

$$\Gamma: AdA - \tilde{A}d\tilde{A} = (\omega + \frac{e}{l})d(\omega + \frac{e}{l}) - (\omega - \frac{e}{l})d(\omega - \frac{e}{l})$$

$$= \frac{2}{l}(e d\omega + \omega de) = \frac{2}{l}(ed\omega) - \frac{1}{l}(ed\omega - d\omega e)$$

$$= \frac{2}{l}(ed\omega) - \frac{2}{l}d(\omega e)$$

Simply  $\frac{2}{3}(A^3 - \tilde{A}^3) = \frac{2}{3}[(\omega + \frac{e}{l})^3 - (\omega - \frac{e}{l})^3] = \frac{2}{3} \times [3e\omega^2 + e^3]$

$$\therefore S_{CS}[A] - S_{CS}[\tilde{A}] = \frac{k}{4\pi} \frac{4}{l} \int_M d^3x \left[ e \wedge \left( R + \frac{1}{3l^2} e^2 \right) \right] - \frac{k}{4\pi} \frac{2}{l} \int_{\partial M} \omega \wedge e$$

$$(R = d\omega + \omega \wedge \omega)$$

$$\Rightarrow R^a = d\omega^a + \frac{1}{2} e^a{}_b \omega^b \wedge \omega^c$$

The general gauge transf. which preserves the form of the flat conn. as  $a(z) = (L_1 + \frac{2\alpha}{r} \lambda(z) L_{-1}) dz$  is  $\delta A = d\lambda + [A, \lambda]$  where  $\lambda = b^{-1} \chi(z) b$  where  $\chi(z)$  must take the form  $\chi(z) = \epsilon(z) L_1 - \partial_z \epsilon(z) L_0 + (\frac{1}{2} \partial_z^2 \epsilon - \frac{2\alpha}{l} \lambda(z) \epsilon) L_{-1}$

so that  $a(z)$  retains the above form. In this case under such a gauge transf.  $\delta \lambda(z) = \epsilon(z) \partial_z^2 z + 2 \partial_z \epsilon(z) - \frac{k}{4\pi} \partial_z^3 \epsilon(z)$  e.g. as

the stress tensor transf. under a coord. transf. (diffm.). (and also identifies the central charge)

A similar logic applies in the Spin-3 generalisation which gives the transf. of  $\lambda(z) + \tilde{\lambda}(z)$ . (See Campoleoni et al.)

This demonstrates the asymptotic  $W_3$ -algebra structure.

In terms of  $A + \tilde{A}$  there read on

$$0 = \tilde{A} \wedge A \wedge \tilde{A} = \tilde{A} \wedge \frac{1}{2} (A + \tilde{A}) \wedge (A + \tilde{A})$$

$$0 = \tilde{A} \wedge A \wedge A + \tilde{A} \wedge A \wedge \tilde{A} + \tilde{A} \wedge \tilde{A} \wedge A + \tilde{A} \wedge \tilde{A} \wedge \tilde{A}$$

$$[ \tilde{A} ]_{002} - [ A ]_{002} = 2$$

$$- (A \wedge A \wedge A + \tilde{A} \wedge \tilde{A} \wedge \tilde{A}) \cdot \frac{1}{\pi l^2} = [ A ]_{002}$$

The relative spin is fixed by seeing that under parity the relative value of  $[ \tilde{A} ]_{002} - [ A ]_{002}$  is fixed by comparing  $[ \tilde{A} ]_{002}$  to  $[ A ]_{002}$

ENTROPY  $\frac{1}{\hbar} \ln Z$  is related to a real action which has a real factor of  $\frac{1}{\hbar}$  and the imaginary part of the action is the entropy  $S$ .

$$Z(z, \bar{z}) = \text{Tr} [ e^{z \alpha i z \bar{z} + \bar{z} \alpha i z \bar{z}} ]$$

For the BTZ BH w/ entropy  $S$ , we have  $S = 4\pi^2 i z \bar{z} - 4\pi^2 i \bar{z} z$

Also  $S = \ln Z - 4\pi^2 i z \bar{z} + 4\pi^2 i \bar{z} z$

Also  $z = -\frac{i}{4\pi^2} \frac{\partial \ln Z}{\partial \bar{z}}, \quad \bar{z} = +\frac{i}{4\pi^2} \frac{\partial \ln Z}{\partial z}$

Since we know  $z(z), \bar{z}(\bar{z})$ , we can integrate these

eqns. to obtain  $\ln Z$  and hence  $S$ .

~~Handwritten notes and calculations, including various equations and diagrams, mostly illegible due to being written upside down or rotated.~~

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The flat connections have... identified by...  
 $A(z) = (z + \frac{z_0}{z}) dz$   
 $\bar{A}(\bar{z}) = (\bar{z} + \frac{\bar{z}_0}{\bar{z}}) d\bar{z}$   
 to correspond to... matrices.  
 $A = (e^{\frac{1}{z}} dz - \frac{z_0}{z^2} dz) e^{-\frac{1}{z}}$   
 $\bar{A} = (e^{\frac{1}{\bar{z}}} d\bar{z} - \frac{\bar{z}_0}{\bar{z}^2} d\bar{z}) e^{-\frac{1}{\bar{z}}}$   
 The holonomy for the BTZ black hole, along the time-like circle...  
 The matrix has eigenvalues...  
 $L_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$   
 Note  $L_0^2 = -1, L_1^2 = 0, L_2^2 = 0$   
 Thus the holonomy  $e^{-\int A}$  is  $e^{-\int (L_0 + L_1 + L_2)}$



$\vec{J}_a + \vec{T}_{ab}$  can be expressed in terms of  $L_m, W_n$  via

$$\vec{J}_0 = \frac{1}{2}(L_1 + L_{-1}), \vec{J}_1 = \frac{1}{2}(L_1 - L_{-1}), \vec{J}_2 = L_0.$$

$$\vec{T}_{00} = \frac{1}{4}(W_2 + W_{-2} + 2W_0), \vec{T}_{01} = \frac{1}{4}(W_2 - W_{-2})$$

$$\vec{T}_{02} = \frac{1}{2}(W_1 + W_{-1}), \vec{T}_{11} = \frac{1}{4}(W_2 + W_{-2} - 2W_0)$$

$$\vec{T}_{22} = W_0, \vec{T}_{12} = \frac{1}{2}(W_1 - W_{-1}).$$

The corresponding metric like formulation of these e.o.m. is very complicated and not fully known of the non-linear level even in d=2.

We can again look for solutions which are asymptotically AdS.

$$\vec{A}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

$$\vec{B}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

$$\vec{A}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

+ simply for  $\vec{A}$

Here we have used a basis for the  $T^{d,d}$  which takes the explicit form

$$W_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; W_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$W_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; W_{-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; L_{-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[W_m, W_n] = (m-n)W_{m+n}, [L_m, L_n] = (m-n)L_{m+n}$$

Thus  $\vec{J}(z), \vec{T}(z)$  can be viewed as the exp. value for  $\vec{T}(z), \vec{J}(z)$  (stress tensor) on the boundary, the  $W(z), L(z)$  are exp. values for spin-3 currents on the boundary.