Γ: $A\,dA - \tilde{A}\,d\tilde{A} = (\omega + \frac{e}{l})\, d(\omega + \frac{e}{l}) - (\omega - \frac{e}{l})\, d(\omega - \frac{e}{l})$

$$= \frac{2}{l}(e\,d\omega + \omega\, de) = \frac{4}{l}(e\,d\omega) - \frac{1}{l}(e\,d\omega - de\,\omega)$$

$$= \frac{4}{l}(e\,d\omega) - \frac{2}{l}\, d(\omega e)$$

Simly $\frac{2}{3}(A^3 - \tilde{A}^3) = \frac{2}{3}\left[(\omega + \frac{e}{l})^3 - (\omega - \frac{e}{l})^3\right] = \frac{4}{3} \times \left[3 e \wedge \omega^2 + e^3\right]$

$$\therefore \quad S_{cs}[A] - S_{cs}[\tilde{A}] = \frac{k}{4\pi} \times \frac{4}{l} \int_M d^3x\, \mathrm{Tr}\left[e \wedge \left(R + \frac{1}{3l^2} e^2\right)\right] - \frac{k}{4\pi} \times \frac{2}{l} \int_{\partial M} \omega \wedge e$$

$(R = d\omega + \omega \wedge \omega)$

$\Rightarrow R^a = d\omega^a + \frac{1}{2} \epsilon^{a}{}_{bc}\, \omega^b \wedge \omega^c$.

[The general gauge transf. which preserves the form of the flat conn. as $a(z) = (L_1 + \frac{2\pi}{k} \mathcal{L}(z) L_{-1})\, dz$ is $\delta A = d\Lambda + [A, \Lambda]$ where $\Lambda = b^{-1} \lambda(z)\, b$ where $\lambda(z)$ must take the form $\lambda(z) = \epsilon(z) L_1 - \partial_z \epsilon(z) L_0 + (\frac{1}{2}\partial_z^2 \epsilon - \frac{2\pi}{k} \mathcal{L}(z) \epsilon(z)) L_{-1}$ so that $a(z)$ retains the above form. In this case under such a gauge transf. $\delta \mathcal{L}(z) = \epsilon(z) \partial_z \mathcal{L} + 2 \partial_z \epsilon(z) - \frac{k}{4\pi} \partial_z^3 \epsilon(z)$ e.g. as the stress tensor transf. under a coord. transf. (diffeom.). (and also identifies the central charge)
A similar logic applies in the Spin-3 generalisation which gives the transf. of $\mathcal{L}(z) + \mathcal{W}(z)$. (See Campoleoni et. al.). This demonstrates the asymptotic W_3-algebra structure.]

<u>ENTROPY</u>

$$Z(z,\bar{z}) = \mathrm{Tr}\left[e^{2\pi i z L_0 + 2\pi i \bar{z} \bar{L}_0} \right]$$

$$= e^{S + 4\pi^2 i z \mathcal{L} - 4\pi^2 i \bar{z} \bar{\mathcal{L}}} \quad \text{(for the BTZ BH w/ entropy } S\text{)}$$

$$\Rightarrow \quad S = \ln Z - 4\pi^2 i z \mathcal{L} + 4\pi^2 i \bar{z} \bar{\mathcal{L}}$$

Also $\quad \mathcal{L} = -\dfrac{i}{4\pi^2} \dfrac{\partial \ln Z}{\partial z}, \quad \bar{\mathcal{L}} = +\dfrac{i}{4\pi^2} \dfrac{\partial \ln Z}{\partial \bar{z}}$

Since we know $\mathcal{L}(z)$, $\bar{\mathcal{L}}(\bar{z})$, we can integrate these eqns. to obtain $\ln Z$ and hence S.

~~In terms of~~ J_a + T_{ab} can be expressed in terms of L_m, W_n via

$$J_0 = \tfrac{1}{2}(L_1 + L_{-1}),\ J_1 = \tfrac{1}{2}(L_1 - L_{-1}),\ J_2 = L_0.$$

$$T_{00} = \tfrac{1}{4}(W_2 + W_{-2} + 2W_0),\ T_{01} = \tfrac{1}{4}(W_2 - W_{-2})$$

$$T_{02} = \tfrac{1}{2}(W_1 + W_{-1}),\ T_{11} = \tfrac{1}{4}(W_2 + W_{-2} - 2W_0).$$

$$T_{22} = W_0,\ T_{12} = \tfrac{1}{2}(W_1 - W_{-1}).$$