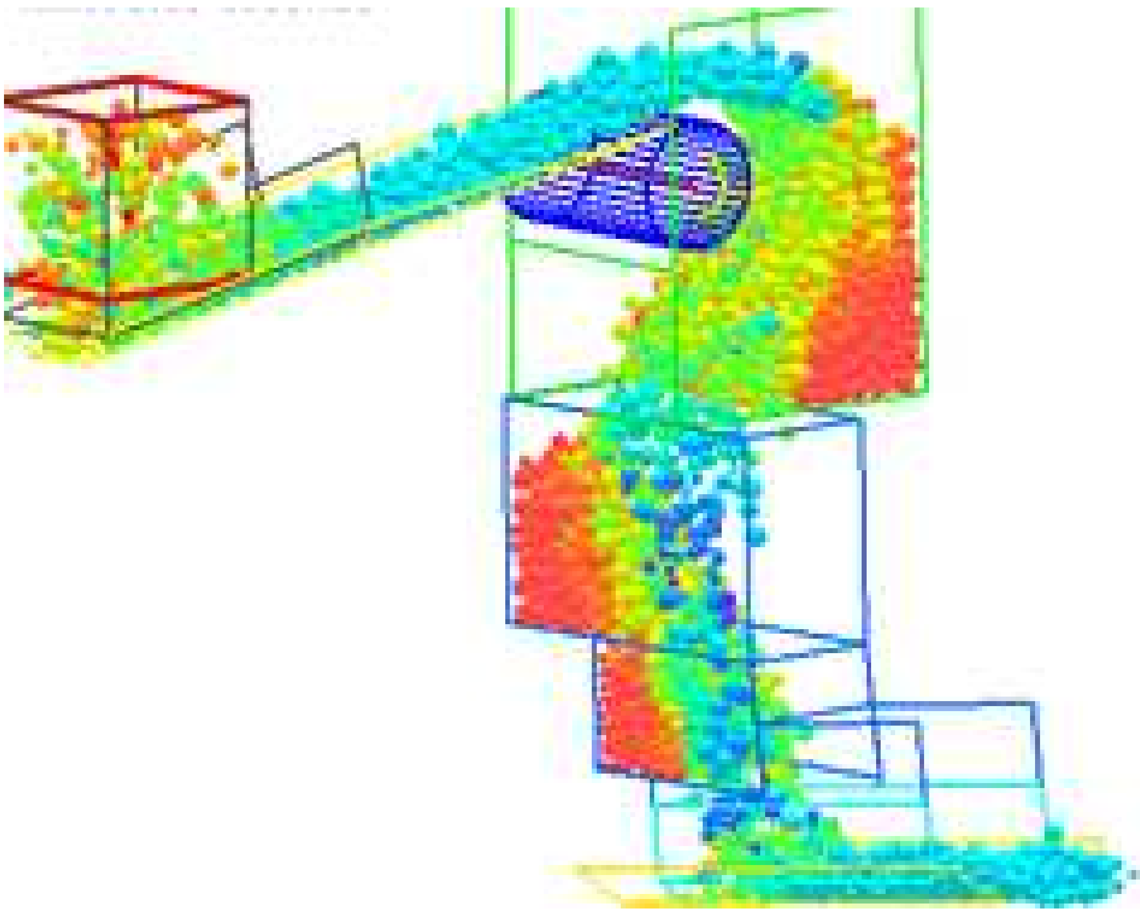
Industrial applications — fluidised beds.



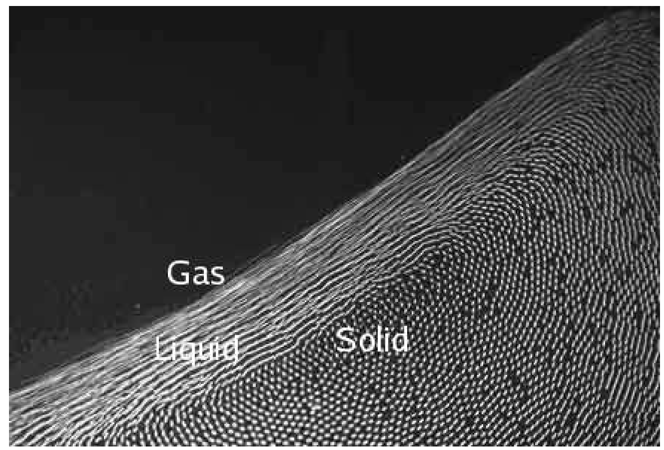
Industrial applications — Rotating drum mixer:

Industrial applications — Chute flow:

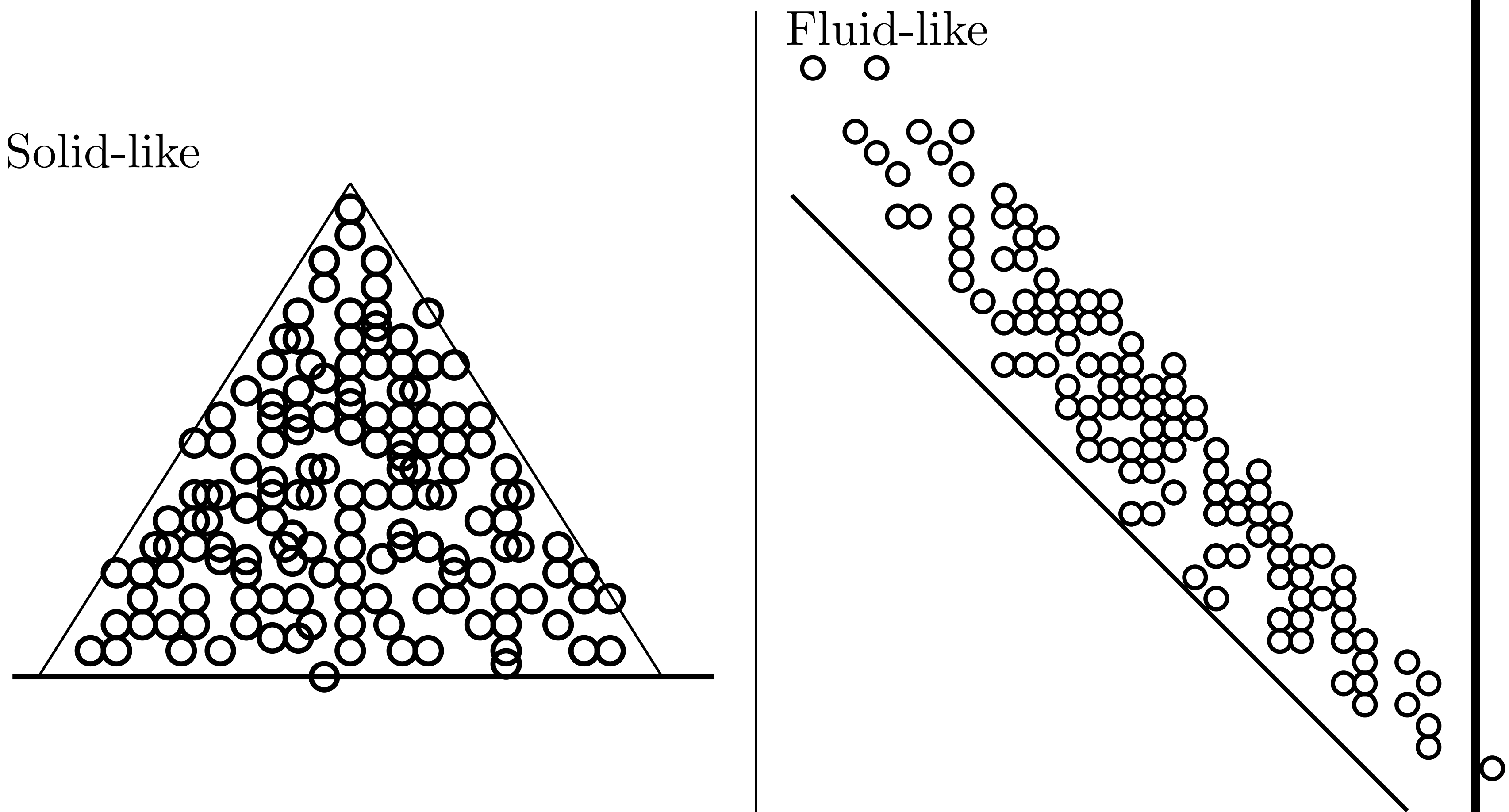


Silo collapse:

*www.civil.usyd.edu.au

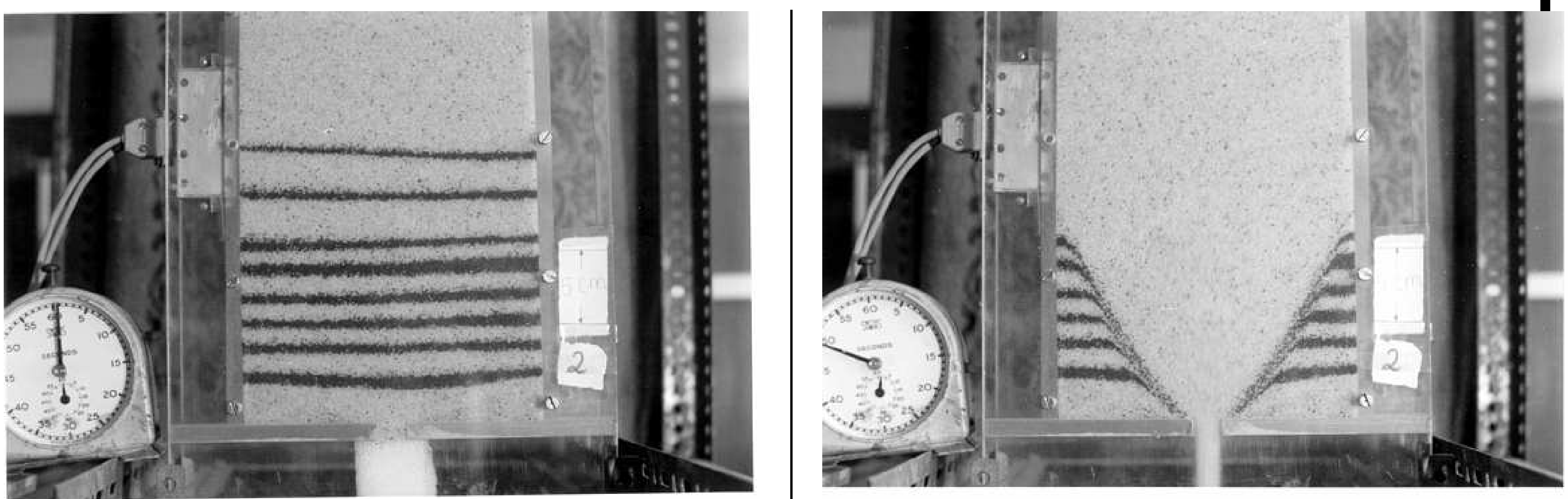


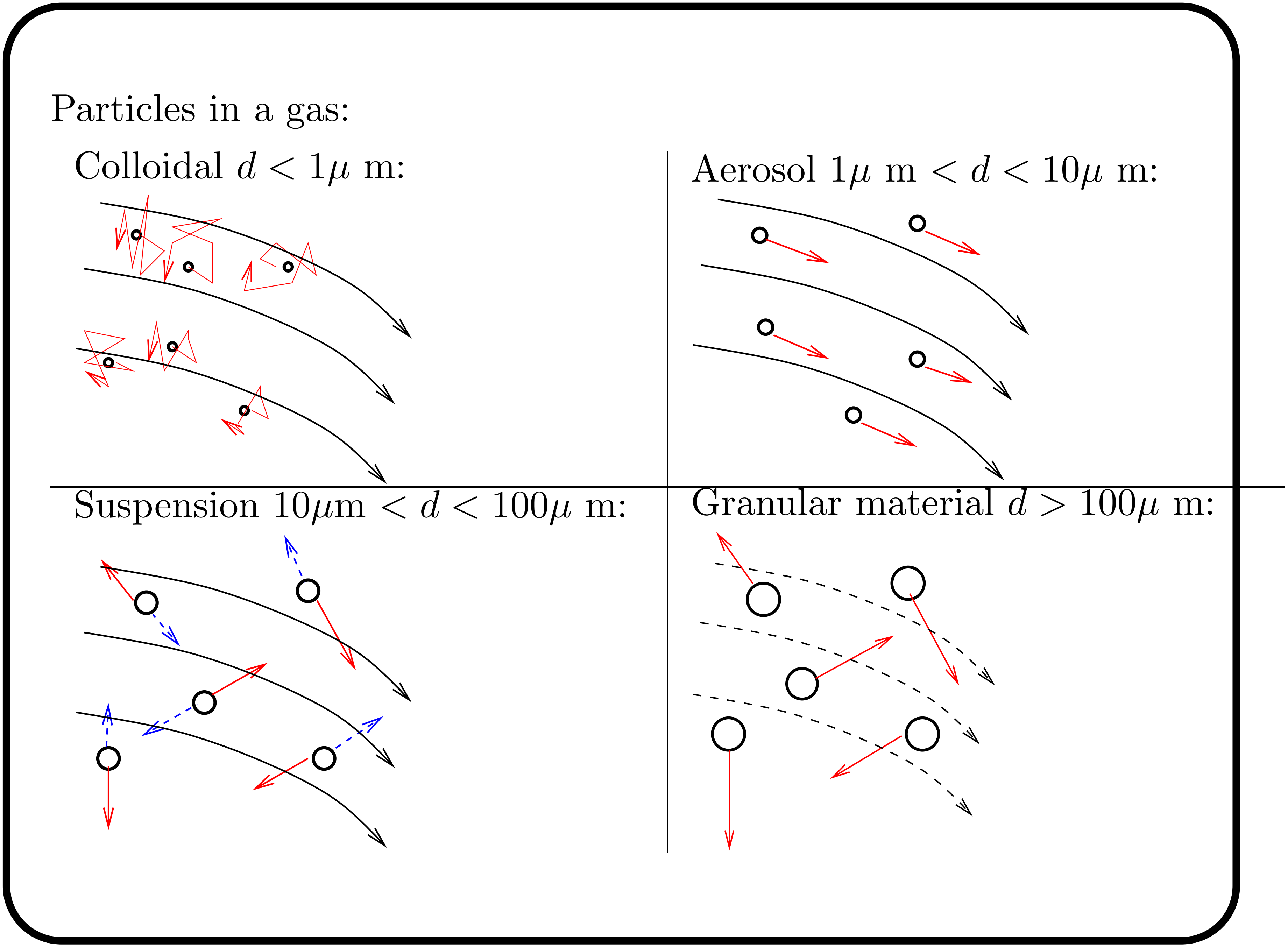
Granular materials



Granular materials

Creation of interfaces between solid-like & fluid-like




Particles in a gas:
Colloidal $d < 1\mu$ m:
Aerosol 1μ m $< d < 10\mu$ m:
Suspension 10μm $< d < 100\mu$ m:
Granular material $d > 100\mu$ m:

Gas-particle suspensions:

Parameters:

- Terminal velocity $U_t = (mg/3\pi\mu d) \sim d^2$.
- Brownian diffusivity $D_B = (k_B T/3\pi\mu d) \sim d^{-1}$.
- Peclet number (convection/Brownian diffusion)

  $\mathrm{Pe} = (U_t d/D_B) \sim d^{-4}$.
- Reynolds number (fluid inertia/fluid viscosity)

  $\mathrm{Re} = (\rho_g U_t d/\mu) \sim d^3$.
- Stokes number (particle inertia/fluid viscosity)

  $\mathrm{St} = (\rho_p U_t d/\mu) \sim d^3$.

$\rho_g = 1 kg/m^3$, $\rho_p = 10^3 kg/m^3$, $\mu = 1.8 \times 10^{-5} kg/m/s$, $T = 300K$.

Granular materials:

$1.5 \times 10^{-3} < U_t < 1.5 \times 10^{-1}$

$1.2 \times 10^{-12} < D_B < 1.2 \times 10^{-13}$.

$1.3 \times 10^{8} < (U_t d / D_B) < 1.3 \times 10^{12}$.

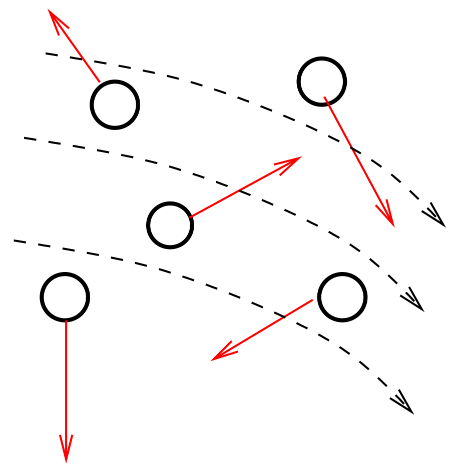
$8.5 \times 10^{-1} < \mathrm{Re} < 8.5 \times 10^{2}$.

$8.5 \times 10^{2} < \mathrm{St} < 8.5 \times 10^{5}$.

Fluid viscosity negligible.

Dominated by particle inertia & contact dissipation.

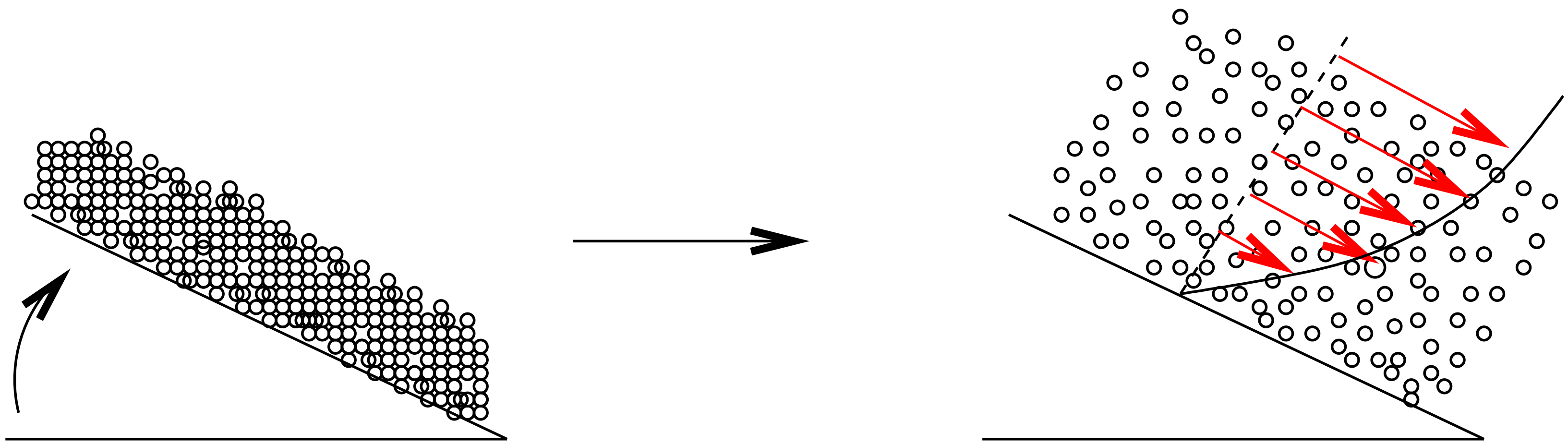
Granular material $d > 100\mu$ m:



Granular materials:

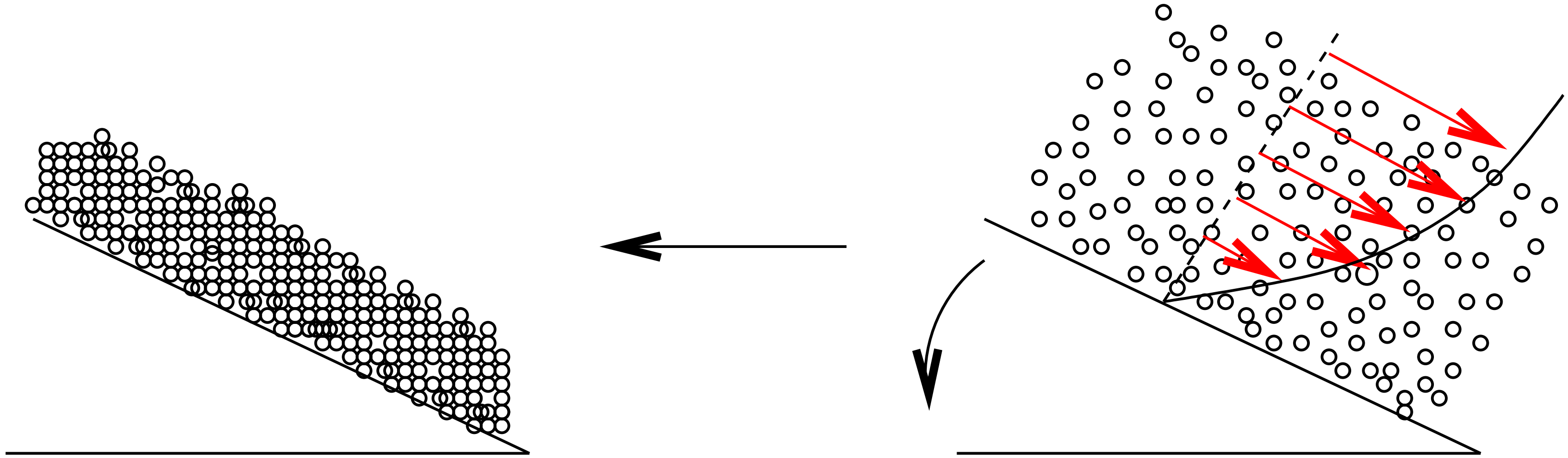
- Thermal velocity $(3k_BT/m)^{1/2} \sim 7.5 \times 10^{-12} m/s$
- Electrostatic, colloidal, disperson forces negligible.
- Fluid forces negligible.
- Energy dissipation due to particle interactions.
- Steady flow requires energy input to 'fluidise' the particles.
- Energy input from boundaries or through distributed forcing (mean shear).

Flow down inclined plane:



$$\sigma_{ij} = \sigma_{ij}^{(y)} + \sigma_{ij}^{(k)} ??$$

Flow down inclined plane:

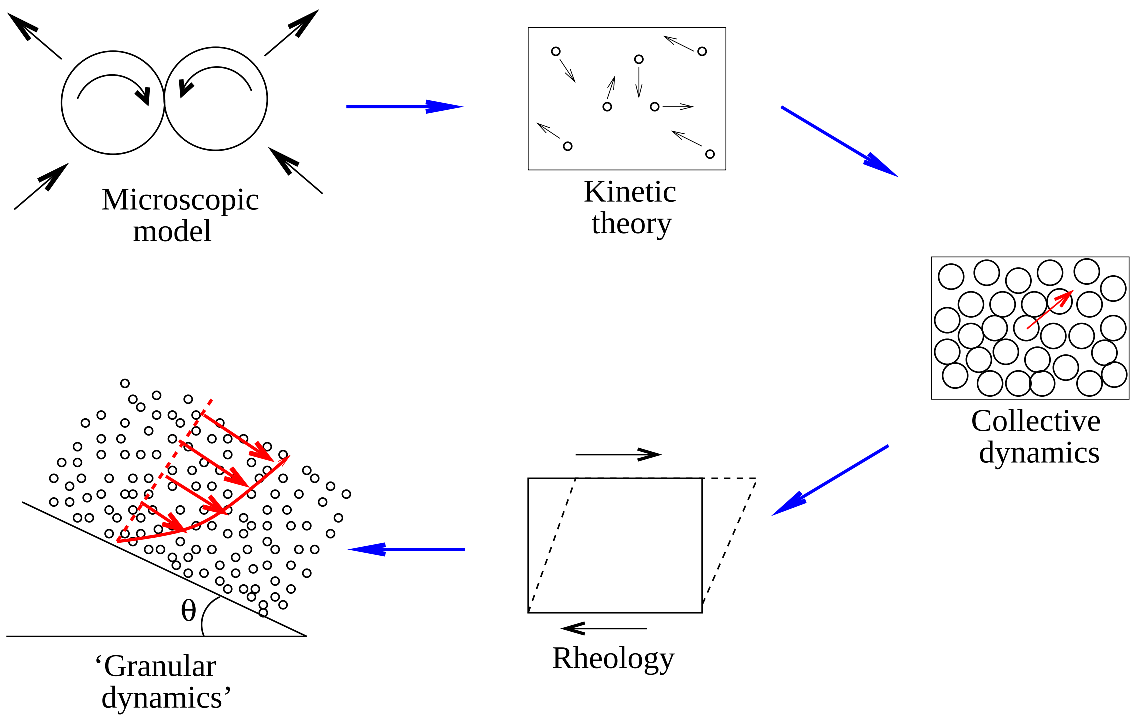


$$\sigma_{ij} = \sigma_{ij}^{(k)}!!$$

Flowing granular materials:

- Yield condition: Flow only when ratio or stresses exceeds critical value.
- Fluid constitutive relations cannot predict yield condition.
- Solid constitutive relations cannot predict flow.
- Fluidisation of particles facilitates flow.
- No thermal energy — fluidisation requires forcing either at boundaries or within flow.

Macroscopic behaviour from microscopic model:

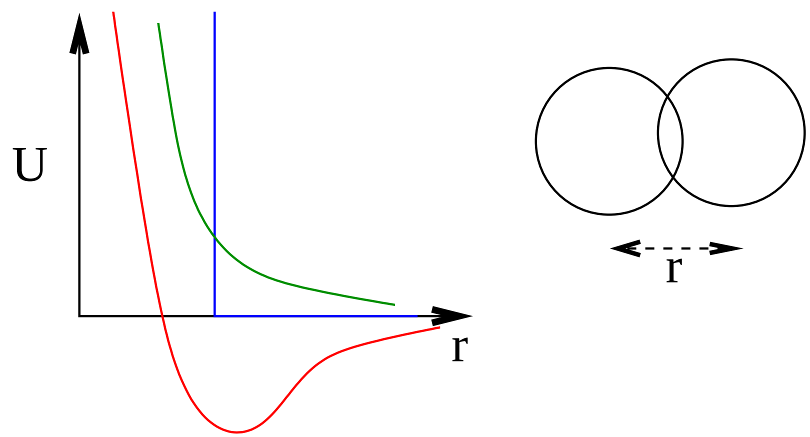


Outline:

- Microscopic models.
- Flow regimes.
- Conservation equations.
- Rate of deformation tensor.
- Constitutive relations.
- From constitutive relations to flow dynamics.

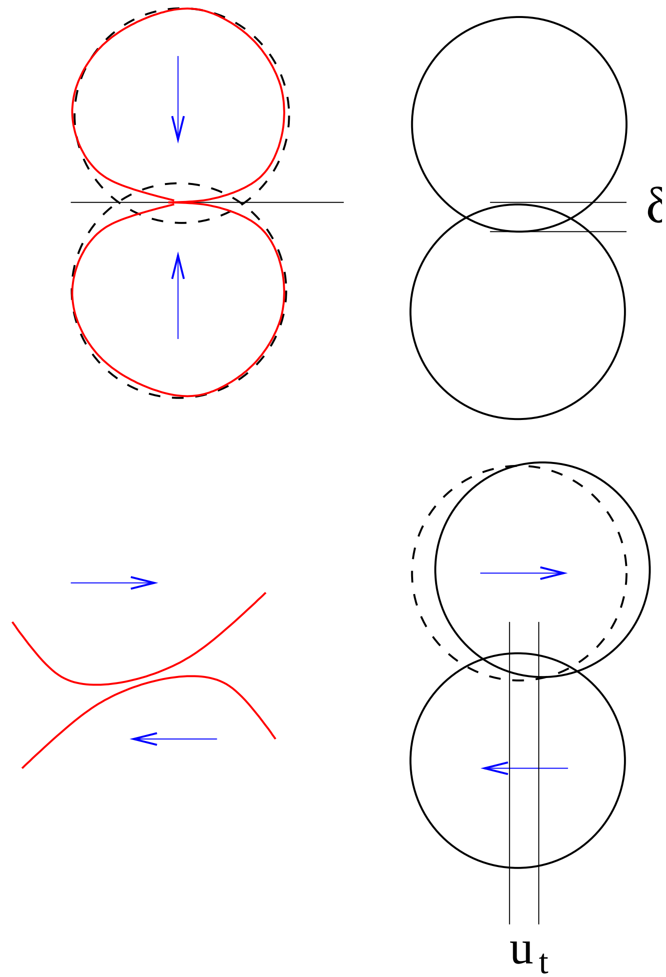
Interaction between particles:

- Forces only on contact.
- Force due to resistance to particle deformation.
- Forces repulsive (no attractive part).
- Forces dissipative.
- Well represented by hard-particles in some limits.



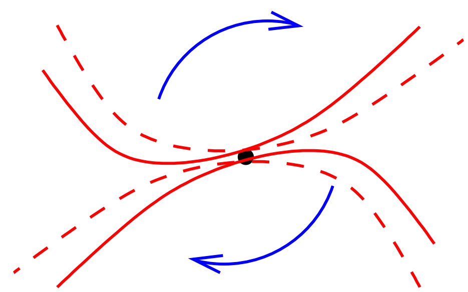
Contact laws:

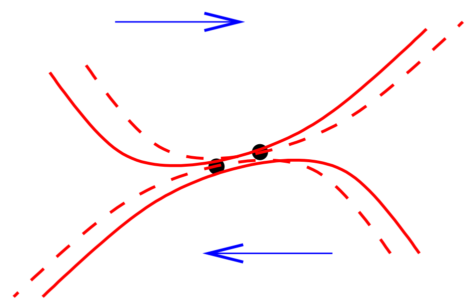
- Normal contact force expressed in terms of 'overlap' δ.
- Tangential contact force in terms of tangential displacement vector $\mathbf{u}_t$.
- Tangential displacement vector initialised to zero at contact.



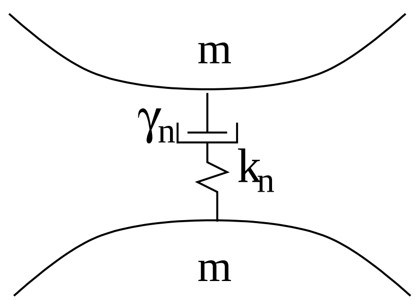
Contact laws:

- Tangential contact — slipping and sticking.
- Sticking — motion of particles around fixed contact point.
- Sliding — contact point moves.





Microscopic contact models: Linear spring-dashpot:

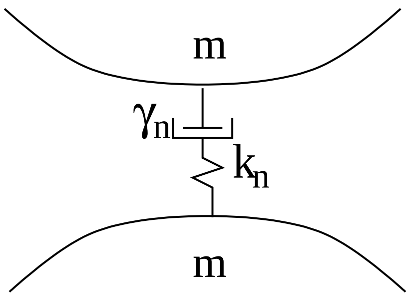


Normal force:

$$F_n = -k_n \delta - m_{\text{eff}} \gamma_n v_n$$

- Normal velocity $v_n = \frac{d\delta}{dt}$
- Spring constant k_n.
- Damping constant γ_n.
- Effective mass $m_{\text{eff}} = (m_i m_j / (m_i + m_j))$

Microscopic contact models: Linear spring-dashpot:



Two-body interaction:

$$m\frac{d^2\delta}{dt^2} = -k_n\delta - m_{\text{eff}}\gamma_n\frac{d\delta}{dt}$$

Initial conditions

$\delta = 0$, $(d\delta/dt) = -v_0$ at $t = 0$.

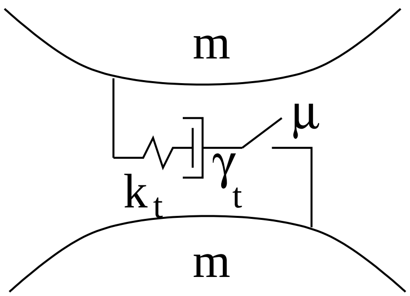
Solution:

$$\delta = \frac{v_0 e^{(-\gamma_n t/2)} \sin\left(t\sqrt{2k_n/m - \gamma_n^2/4}\right)}{\sqrt{(2k_n/m) - (\gamma_n^2/4)}}$$

$$t_{col} = (\pi/\sqrt{2k_n/m - \gamma_n^2/4})$$

$$(-v_f/v_0) = \exp\left(-\gamma_n t_{col}/2\right) = e$$

Microscopic contact models: Linear spring-dashpot:



Tangential motion:

Tangential displacement $\mathbf{u}_t$.

Tangential relative velocity $\mathbf{v}_t = \mathbf{v} - \mathbf{v}_n$

$$\frac{d\mathbf{u}_t}{dt} = \mathbf{v}_t - \frac{\mathbf{u}_t.\mathbf{v}(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2}$$

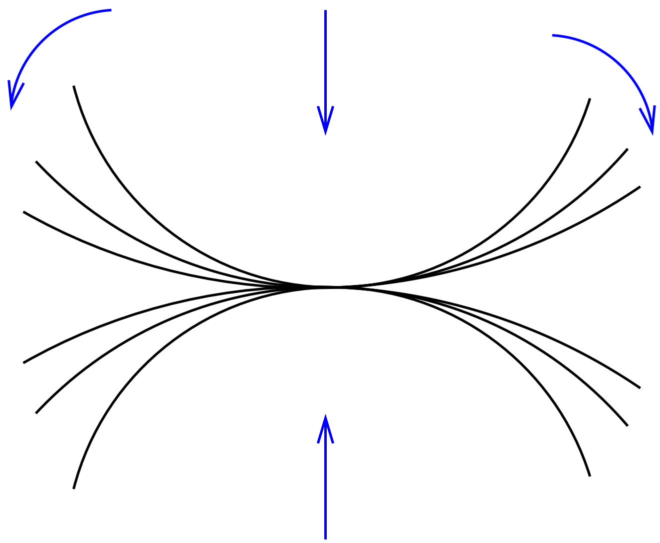
Sticking contacts $(|\mathbf{F}_t| < \mu|\mathbf{F}|_n$:

$$\mathbf{F}_t = -k_n \mathbf{u}_t - \gamma_n m_{eff} \mathbf{v}_t$$

Sticking contacts $(|\mathbf{F}_t| > \mu|\mathbf{F}|_n$:

$$|\mathbf{F}_t| = \mu|\mathbf{F}_n|$$

Microscopic contact models: Hertzian spring-dashpot:

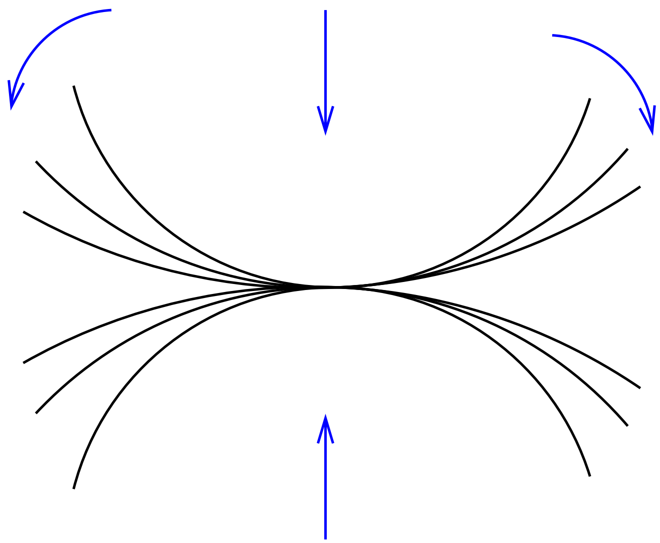


Microscopic interaction between smooth surfaces:

- Area of interaction increases as overlap increases.
- Stiffness should increase as overlap increases.

Hertz contact law:
$F_n = \sqrt{(\delta/d)}(-k_n\delta - \gamma_n m_{eff} v_n)$
Sphertical particles exact result:
$k_n = \frac{Ed^{1/2}}{3(1-\nu^2)}$
(Mindlin & Deresiewicz 1953).
Young's modulus E, Poisson ratio ν.

Microscopic contact models: Hertzian spring-dashpot:



Hertz contact law: Normal:

$$F_n = \sqrt{(\delta/d)}(-k_n \delta - \gamma_n m_{eff} v_n)$$

Tangential sticking: For $|\mathbf{F}_t| < \mu |\mathbf{F}_n|$

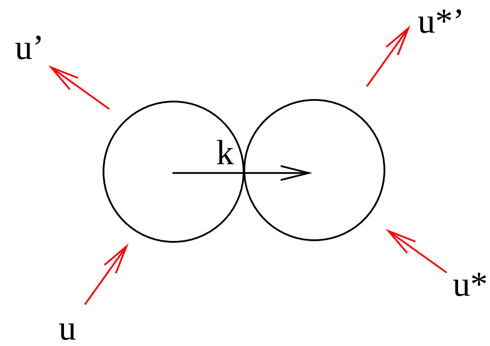
$$\mathbf{F}_t = \sqrt{(\delta/d)}(-k_t \mathbf{u}_t - \gamma_t m_{eff} \mathbf{v}_t)$$

Tangential sliding: For $|\mathbf{F}_t| > \mu |\mathbf{F}_n|$

$$|\mathbf{F}_t = \mu |\mathbf{F}_n|$$

Microscopic model — hard particles:

Smooth inelastic particles.



Relative velocity $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$

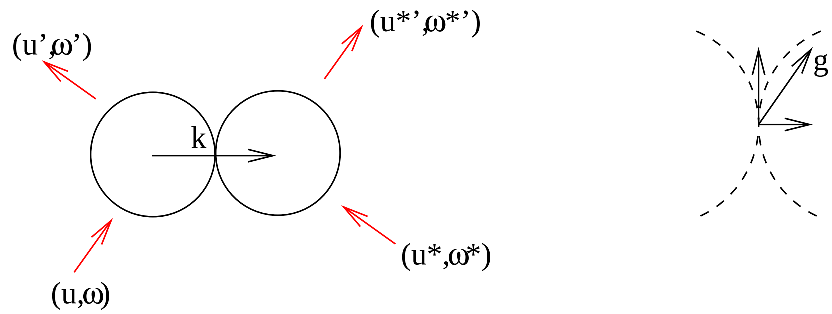
$$w'_k = -e_n w_k = -(1 - \varepsilon_n^2) w_k$$

$$w'_t = w_t$$

Energy conserved for $\varepsilon_n = 0$.

Microscopic model — hard particles:

Rough inelastic particles:



$$\mathbf{g} = \mathbf{v} - \mathbf{v}^* - \mathbf{k} \times (\omega + \omega^*)$$

$$\mathbf{g}'.\mathbf{k} = -e_n \mathbf{g}.\mathbf{k}$$

$$\mathbf{k} \times .\mathbf{g}' = -e_t \mathbf{k} \times .\mathbf{g}$$

Energy conserved for $e_n = 1$ and $e_t = \pm 1$.

Smooth inelastic particles:

$e_t = -1; (1 - e_n^2) = \varepsilon_n^2 \ll 1$

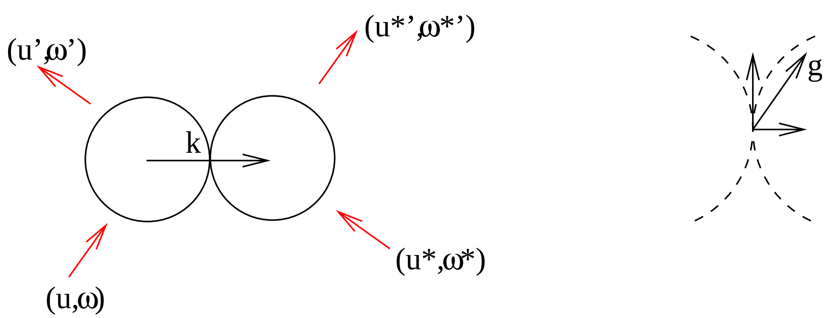
Rough inelastic particles:

$e_t = 1;$

$(1 - e_n^2) = \varepsilon_n^2 \ll 1$

Microscopic model — hard particles:

Rough inelastic particles:



$$\mathbf{g} = \mathbf{v} - \mathbf{v}^* - \mathbf{k} \times (\omega + \omega^*)$$

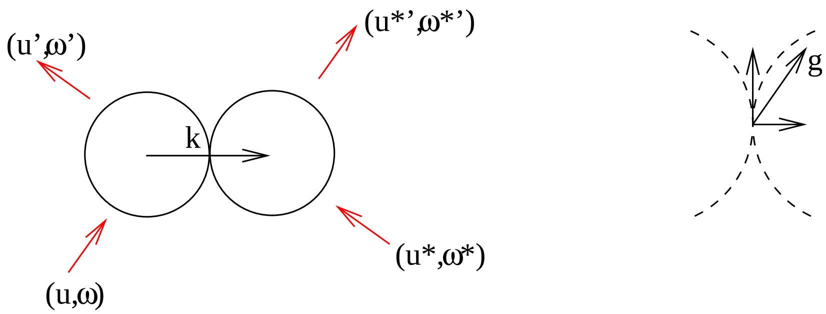
$$\mathbf{g}'.\mathbf{k} = -e_n \mathbf{g}.\mathbf{k}$$

$$\mathbf{k} \times .\mathbf{g}' = -e_t \mathbf{k} \times .\mathbf{g}$$

Tangential impulse $\mathbf{I}_t = m_{eff}(\mathbf{g}_t' - \mathbf{g}_t)$

If $|\mathbf{I}_t| > \mu|\mathbf{I}_n|$, then $|\mathbf{I}_t| = \mu|\mathbf{I}_n|$.

Microscopic model — hard particles:



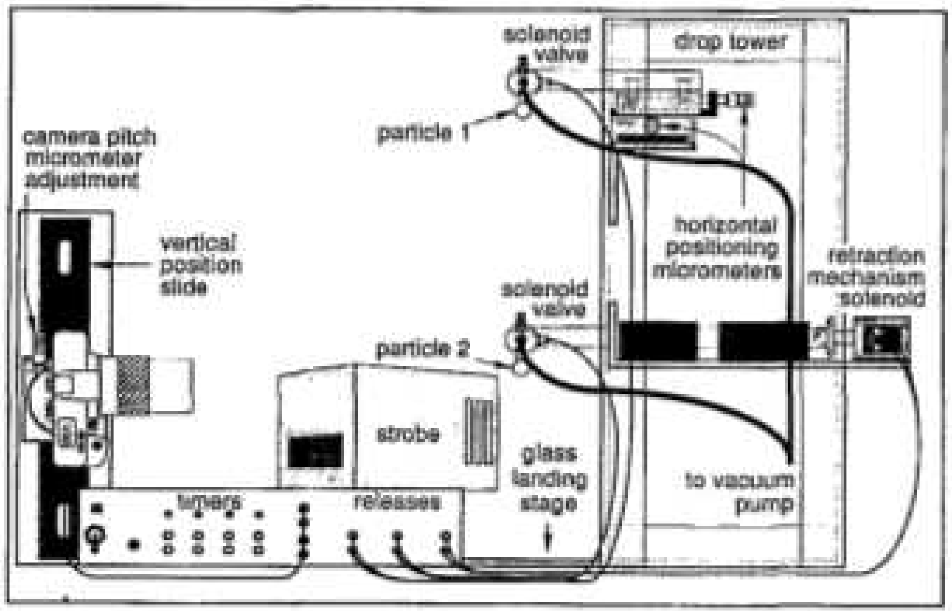
Microscopic model based on hard visco-elastic particles (Ramirez et al PRE 60, 4465, 1999).

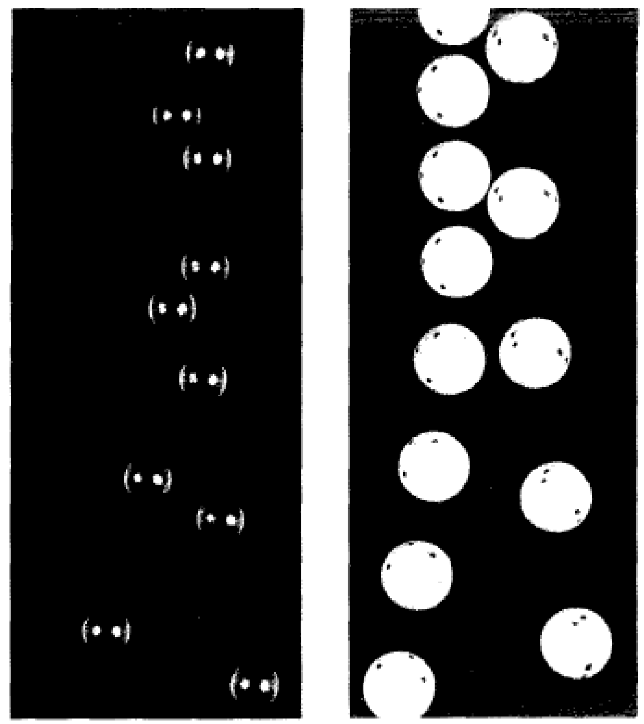
$$e = 1 - C_1 A \frac{\rho}{m_{eff}} |\mathbf{g}.\mathbf{k}|^{1/5} - C_2 \left(\frac{A\rho}{m_{eff}} \right)^2 |\mathbf{g}.\mathbf{k}|^{2/5} + \dots$$

$$A = \frac{1}{2} \frac{(3\eta_b - \eta)^2}{(3\eta_b + 2\eta)} \frac{(1-\nu^2)(1-2\nu)}{\nu^2}$$

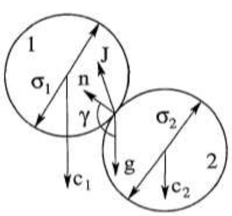
Microscopic laws: Experiments:

Forster et al, Phys Fluids **6**, 1108, 1994:





(a) (b)



Microscopic laws: Experiments: Forster et al, Phys Fluids **6**, 1108, 1994:

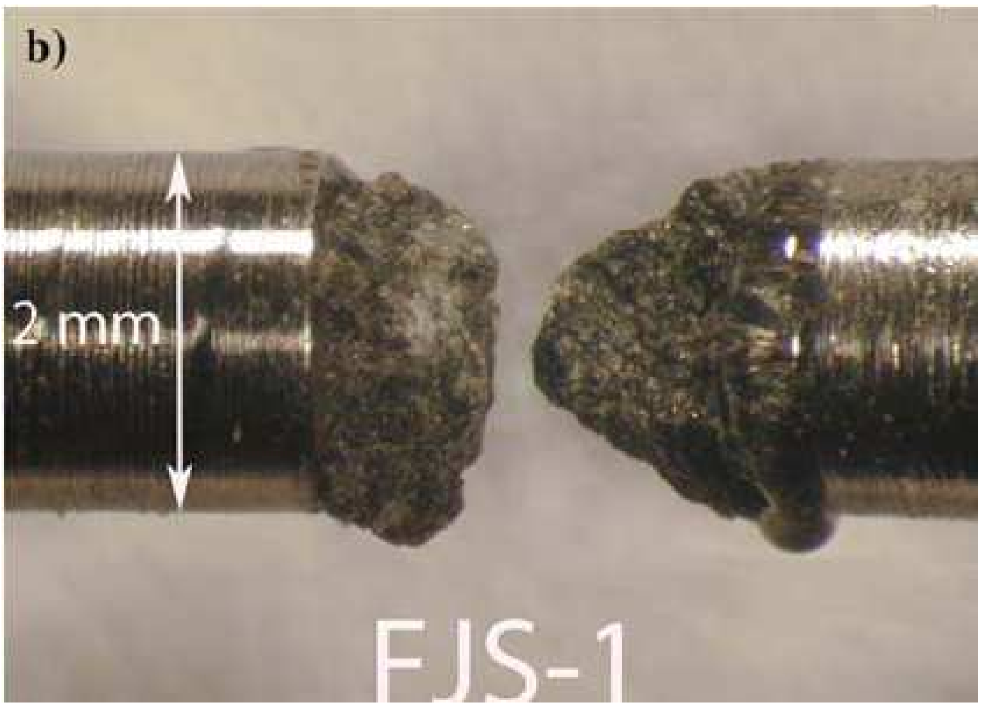
TABLE I. Sphere properties.

Material		Soda lime glass	Cellulose acetate
Finish		polished, grade "200"	ashed
Diameter (mm)		3.18 ± 0.03	5.99 ± 0.03
Density (g/cm^3)		2.5	1.319
Poisson's ratio		0.22	0.28[a]
Young's modulus (N/m^2)		$7.1\ 10^{10}$	3.2×10^9 [a]
	e	0.97 ± 0.01	0.87 ± 0.02
Binary collisions	μ	0.092 ± 0.006	0.25 ± 0.02
	β_0	0.44 ± 0.07	0.43 ± 0.06
Relative contact velocities		$0.64 < \lvert \mathbf{g} \cdot \mathbf{n} \rvert < 1.2$ m/s	$0.29 < \lvert \mathbf{g} \cdot \mathbf{n} \rvert < 1.2$ m/s
		$0.06 < \lvert \mathbf{g} \cdot \mathbf{t} \rvert < 0.41$ m/s	$0.14 < \lvert \mathbf{g} \cdot \mathbf{t} \rvert < 0.86$ m/s
	e	0.831 ± 0.009	0.891 ± 0.003
Wall collisions	μ	0.125 ± 0.007	0.208 ± 0.007
	β_0	0.31 ± 0.06	0.39 ± 0.07
Relative contact velocities		$1.0 < \lvert \mathbf{g} \cdot \mathbf{n} \rvert < 1.7$ m/s	$0.67 < \lvert \mathbf{g} \cdot \mathbf{n} \rvert < 1.7$ m/s
		$0.24 < \lvert \mathbf{g} \cdot \mathbf{t} \rvert < 0.81$ m/s	$0.06 < \lvert \mathbf{g} \cdot \mathbf{t} \rvert < 1.2$ m/s
Manufacturer		Winsted Precision Ball Co.	Engineering Laboratories
Aluminum plate		Density = 2.7 g/cm^3 Young's modulus = 6.9×10^{10} N/m^2 Poisson's ratio = 0.33 Machine finish	

[a]Estimates, see Drake.[6]

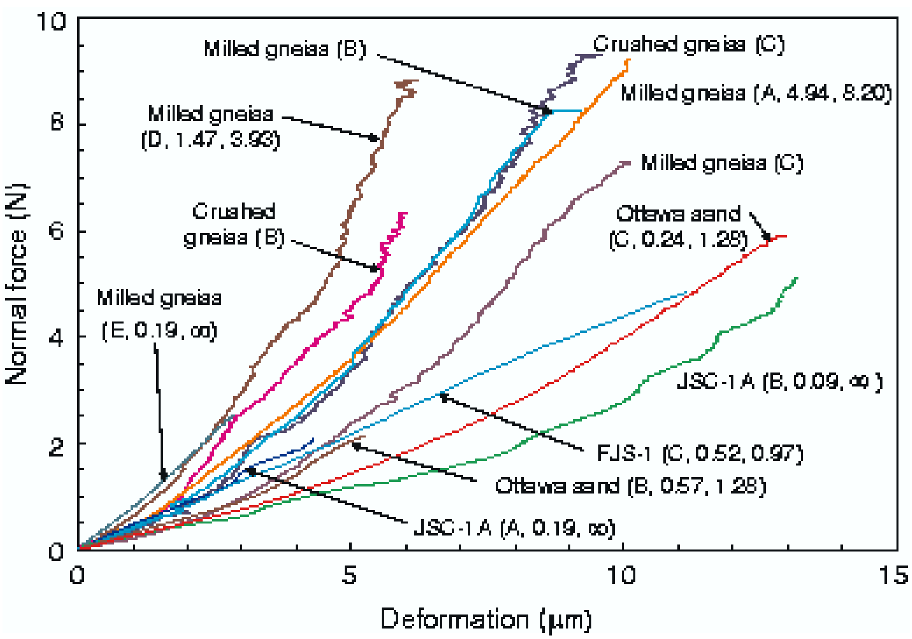
Contacts between real particles: Experiments.

Cole & Peters (*Gran. Matt.* **10**, 171, 2008; **9**, 309, 2007).



- Particles mounted on pins.
- Pins pressed against each other.
- Measure force and displacement.

Contacts between real particles: Experiments.



- Force-displacement curves.
- Back out spring constant.

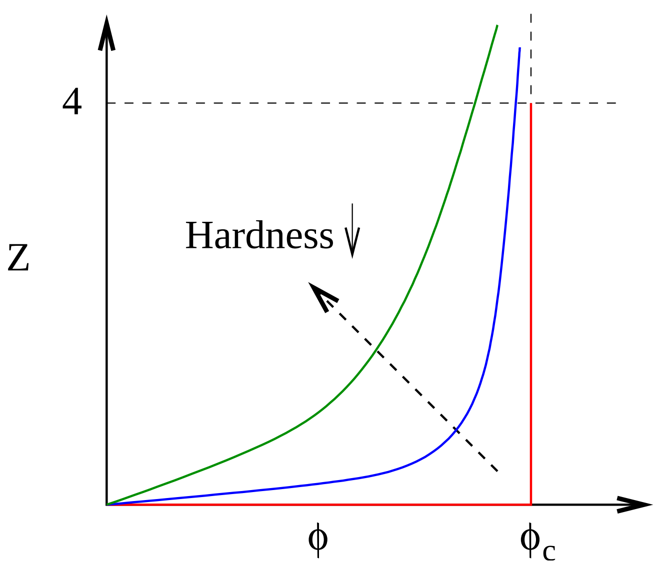
Rough sand particles linear contact law due to asperities,
$d = 0.2 - 2mm$ $k_n = 10^6 N/m$.

Smooth particles — Hertzian contact law,

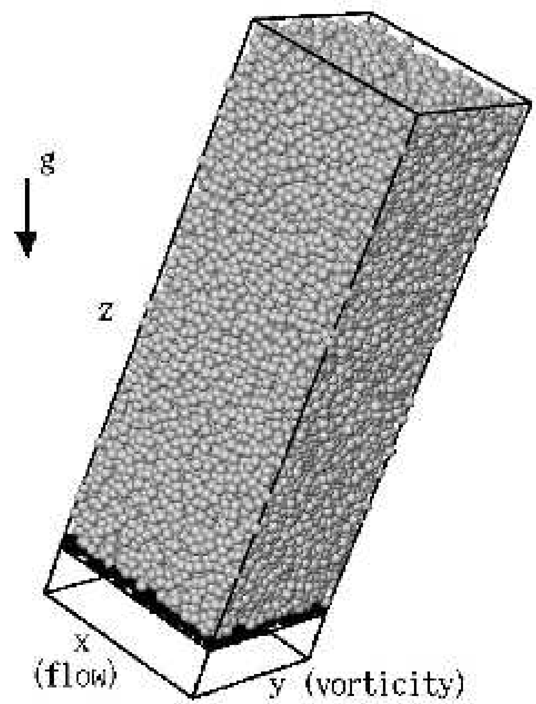
$k_n \approx 0.8$ times Mindlin-Deresiewicz prediction, $k_n \sim Ed^{1/2}$.

Flowing hard particles: Realistic?

- Collisions instantaneous.
- Av. co-ordination number $Z \ll 1$.



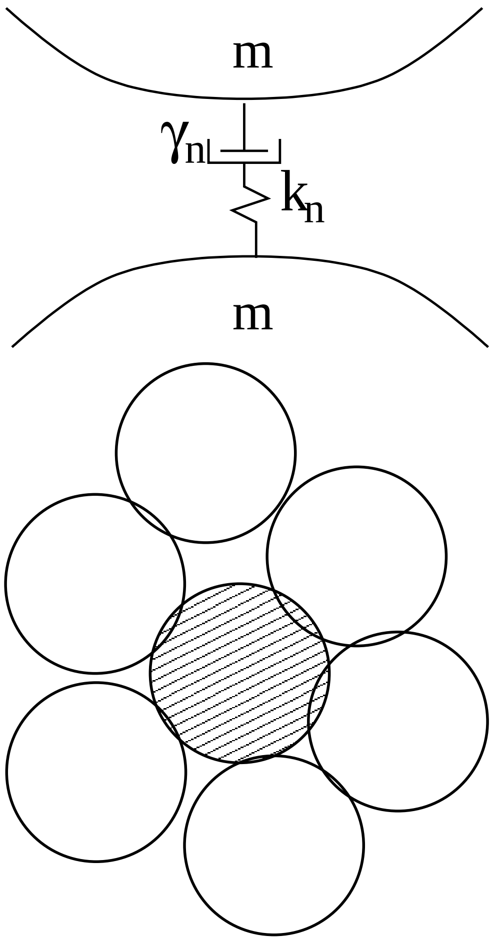
Inclined plane flow (Silbert et al 2001)

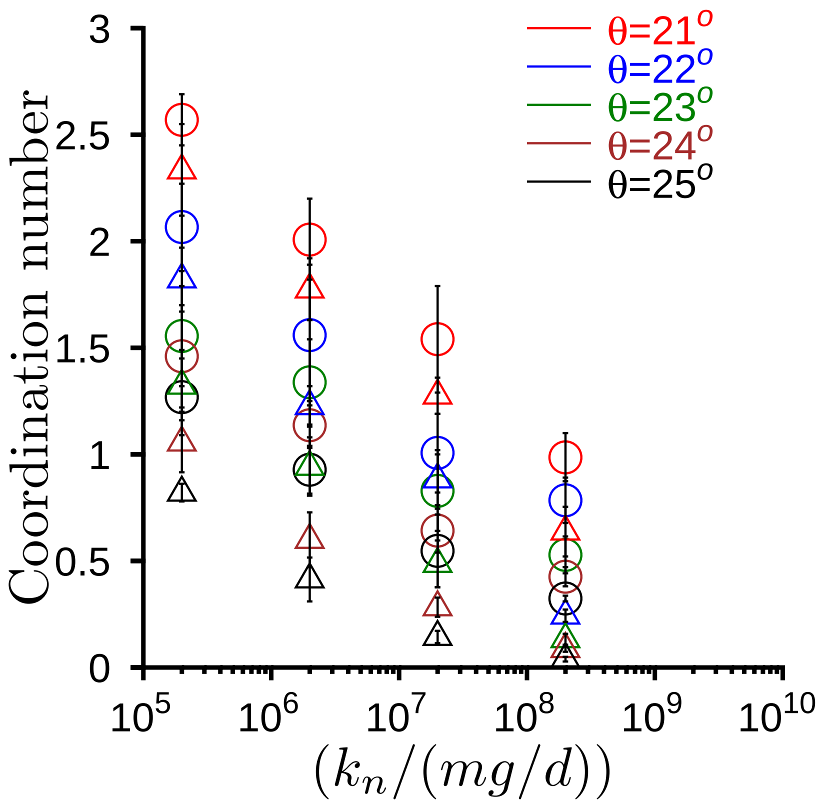


Flow starts at $\theta = 21^o$,
Stable flow till $\theta = 25^o$.

Flowing hard particles: Realistic?

Average co-ordination number:



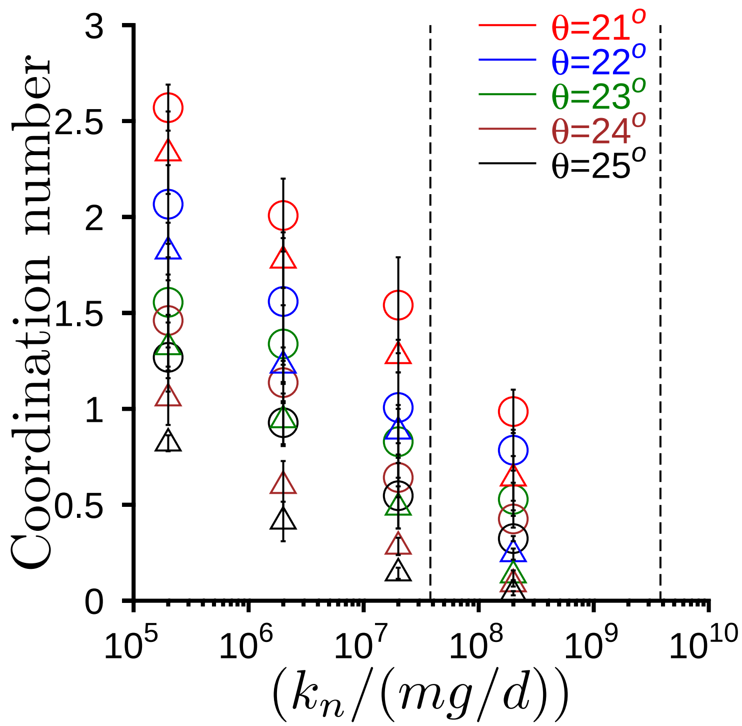


$\circ e_n = 0.5$; $\triangle e_n = 0.9$.

Flowing hard particles: Contact model.

Rough particles:

- Linear contact law
  $F = k_n \delta$
  due to compression of asperities.
- $k_n = 10^6 N/m$ for particles in 0.2-2mm size.
- Scaled spring constant
  $k_n/(mg/d^{3/2}) \sim 7.6 \times 10^7 - 7.6 \times 10^9$.
  for $d = 1 - 0.1mm$.

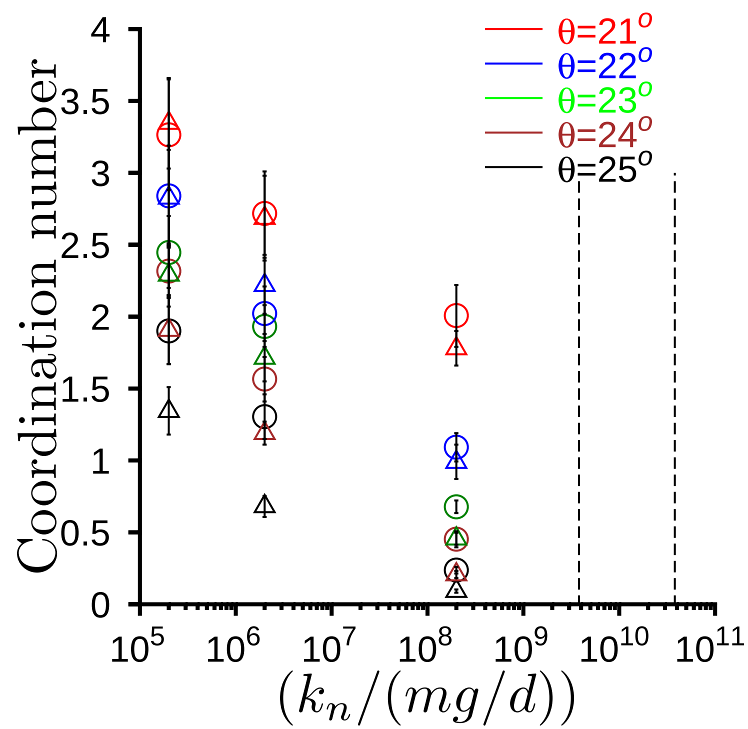


$\circ e_n = 0.9$; $\triangle e_n = 0.5$.

Flowing hard particles: Contact model.

Smooth particles:

- Hertzian contact law
  $F = k_n \delta^{3/2}$
- Value of k_n from dimensional analysis.
  $k_n \sim E d^{1/2}$.
- Sand, glass, $E \sim 10^{11} N/m^2$.
- Hertzian spring constant $k_n = 10^7 - 10^8 N/m^{3/2}$.
- Scaled spring constant
  $k_n/(mg/d^{1/2}) \sim 7.6 \times 10^9 - 7.6 \times 10^{10}$
  for $d = 1 - 0.1mm$.



○ — $\gamma_n = 375, 12000$.

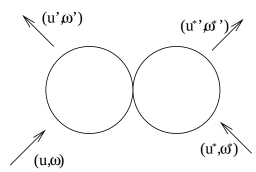
△ — $\gamma_n = 55, 1850$.

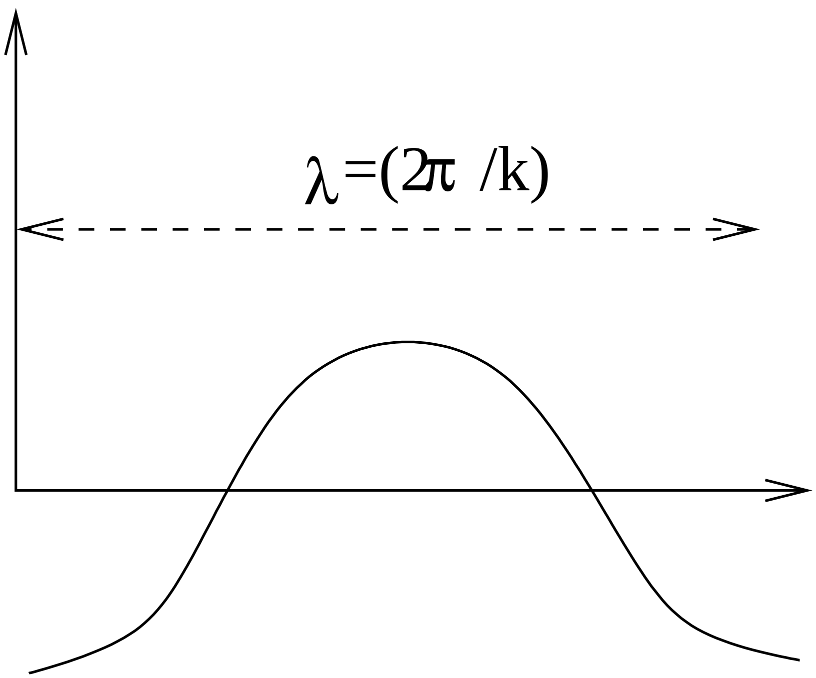
Contact models summary:

- Linear contact model — constant coefficient of restitution.
- Hertzian contact model — accounts for variation in area of contact, coefficient of restitution depends on velocity.
- Hard particles — instanteneous collisions.
- Calculation of particle impacts show that coefficient of restitution is 1 for low relative velocities, decreases as velocity increases.
- Experiments on instanteneous collisions — described by normal and tangential restitution, and friction.
- Experiments on spring stiffness between particles — linear for small deformations due to asperities, Hertzian for larger deformations with spring constant close to Mindlin-Deresiewicz value.

Conservation equations: Energy conserving:

Slow variables:





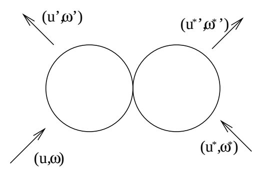
- Mass, momentum, energy conserved in individual particle interactions.
- Net addition — increases value of slow variable.
- Perturbation of wavelength $\lambda = (2\pi/k)$ long compared to microscopic scale decays diffusively.

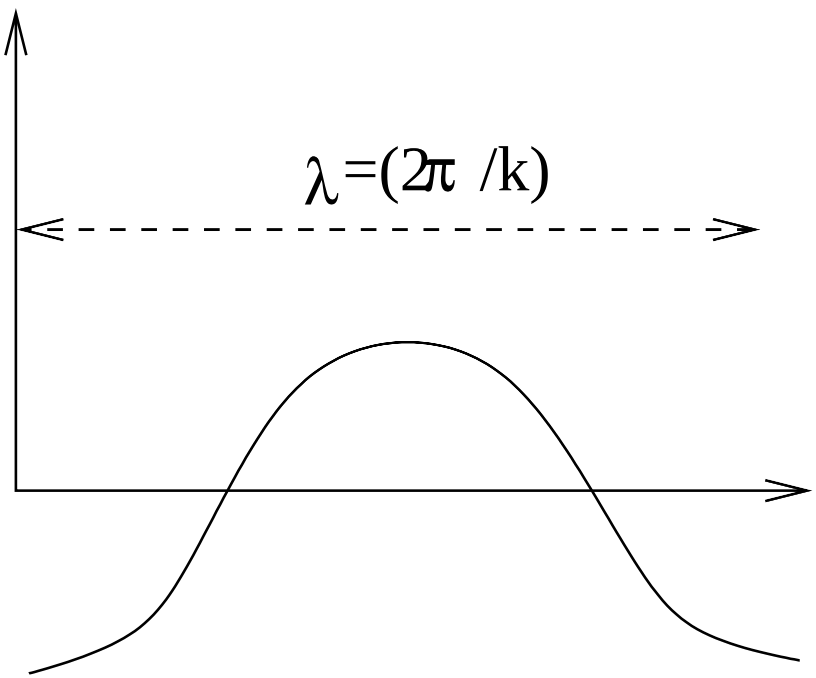
$$\frac{\partial c}{\partial t} = D\nabla^2 c$$

$$c = c_0 \exp\left(-Dk^2 t\right)$$

Conservation equations: Energy conserving:

Fast variables:





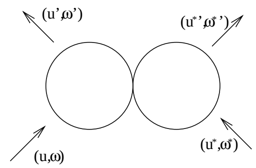
- Angular momentum, all higher velocity moments.
- Net addition — decreases to steady-state value.
- Perturbation of wavelength $\lambda = (2\pi/k)$ long compared to microscopic scale decays reactively.

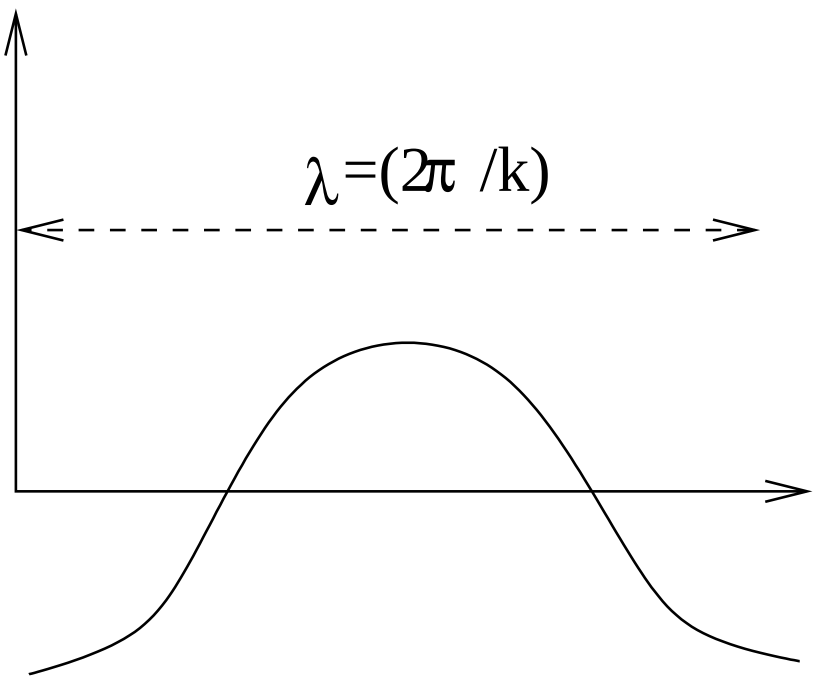
$$\frac{\partial c}{\partial t} = -Kc + D\nabla^2 c$$

$$c = c_0 \exp\left(-(K + Dk^2)t\right)$$

Conservation equations: Granular matter:

Slow and fast variables:



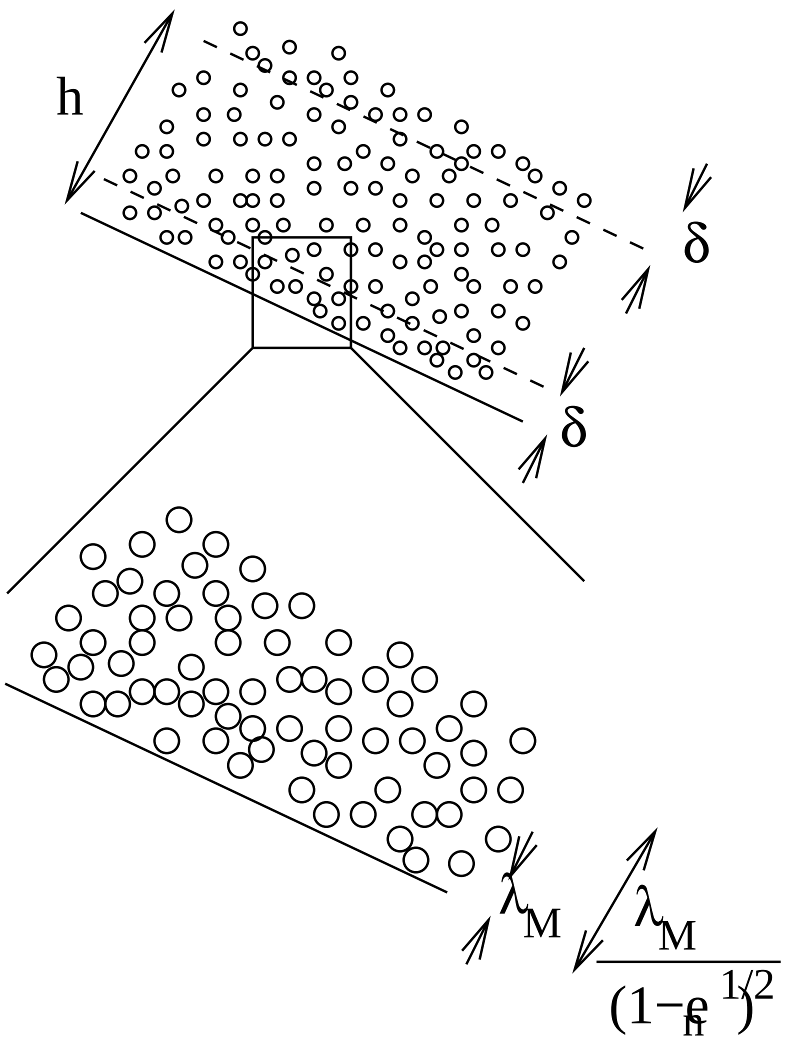


- Mass and momentum conserved in particle interactions.
- Energy dissipated in particle interactions.

$$\frac{dE}{dt} = -\alpha(1 - e^2)E + D_E \nabla^2 E$$

Conservation equations: Granular matter:

Slow and fast variables:



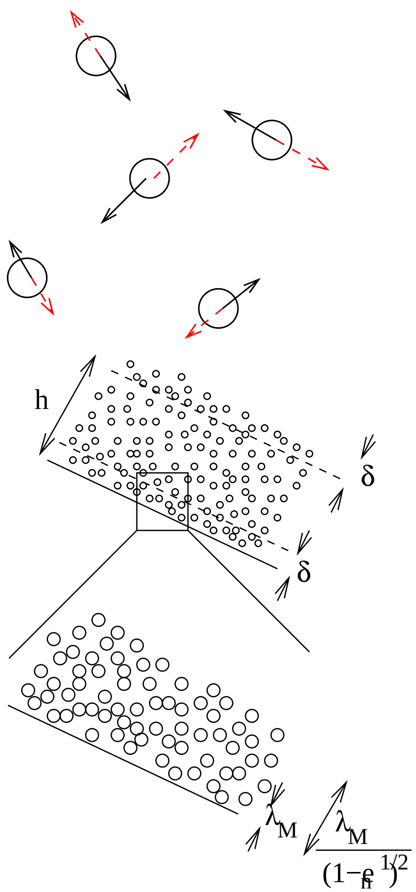
$$\frac{dE}{dt} = -\alpha(1-e^2)E + D_E\nabla^2 E$$

$$E = E_0 \exp\left(-\alpha(1-e^2) - D_E k^2\right)t$$

- Energy slow variable for $\alpha(1-e^2) < (D_E/L^2)$;
- Fast variable for $\alpha(1-e^2) > (D_E/L^2)$.
- Conduction & dissipation comparable for $L = (D_E/\alpha(1-e^2))^{1/2}$.

Conservation equations: Suspensions:

Slow and fast variables:

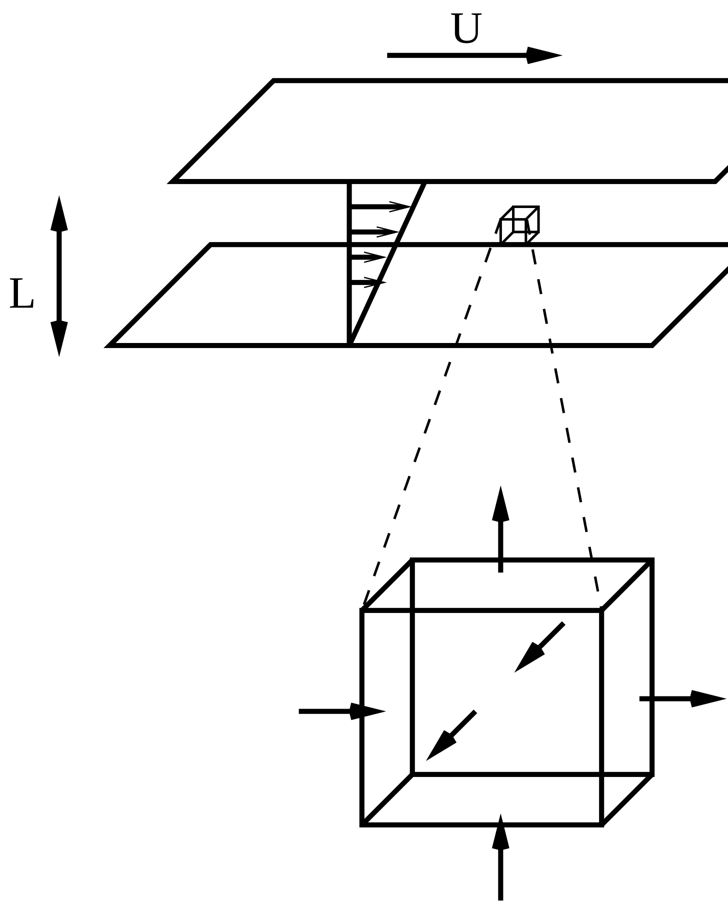


- Particle drag transfers momentum to fluid.
- Momentum not conserved due to fluid drag.
- Momentum fast variable.
- Only slow variable is mass.

$$\frac{d\mathbf{p}}{dt} = -\mu\mathbf{p} - \eta\nabla^2\mathbf{p}$$

Length scale $L = (\eta/\mu)$.

Conservation equations: Mass:



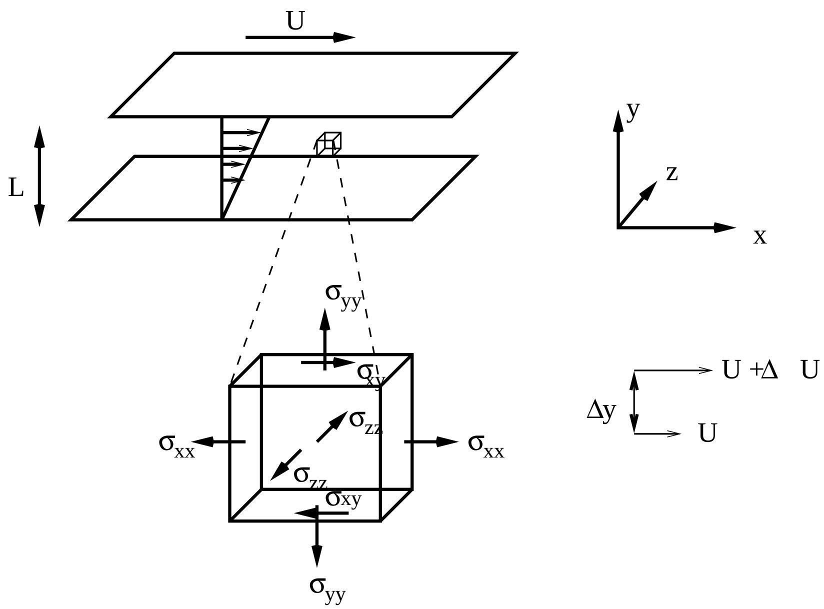
Mass conservation:

$$\begin{pmatrix} \text{Rate of mass} \\ \text{accumulation} \end{pmatrix}$$

$$= \begin{pmatrix} \text{Rate of} \\ \text{mass IN} \end{pmatrix} - \begin{pmatrix} \text{Rate of} \\ \text{mass OUT} \end{pmatrix}.$$

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{v}) = 0$$

Conservation equations: Momentum:



$$\text{Forces} = \int dV\,\mathbf{f} + \int dS\,\mathbf{R}$$

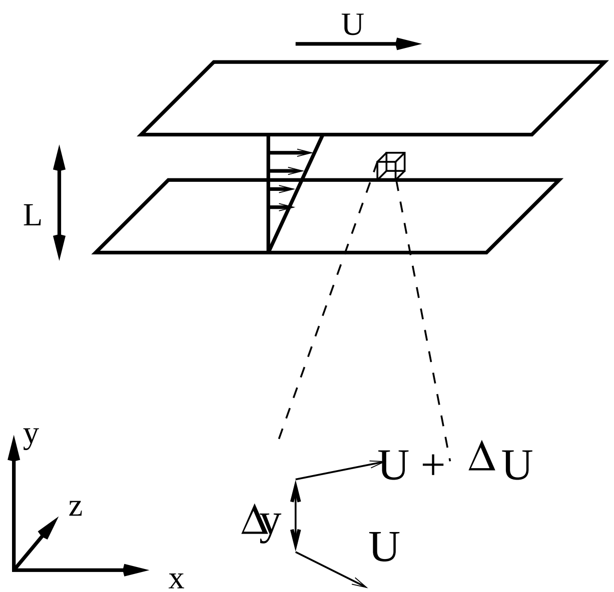
$$\mathbf{R} = \sigma.\mathbf{n}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \nabla.(\rho\mathbf{v}\mathbf{v}) = \nabla.\sigma + \mathbf{f}$$

$$\begin{pmatrix}\text{Rate of momentum} \\ \text{accumulation}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \text{momentum IN}\end{pmatrix}$$

$$- \begin{pmatrix}\text{Rate of} \\ \text{momentum OUT}\end{pmatrix} + \begin{pmatrix}\text{Sum of} \\ \text{forces}\end{pmatrix}$$

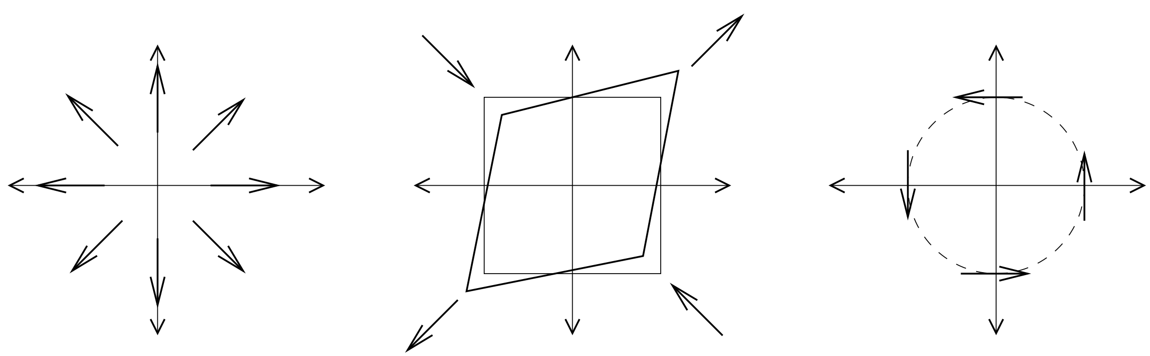
Rate of deformation:



$$\begin{pmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{pmatrix} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

$$\mathbf{\Delta v} = \mathbf{\Delta x}.\nabla\mathbf{v}$$

$$\nabla\mathbf{v} = (\mathbf{I}/3)(\nabla.\mathbf{v}) + \mathbf{S} + \mathbf{A}$$



Constitutive relation:

- Newtonian fluids: Linear stress-strain rate relationship.

$$\sigma = 2\mu\mathbf{S} + \mu_b\mathbf{I}(\nabla.\mathbf{u})$$

- Non-Newtonian fluids: Invariants of the rate of deformation tensor.

$$I_1 = \mathrm{Trace}(\nabla\mathbf{v}) = \nabla.\mathbf{v}$$

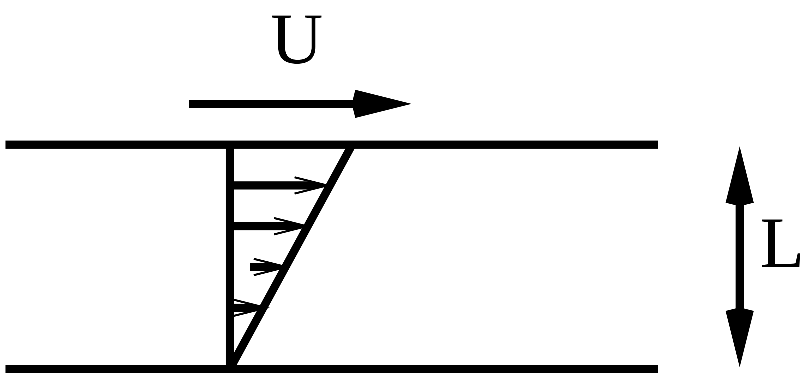
$$I_2 = \mathbf{S} : \mathbf{S}$$

$$I_3 = \mathrm{Det}(\mathbf{S})$$

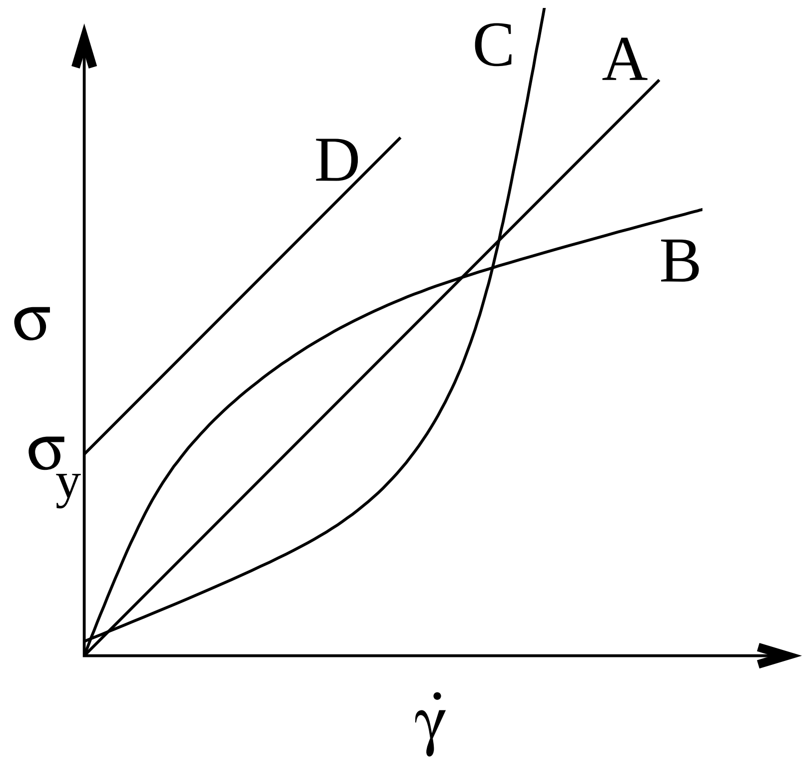
$$\sigma = 2\mu(I_1, I_2, I_3)\mathbf{S} + \mu_b(I_1, I_2, I_3)\mathbf{I}(\nabla.\mathbf{u})$$

Non-Newtonian fluids:

Simple shear:

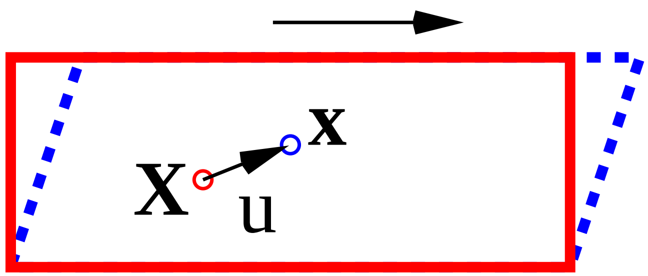


$\mu = \mu(\dot{\gamma})$



Newtonian (A), shear thinning (B), shear thickening (C) and yield stress (D) fluids.

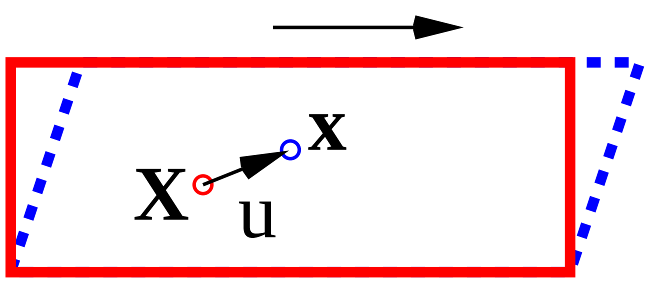
Deformation: Solid mechanics:



- Displacement field $\mathbf{u}$ — displacement of material points from steady state positions.
- Strain measure:
- Initial reference $\mathbf{X}$.
- Strained position $\mathbf{x}(t)$.
- Displacement $\mathbf{u} = \mathbf{x} - \mathbf{X}$.
- Deformation tensor

$$\mathbf{f} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{I} - \nabla \mathbf{u}$$

Deformation: Solid mechanics:



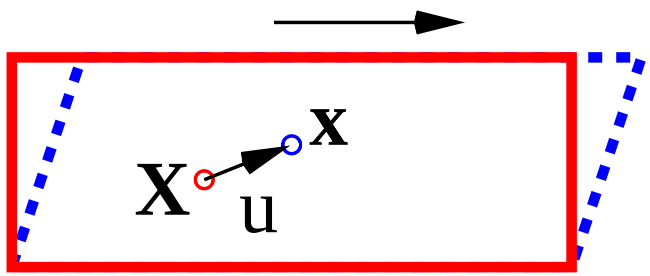
Strain measure:

- Deformation tensor $\mathbf{f} = \mathbf{I} - \nabla \mathbf{u}$.
- Incompressible $\mathrm{Det}(\mathbf{f}) = 0$.
- Eulerian velocity

$$
\begin{aligned}
\mathbf{v} &= \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} \\
&= \frac{\partial \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t)}{\partial t} \\
&= \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} \\
&= \partial_t \mathbf{u} + \mathbf{v}.\nabla \mathbf{u}
\end{aligned}
$$

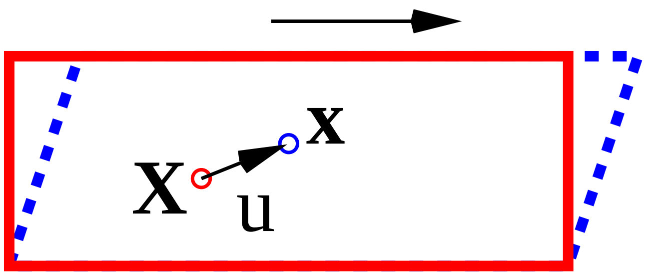
- Incompressible $\nabla.\mathbf{v} = 0$.

Deformation: Solid mechanics:



- Symmetric strain measure $\mathbf{e}$.
  $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{f}^T\mathbf{f})$
  $\mathbf{f} = \mathbf{I} - \nabla\mathbf{u}$.
- Hookean strain: $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - (\nabla\mathbf{u}).(\nabla\mathbf{u})^T)$
- Linearisation approximation: $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$
- Linear strain is not rotational frame invariant

Deformation: Solid mechanics:



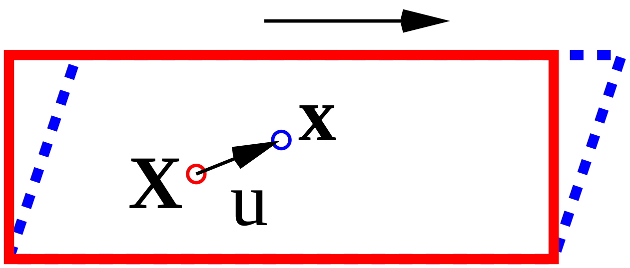
Linear & Hookean strain:

- Sheared state with strain $\mathbf{u}$.

- Symmetric strain measure $\mathbf{e}$.
  $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - (\nabla\mathbf{u}).(\nabla\mathbf{u})^T)$

- Normal stress differences.

Deformation: Solid mechanics:



Linear stress-strain relation:

$$\sigma = -p\mathbf{I} + 2G\mathbf{e} + 2\eta\dot{\mathbf{e}}$$

- Neo-Hookean:

  $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - (\nabla\mathbf{u}).(\nabla\mathbf{u})^T)$

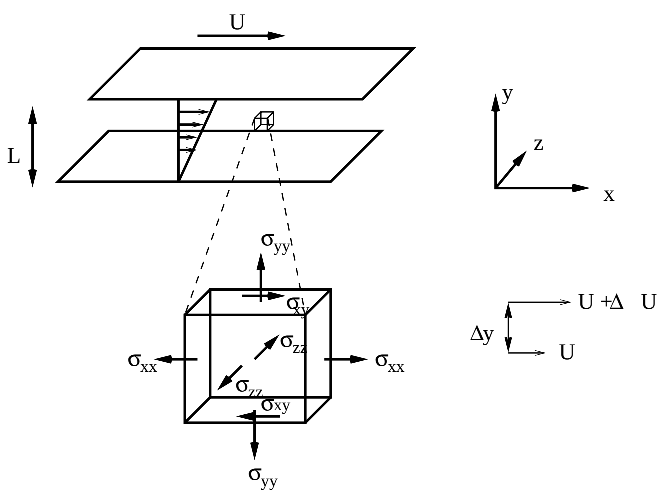
  $\dot{\mathbf{e}} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) - (\mathbf{e}.(\nabla\mathbf{v})^T + (\nabla\mathbf{v}).\mathbf{e})$

- Non-linear???

Deformation: Solid mechanics:

- Elasticity relations applicable for small deformations.
- For large deformations, microstructure changes, and elasticity equations cannot be used.
- Modifications: Yield stress: $\sigma = \sigma_y - p\mathbf{I} + 2G\mathbf{e} + 2\eta_g\dot{\mathbf{e}}$

Granular rheology: Mohr-Coulomb analysis.



Symmetric Stress tensor:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

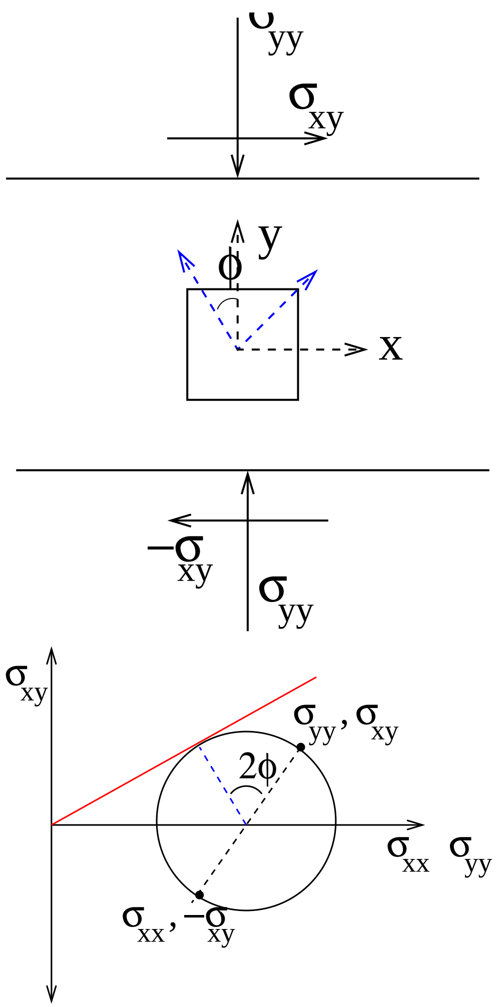
$$\sigma = \mathbf{E \Gamma E}^{-1}$$

Principal stress:

$$\Gamma = \begin{pmatrix} \Gamma_{xx} & 0 & 0 \\ 0 & \Gamma_{yy} & 0 \\ 0 & 0 & \Gamma_{zz} \end{pmatrix}$$

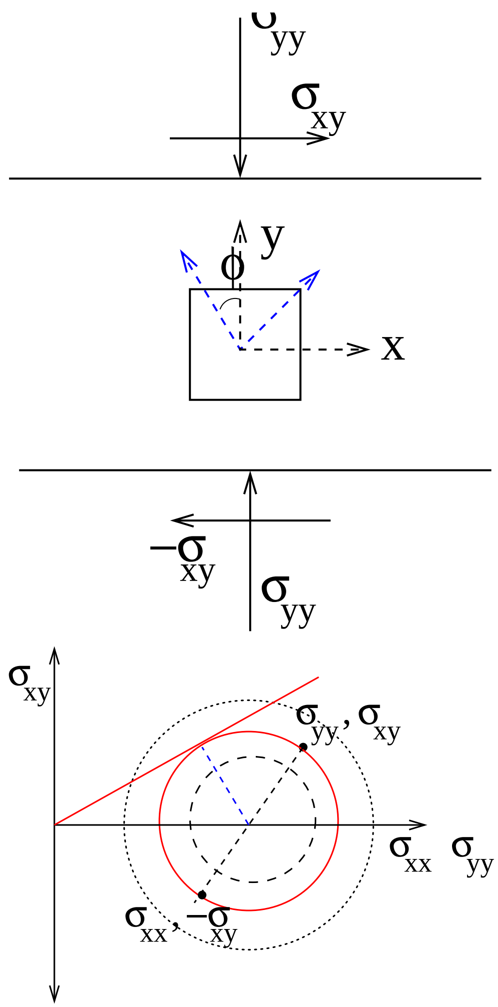
Principal directions rows of **E**.

Granular rheology: Mohr-Coulomb analysis.



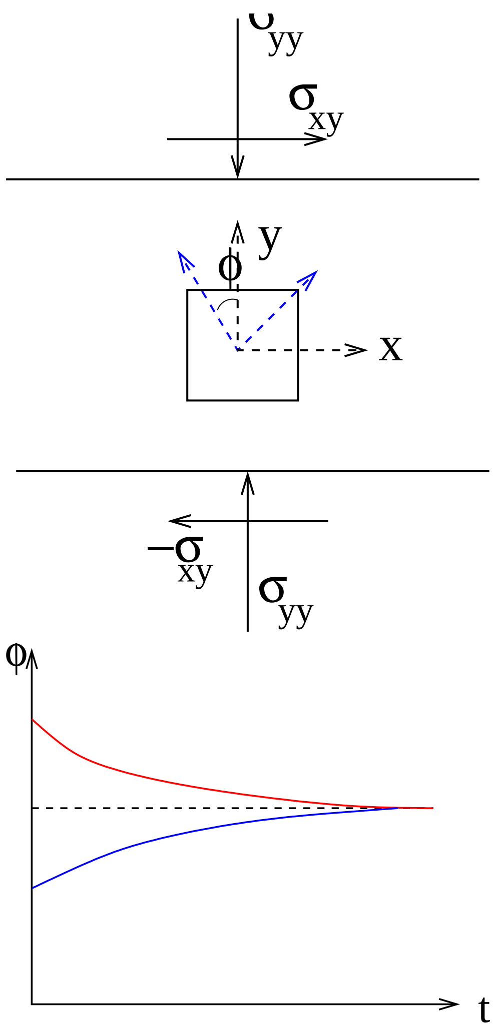
- Normal stress — shear stress axis.
- Circle through $(\sigma_{yy}, \sigma_{xy})$, $(\sigma_{xx}, -\sigma_{xy})$.
- Points on circle — stresses in rotated reference frame.
- Highest ratio of (normal/shear) stress at tangent from origin.

Granular Rheology: Mohr-Coulomb analysis.



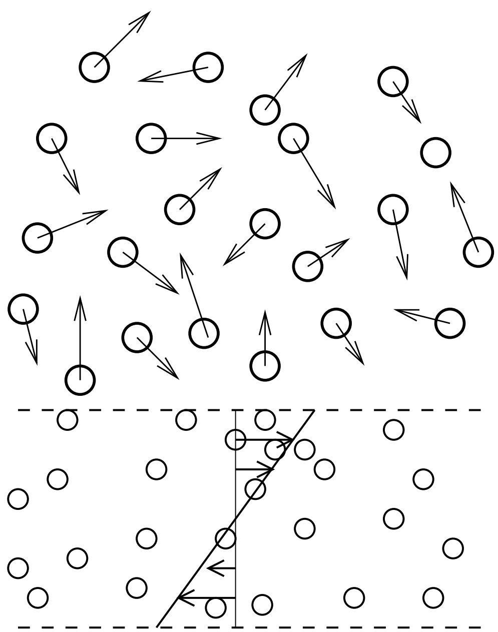
- Yield surface. If stress is on this surface, material flows.
- Inside yield surface — material jammed.
- Points outside yield surface not accessible, because material flows.
- Ratio of (shear/normal) stress on yield surface — tangent from origin to yield surface.

Granular Rheology: Critical state theory.



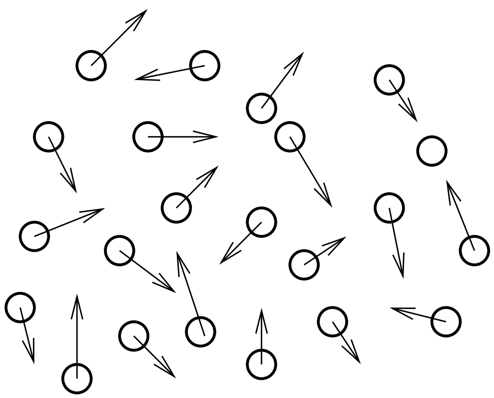
- Critical state — uniform flowing state for fixed σ_{yy}, σ_{xy} and volume fraction ϕ.
- $\sigma_{xy} = M\sigma_{yy}$
- $\phi = A + B \log\left(\sigma_{yy}/p_0\right)$
- Strain rate undefined!

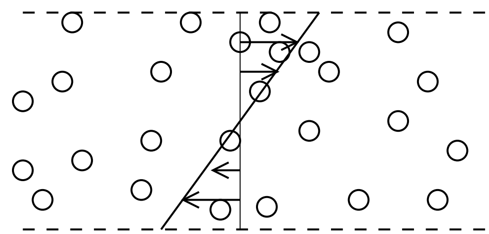
Granular rheology: Kinetic theory.



- Dilute granular flows.
- Flow is due to ‘fluidisation’ of particles due to boundary energy input or internal energy production.
- Define ‘granular temperature’ $T = 1/2m\langle v^2\rangle$ to quantify forcing.

Granular rheology: Kinetic theory.





- Homogeneous cooling state.
- System initiated at uniform temperature, inelastic particles.
- Evolution of temperature with time.

- Homogeneous sheared state.
- Production of energy due to mean shear, dissipation due to inelastic collisions.

Granular rheology: Kinetic theory.

Conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} \right) = -\nabla.\sigma$$

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = -\nabla.\mathbf{q} + \sigma :(\nabla \mathbf{u}) - D$$

$$\sigma = -p_\phi T \mathbf{I} + \mu_\phi T^{1/2} \mathbf{S} + B_1 \mathbf{S}.\mathbf{S} + B_2 (\mathbf{S}.\mathbf{A} - \mathbf{A}.\mathbf{S}) + B_3 \mathbf{A}.\mathbf{A}$$

Energy non-conserved.

Granular rheology: Kinetic theory.

Constitutive relations: Stress:

$$\sigma = -pT\mathbf{I} + \mu\mathbf{S} + B_1\mathbf{S}.\mathbf{S} + B_2(\mathbf{S}.\mathbf{A} - \mathbf{A}.\mathbf{S}) + B_3\mathbf{A}.\mathbf{A}$$

$$\mathbf{q} = -K\nabla T$$

$$D = D_\phi \rho^2 d^2 T^{3/2}$$

Kinetic theory for hard particles:

$$p = p_\phi(\phi)T$$

$$\mu = \mu_\phi(T^{1/2}/d^2)$$

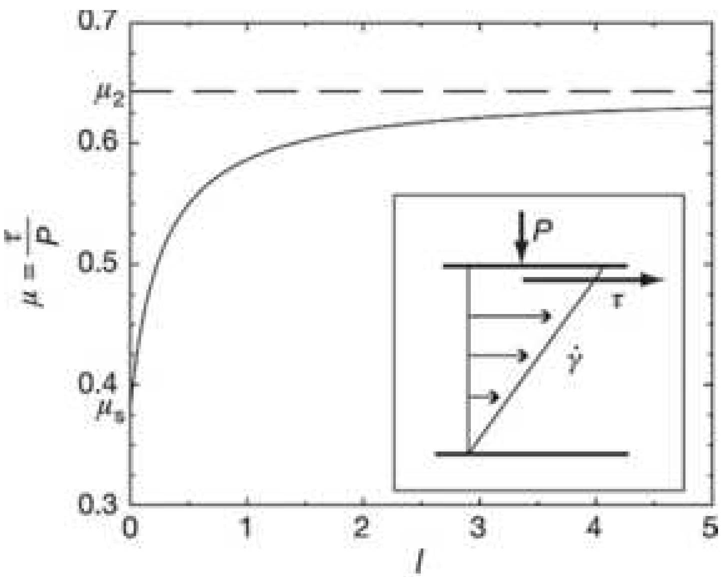
$$B_i = (B_{i\phi}/d)$$

$$K = K_\phi(T^{1/2}/d^2)$$

Granular rheology: Frictional-kinetic models:

- $$\sigma = \sigma_k + \sigma_f$$
- σ_k is given by kinetic theory.
- Temperature determined from energy balance equation.
- σ_f frictional part.
- Frictional stress components assumed to be on Mohr's circle.

Granular rheology: Inertia parameter model:

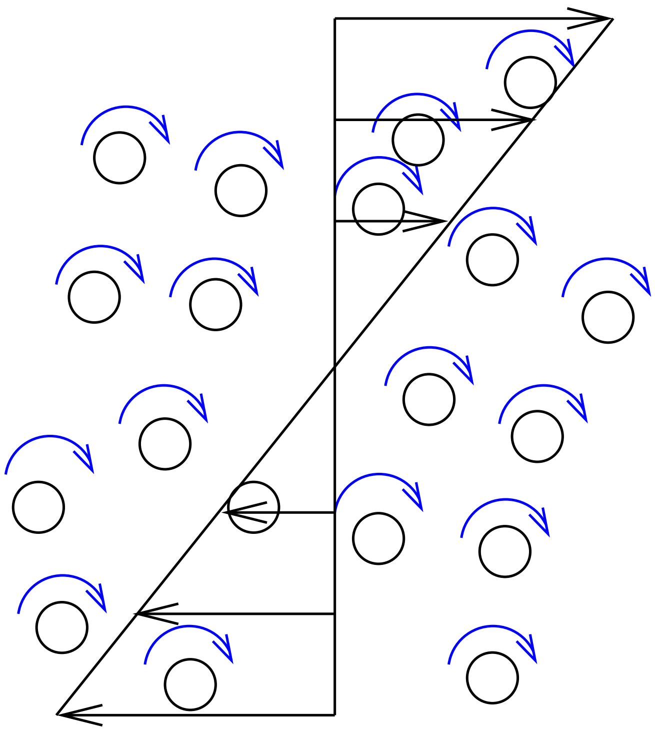


$$\sigma_{xy} = \mu(I)p$$

$$I = \dot{\gamma}d/\sqrt{p/\rho}$$

$$\mu(I) = \mu_s + (\mu_2 - \mu_s)/(I_0/I + 1)$$

Granular rheology: Micropolar models:



- Difference between local material rotation and particle spin.
- Angular momentum:

$$I\frac{D\omega}{Dt} = \epsilon : \sigma - \nabla.\mathbf{M}$$

- Couple stress $\mathbf{M}$.
- $$\sigma^a = 2\beta\left(\frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^T)\right) - \epsilon : \omega)$$
- $$\mathbf{M} = \alpha\mathbf{I}\nabla.\omega + 2\gamma\nabla\omega + 2\kappa(\nabla\omega)^T$$