(1) HIGHER SPIN THEORIES ON AdS_3

Theories of interacting higher spin gauge fields in 3 dim. are simple as well as special compared to their higher dim. counterparts. This is essentially due to the fact that such fields carry no propagating d.o.f. in 3d. As we will see, they are essentially topological, like pure gravity itself.

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ FORMULATION OF AdS_3 GRAVITY (Achucarro-Townsend; Witten)

In the frame formulation of gravity, we have vielbeins e_μ^a ($a=1,2,3$) (a → tgt. space, μ → coord. sp.) and spin connection $\omega_\mu^{[ab]} \equiv \epsilon^{abc}{}_c\omega_\mu{}^c$ (indices lowered and raised w/ η_{cd}). Thus the nice feature of 3d is that we have (e_μ^a, ω_μ^a) which can be combined into two sets of $SL(2,\mathbb{R}) \cong SO(2,1)$ - tgt. space group in 3d gauge fields

$$A_\mu^a = (\omega_\mu^a + \frac{1}{\ell} e_\mu^a) \quad ; \quad \tilde{A}_\mu^a = (\omega_\mu^a - \frac{1}{\ell} e_\mu^a).$$

The conventional torsion condn. + Einstein e.o.m. in the frame formulation are given by.

$$T^a \equiv de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0. \qquad (e^a \equiv e_\mu^a dx^\mu)$$

$$R^a \equiv d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\,\omega^b \wedge \omega^c = \frac{-1}{2\ell^2}\underbrace{\epsilon^a{}_{bc}\,e^b \wedge e^c}_{\text{Cosm. constant term}} \qquad (\omega^a \equiv \omega_\mu^a dx^\mu)$$

In terms of A^a & $\tilde{A}^a$, these read as

$$dA^a + \frac{1}{2}\epsilon^a{}_{bc}A^b \wedge A^c = 0 \quad ; \quad d\tilde{A}^a + \frac{1}{2}\epsilon^a{}_{bc}\tilde{A}^b \wedge \tilde{A}^c = 0.$$

i.e. $F^a(A) = F^a(\tilde{A}) = 0$. where $F^a \equiv F^a_{\mu\nu}dx^\mu \wedge dx^\nu$ is the usual $SL(2,\mathbb{R})$ field strength. These e.o.m. follow from an action

$$S = S_{CS}[A] - S_{CS}[\tilde{A}] \quad \text{where}$$

$$S_{CS}[A] = \frac{\hat{k}}{4\pi}\int \mathrm{Tr}\,(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \quad - \text{Chern-Simons action}$$

The relative sign is fixed by seeing that under parity / time reversal $S_{CS}[A] \to S_{CS}[-\tilde{A}] = -S_{CS}[\tilde{A}]$. The absolute value $\frac{\hat{k}}{4\pi}$ is fixed by comparing with the Einstein-Hilbert action which has a factor of $\frac{1}{16\pi G_N}$. ②

This gives $\hat{k} = \frac{l}{4G_N}$.

Apart from the global AdS_3 soln. to the e.o.m., there are also BTZ black holes. In the C.S. formulation these are given by family of flat connections. They are conveniently parametrised as follows.

Write $A = b^{-1} a(z) b + b^{-1} db$, $\tilde{A} = b\, \bar{a}(\bar{z}) b^{-1} + b\, db^{-1}$ where

$b = e^{\rho L_0}$ ($L_0, L_{\pm 1}$ are the $SL(2,\mathbb{R})$ generators obeying $[L_m, L_n] = (m-n)L_{m+n}$)

[Here we are also choosing (Fefferman-Graham) coords. (ρ, ϕ, t) for global AdS_3 such that the asymptotically AdS_3 metric takes the form

~~$ds^2 = d\rho^2 + g_{z\bar{z}}(z,\bar{z},\rho)(dz\,d\bar{z})$ where $g_{z\bar{z}} \sim e^{2\rho/l} g^{(0)}_{z\bar{z}}(z,\bar{z}) + g^{(2)}_{z\bar{z}}(z,\bar{z})$~~

$$ds^2 = d\rho^2 + g_{ij}(\omega,\bar{\omega},\rho)\, dx^i dx^j \quad w/ \quad g_{ij}(\omega,\bar{\omega},\rho) \underset{\rho\to\infty}{=} e^{2\rho/l} g^{(0)}_{ij}(\omega,\bar{\omega}) + g^{(2)}_{ij}(\omega,\bar{\omega}) + \cdots$$

$(x^i = (\omega,\bar{\omega})\ i=1,2)$, $\omega = \phi + it/l$. In these coords. the BTZ soln. takes the form

$$ds^2 = d\rho^2 + 8\pi G l\,(\mathcal{L}\, d\omega^2 + \bar{\mathcal{L}}\, d\bar{\omega}^2) + (l^2 e^{2\rho/l} + (8\pi G)^2 \mathcal{L}\bar{\mathcal{L}}\, e^{-2\rho/l})\, d\omega\, d\bar{\omega}$$

$\mathcal{L} = \frac{1}{2\pi} L_0 = \frac{1}{2\pi}\left(\frac{Ml - J}{2}\right)$, $\bar{\mathcal{L}} = \frac{1}{2\pi}\bar{L}_0 = \frac{1}{2\pi}\left(\frac{Ml + J}{2}\right)$. Global AdS has $M = -\frac{1}{8G_N}$, $J = 0$ in these conventions. and Cardy entropy $S_{BH} = 2\pi\left(\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + \sqrt{\frac{c}{6}\left(\bar{L}_0 - \frac{c}{24}\right)}\right)$

The chem. pot. + temp are encoded in $(\tau, \bar{\tau}) = \left(\frac{ik}{2}\cdot\frac{1}{\sqrt{2\pi k \mathcal{L}}}, -\frac{ik}{2}\cdot\frac{1}{\sqrt{2\pi k \bar{\mathcal{L}}}}\right)$.

The flat connections have $\rightarrow$ identified w/ Stress tensor of bdy.

$$a(z) = \left(L_1 + \frac{2\pi}{k}\mathcal{L}(z) L_{-1}\right) dz$$

$$\bar{a}(\bar{z}) = \left(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}(\bar{z}) L_1\right) d\bar{z}$$

(Note $z = \phi + it$ differs from $\omega = \phi + it/l$.)

to correspond to asymp. AdS_3 metrics.

$$\Rightarrow \quad A = (e^{\rho} L_1 - \tfrac{2\pi}{k}\mathcal{L} e^{-\rho} L_{-1})\, dz + L_0\, d\rho.$$

$$\tilde{A} = (e^{\rho} L_{-1} - \tfrac{2\pi}{k}\bar{\mathcal{L}} e^{-\rho} L_1)\, d\bar{z} - L_0\, d\rho.$$

(For. $\mathcal{L}(z) = \mathcal{L}$, $\bar{\mathcal{L}}(\bar{z}) = \bar{\mathcal{L}}$ i.e. BTZ BH).

The Holonomy of $(A, \tilde{A})$, for the BTZ black hole, along the time-like circle is $e^{2\pi\tau a(z)}$, $e^{2\pi\bar{\tau}\bar{a}(\bar{z})}$. This matrix has eigenvalues

$e^{\pm\sqrt{\frac{8\pi^3\mathcal{L}}{k}}\tau}$, $\left\{\text{Note } L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, L_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right\}$

$= e^{\pm i\pi}$. $\left(\because \tau = \frac{ik}{2}\cdot\frac{1}{\sqrt{2\pi k\mathcal{L}}}\right)$. Thus the holonomy $\sim \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{Z}_2$ - centre of $SL(2,\mathbb{R})$

$SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ FORMULATION OF HIGHER SPIN THEORY

It turns out that the $SL(N,\mathbb{R})$ generalisation of the ~~above~~ pure gravity case corresponds to gravity coupled to higher spin gauge fields of spin $s = 3, 4, \ldots N$. (This assumes that the $SL(2,\mathbb{R})$ subgp. corresponding to gravity is embedded inside $SL(N,\mathbb{R})$ in a particular way – principal embedding. Here one takes $SL(2,\mathbb{R})$ generators J^a in the N dim. repn. of $SL(2,\mathbb{R})$ i.e. as $N \times N$ traceless matrices $\subset SL(N,\mathbb{R})$. Then one constructs $T^{ab} = J^{(a} J^{b)}$, $T^{abc} = J^{(a} J^{b} J^{c)}, \ldots$ (symm. traceless) These are all $N \times N$ traceless matrices and all the way to $T^{a_1 \ldots a_{N-1}} = J^{(a_1} J^{a_2} \ldots J^{a_{N-1})}$ for sizes $< N$ ($\because$ there can't be any relns.) give a total of $3 \oplus 5 \oplus 7 \oplus \ldots \oplus (2N-1) = (N^2-1)$ independent matrices – generate $SL(N,\mathbb{R})$.)

Let's focus on the $SL(3,\mathbb{R})$ case. We now have gen. $J_a \to s=2$ and $T_{ab} \propto J_a J_b - \frac{2}{3}\eta_{ab}(J^2) \to s=3$. This is because we can now define generalised vielbeins / conn. from the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ gauge fields $A, \tilde{A}$ via. $A = \left(\omega + \frac{e}{l}\right)$; $\tilde{A} = \left(\omega - \frac{e}{l}\right)$ as before.

but: $e_\mu = e^a_\mu J_a + e^{ab}_\mu T_{ab}$. and simly for $\omega^a_\mu, \omega^{ab}_\mu$.

We see that e^{ab}_μ w/ two tangent space indices behaves like a spin 3 field. $\varphi_{\mu\nu\lambda} \sim e^{ab}_\mu \bar{e}_{\nu a} \bar{e}_{\lambda b}$. ($\bar{e}$ → backgd. vielbein of AdS_3.)

The $SL(3,\mathbb{R})^2$ gauge transf. contain the usual $SL(2,\mathbb{R})^2$ of gravity (diffmm. + local Lorentz rotation) but also additional gauge freedom which correspond to transf. of the spin 3 field (parametrised by $\Lambda^{ab} T_{ab} \to s=2$). These act (at linearised level) as $\varphi_{\mu\nu\lambda} \sim \varphi_{\mu\nu\lambda} + \partial_{(\mu}\xi_{\nu\lambda)}$ (with $\xi_\lambda{}^\lambda = 0$). Notice that these transformations also act on the usual gravity field – thus usual diff. invt. quantities of gravity are not invt. under this bigger set of gauge invariances.

Taking the action for this new theory to be again
$S = S_{CS}[A] - S_{CS}[\tilde{A}]$ we have the e.o.m. giving flat connections $dA + A\wedge A = 0 = d\tilde{A} + \tilde{A}\wedge\tilde{A}$. In addition to the torsion + Einstein eqns. (w/ spin 3 source terms) (the J^a pieces) there are generalised torsion condns. (which together fix $\omega_\mu^a, \omega_\mu^{ab}$ in terms of e_μ^a, e_μ^{ab}) as well as the generalised spin-3 eqn. of motion. (the T^{ab} pieces)

[The corresponding metric like formulation of these e.o.m. is very complicated and not fully known at the non-linear level even in d=3 even if we assume $g_{\mu\nu} \propto \mathrm{Tr}[e_\mu e_\nu]$, $\varphi_{\mu\nu\lambda} = \mathrm{Tr}[e_{(\mu}e_\nu e_{\lambda)}]$] $^{*}(A_\mu - \tilde{A}_\mu)$

We can again look for solns. which are asymptotically AdS. by going to $a(z)$, $\bar{a}(\bar{z})$ which are of the form

$$a(z) = \left(L_1 - \frac{2\pi}{k}\mathcal{L}(z)L_{-1} - \frac{\pi}{2k}W(z)W_{-2}\right)dz$$

$$\bar{a}(\bar{z}) = \left(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}(\bar{z})L_1 - \frac{\pi}{2k}\overline{W}(\bar{z})W_2\right)d\bar{z}$$

[$\rightarrow A(z) = \left(e^{\rho}L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}(z)L_{-1} - \frac{\pi}{2k}e^{-2\rho}W(z)W_2\right)dz + L_0 d\rho$.
+ simly. for $\tilde{A}(\bar{z}, \rho)$]

Here we have used a basis $W_{0,\pm1,\pm2}$ for the T^{ab} which take the explicit form

$$W_2 = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 2 & 0 & 0 \end{pmatrix};\quad W_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & -1 & 0 \end{pmatrix};\quad W_0 = \frac{2}{3}\begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix}$$

$$W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & 2 \\ & & 0 \end{pmatrix};\quad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ & 0 & 0 \\ & & 0 \end{pmatrix};\quad L_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_0 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix};\quad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & -2 \\ & & 0 \end{pmatrix}$$

and obey $[L_m, L_n] = (m-n)L_{m+n}$, $[L_m, W_n] = (2m-n)W_{m+n}$

$$[W_m, W_n] = -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}$$

Thus $\mathcal{L}(z)$, $\bar{\mathcal{L}}(\bar{z})$ can be viewed as the exp. value for $T(z)$, $\bar{T}(\bar{z})$ (stress tensor) on the boundary, the $W(z)$, $\overline{W}(\bar{z})$ are exp. values for spin-3 currents on the boundary.

Under the most general gauge transf. which preserve the form of these solns., one finds that $\mathcal{L}(z)$, $\bar{\mathcal{L}}(\bar{z})$ transform as the stress tensor while $W(z)$ transforms like a spin-3 current. One can read off the singular part of the OPE from this and. See that these obey a classical W_3 algebra (Campoleoni et. al/ Henneaux-Rey / Gutperle-Kraus)

This is generalised in a straight forward way to $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$. We have. $e_\mu = e_\mu^a J_a + e_\mu^{ab} T_{ab} + \dots + e_\mu^{a_1 \dots a_{N-1}} T_{a_1 \dots a_{N-1}}$ corresp. to fields of spin $2 \dots N$ (together with their generalised spin conn. $\omega_\mu^{a_1 \dots a_{s-1}}$ $(s = 2 \dots N)$). The action is the difference of two $SL(N,\mathbb{R})$ C.S. actions. We can again consider asymp. AdS_3 solns. of the form

$$a(z) = (L_1 - \sum c_{(s)} W^{(s)}(z) W^{(s)}_{-(s-1)}).$$

The $W^{(s)}(z)$ obey a classical W_N algebra $(s=2 \dots N)$. (w/ constant $W^{(s)}(z)$)

We find an interesting class of smooth solns — those which have a trivial holonomy (in the centre of $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ or $SL(N,\mathbb{C})$ in Eucl. sign.) in the spatial (ϕ) circle. These are conical surplus like solutions (Castro et. al.) The eigenvalues of $a(z)$ are not nece. those of vac. AdS_3 which corresponds to $\lambda_j = \frac{1}{2}(N+1-2j)$.

We can also consider B.H. solns. which carry higher spin charge. Since these are to be viewed as thermodynamic entities - they also have the canonical variables - temperature and a (for e.g.) a spin-3 chemical potential (coming back to $SL(3,\mathbb{R})$). In the Euclidean theory these are encoded in terms of the periodicity Int. of the time direction as well as an extra piece $\mu(z,\bar{z})$ in the connection. However, this now goes out of the class of asymp. AdS_3 solns.

considered earlier (we are adding a source term for an irrelevant operator $W^{(3)}$ - of dim 3 from a dual CFT point of view). ~~Nonetheless, we can find a sensible description of such solutions~~. It turns out that the soln. is of the form. (6)

$$a = (L_1 - \frac{2\pi}{k}\mathcal{L} L_{-1} - \frac{\pi}{2k} \mathcal{W} W^{(3)}_{-2})dz - \mu(W_2 - \frac{4\pi \mathcal{L}}{k} W_0 + \frac{4\pi^2 \mathcal{L}^2}{k^2} W_{-2} + \frac{4\pi}{k}\mathcal{W} L_{-1})d\bar{z}$$

(and simly. for $\bar{a}$). Notice we now have a $d\bar{z}$ piece. we can check that $a_{\bar{z}} = -2\mu\left[a_z^2 - \frac{1}{3}\mathrm{Tr}(a_z^2)\right]$ which ensures that $[a_z, a_{\bar{z}}] = 0$ and hence a is a flat conn.

The thermodynamic relns. $\mathcal{L}(\tau,\mu)$, $\mathcal{W}(\tau,\mu)$ are fixed by demanding that the holonomy of $a(\bar{a})$ (and $\bar{a}$) along the time direction be trivial ($= B\mathbb{1}Z$). This gives two condns relating $(\mathcal{L}, \mathcal{W}, \tau, \mu)$. This is the analogue of demanding absence of conical singularity at horizon in the usual Einstein BH soln. The ~~correct~~ soln. that is studied is the one which has $\mathcal{W} \to 0$ when $\mu \to 0$. Most importantly it ~~must~~ satisfies the integrability condn. $\frac{\partial \mathcal{L}}{\partial \mu} = \frac{\partial \mathcal{W}}{\partial \tau}$ ($\because \mathcal{L} \propto \frac{\partial \ln Z}{\partial \tau}$ and $\mathcal{W} \propto \frac{\partial \ln Z}{\partial \mu}$.)

It turns out that the natural dual to 2d CFTs is not the Vasiliev theory w/ gauge gp. $SL(N,\mathbb{R})^2$, but rather the most general bosonic higher spin theory which typically contains $s = 2,3 \ldots \infty$ and is characterised by a continuous parameter $0 \leq \lambda \leq \infty$. This is based on a gauge gp. $hs[\lambda]^2$ - infinite dim. lie algebra. For this we consider again J_a, $J_{a_1} J_{a_2}, \ldots J_{a_1} \ldots J_{a_{s-1}}$ - all possible monomials built from J_a modulo the relation $C_2(J) \equiv J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{4}(\lambda^2 - 1)\mathbb{1}$. This Lie Algebra $B[\lambda] \equiv \frac{U[SL(2)]}{\langle C_2 - \frac{1}{4}(\lambda^2-1)\mathbb{1}\rangle} = hs[\lambda] \oplus \mathbb{1}$.

A convenient basis for this Lie Alg. turns out to be

$$V_n^{(s)} = (-1)^{s-1-n}\frac{(n+s-1)!}{(2s-2)!}\underbrace{[J_-,[J_-, \ldots [J_-, J_+^{s-1}]]]}_{(s-1-n)\text{ terms}} \quad (\text{w/ } |n| \leq s-1 \;\&\; s = 2,3,\ldots\infty)$$

The structure consts. of the $V_n^{(s)}$ can be written down explicitly. For $\lambda = N$, the algebra can be truncated further to $SL(N,\mathbb{R})$ - remove all $s \geq N$ generators.

We can write the general Vasiliev theory in 3d in terms of $(A, \hat{A})$ ~~obeying~~ $\in hs[\lambda] \times hs[\lambda]$ w/ the usual Chern-Simons action. The general soln. which is asymp. AdS_3 gives a $W_\infty^{cl}[\lambda]$ algebra as its asymp. algebra.

One can also couple scalar fields to these h-spin gauge fields. This field $C_0(x)$ is part of a bigger field $C(x) \in B[\lambda]$ w/ $C_0(x)$ being the identity part. The other components of $C(x)$ are auxiliary and obey (at lin. level) $dC + A * C - C * \tilde{A} = 0$ ($*$ prod. in $hs[\lambda]$). The lin. e.o.m. for $C_0(x)$ is that of a complex scalar field w/ $m^2 = -1 + \lambda^2$. Thus fixed in terms of λ. Note this is above the BF bound for AdS_3 $(m^2 = -1)$.

2) 2D Coset CFT's w/ W-Symmetries

We saw that higher spin theories on AdS_3 have an asymptotic W_N (more gen. $W_\infty[\lambda]$) symmetry - generalising the Virasoro algebra that a typical gravity theory on AdS_3 possesses. These contain (antiholo.) holomorphic currents on the bdy $\Leftrightarrow$ $W^{(s)}(z)$ $(s=2,3\ldots)$ - i.e. conserved currents of arbitrary spin. This is quite unusual since in QFTs on $d>2$, there are no interacting theories w/ conserved currents of spin >2 ($T_{\mu\nu}$ has $s=2$) since that would imply conserved charges of spin >1 which is prevented by the Coleman-Mandula Thm. There is an analogue of this for CFTs in $d>2$ (Maldacena-Zhiboedov) which seems to suggest only free CFTs can have higher spin conserved currents.

However, as we will see, in $d=2$ it is not difficult to construct interacting theories (CFTs) w/ conserved currents $W^{(s)}(z)$. Such theories are potentially dual to higher spin theories on AdS_3.

Before considering interacting theories, consider the free theories.

(a) Free Boson: For a cplx. free boson define

$$W_B^{(s)}(z) \propto \sum_{k=0}^{s-2} (-1)^k \binom{s-1}{k}\binom{s-1}{k+1} \partial^{s-k-1}\phi^+ \partial^{k+1}\phi \qquad (s=2,3\ldots)$$

$\partial\bar\partial\phi=0 \Rightarrow \bar\partial W_B^{(s)}=0$. The numerical coeff. are chosen so that under the global conf. gp. $z \to \frac{az+b}{cz+d}$, they transform as (quasi-) primaries of weight (s). The free field OPE of these currents is linear and has the structure. $W^{(s)}\cdot W^{(s')} \sim W^{(s+s'-2)} + W^{(s+s'-4)} + \ldots + c_s\delta_{ss'}$

This algebra (after a field redefinition) is $W_\infty[\lambda=1]$

(b) FREE FERMION: Simly, for a free dirac fermion one can define.

$$W_F^{(s)}(z) \propto \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k}^2 \partial^{s-k-1}\bar\psi\, \partial^k \psi$$

They are again quasi-primary of weight (s). Again the OPE is linear as above and is related to $W_\infty[\lambda=0]$ (after truncating out $s=1$ current).

To get interacting theories w/ higher spin currents (and the more general $W_\infty[\lambda]$ algebras) one needs to look at the class of so-called coset CFTs.

We take the WZW theory based on group G and gauge a subgp. H. (without adding a kinetic term for the H-gauge field).

$$S_{G/H}(g,A) = k \times \Big[-\frac{1}{4\pi}\int d^2z\, (g^{-1}\partial g)(g^{-1}\bar\partial g) - \frac{i}{12\pi}\int_{M_3} d^3x\, (g^{-1}\partial g)\wedge(g^{-1}\partial g)\wedge(g^{-1}\partial g)\Big]$$
$$-\frac{k}{2\pi}\int d^2z\,(A\bar\partial g\, g^{-1} - \bar A g^{-1}\partial g + A\bar A - g^{-1}A g \bar A)$$

Thus the A eqns. of motion set the H currents $J^H = [g^{-1}Dg]^H = 0$. (+ antihol.). The g e.o.m. are $F_A = 0$, $[\bar D J]^{G/H} = [D\bar J]^{G/H} = 0$.

The conventional way of solving such theories is to use the algebraic techniques for 2d CFTs. - in particular, use the soln. of the G-WZW model

We can decompose the hilbert space of the G WZW theory into. repns. of H.
$$\mathcal{H}^{(\Lambda)}_G = \bigoplus_{\Lambda'} \left(\mathcal{H}^{(\Lambda;\Lambda')}_{G/H} \otimes \mathcal{H}^{(\Lambda')}_{H}\right)$$
i.e. we take an affine repn. of G and decompose it in terms of Λ' of H WZW model

The multiplicity space $\mathcal{H}^{(\Lambda,\Lambda')}_{G/H}$. span the Hilbert space of the G/H theory - generated by operators in the G-theory which commute with the H currents (i.e. have non-singular OPE). [~~There is a selection rule~~ $\cancel{\Lambda - \Lambda' \in \Lambda_R}$ ~~root lattice of G~~. In particular $T_{G/H} = T_G(z) - T_H(z)$ and therefore. $c_{G/H} = c_G - c_H$. (T_H has non-singular OPE w/ $T_{G/H}$)
($c_G = k\dim G/(k+c_v)$; c_v = dual cox. # = rank+1 for ADE gps.)

The example of most importance to us will be cosets where
$G = SU(N)_k \times SU(N)_1$, $H = SU(N)_{k+1}$ (diagonal) i.e. $J^a_{(1)} + J^a_{(2)}$

$$c = (N^2-1)\left[\frac{k}{N+k} + \frac{1}{N+1} - \frac{k+1}{N+k+1}\right] = (N-1)\left[1 - \frac{N(N+1)}{p(p+1)}\right] \quad (p = k+N).$$

— a discrete series — W_N minimal models. Note the central charge is, in general, not a multiple of a half integer $\Rightarrow$ not a free theory.

For $N > 2$, these theories have conserved currents of spin $s > 2$.
Thus consider the general cubic combination of currents in the numerator (which is $SU(N)$ invt.)

$$d^{abc}\left[a_1 (J^a_{(1)} J^b_{(1)} J^c_{(2)}) + a_2 (J^a_{(2)} J^b_{(1)} J^c_{(1)}) + a_3 (J^a_{(2)} J^b_{(2)} J^c_{(1)}) + a_4 (J^a_{(2)} J^b_{(2)} J^c_{(2)})\right]$$

$\hookrightarrow$ cubic invt. of $SU(N)$ $(N \geq 3)$

By considering the OPE w/ $(J^a_{(1)} + J^a_{(2)})$ we get a general combination of quadratic terms $\propto (b_1 J^b_{(1)} J^c_{(2)} + b_2 J^b_{(2)} J^c_{(1)} + b_3 J^b_{(2)} J^c_{(2)})$ which are potentially singular (from the central term in the $J^a J^b$ OPE. The other term automatically vanishes since the comb. above is $SU(N)$ invt.)
The b_i are homog. fns. of a_i and requiring them to vanish gives 3 condns. on the a_i which fixes them up to overall constant. This generalises the Sugawara current construction.
Similarly, using the higher order Casimirs (invariant symm. tensors) of $SU(N)$ we can get a unique $W^{(s)}(z)$. These $W^{(s)}(z)$ have their OPE define the W_N algebra (w/ a central charge $c = c(N,k)$)

The full CFT has (W_N) primaries labelled by $\Lambda \in SU(N)_k$ and $\Lambda' \in SU(N)_{k+1}$. These form an intricate spectrum (finite for any finite N, k) w/ dimensions

$$h(\Lambda; \Lambda') = \frac{1}{2p(p+1)}\left[\,|(p+1)(\hat\Lambda + \hat\rho) - p(\hat\Lambda' + \hat\rho)|^2 - \hat\rho^2\right]$$

Thus
$$h(0; f) = \frac{(N-1)}{2N}\left(1 - \frac{N+1}{N+k+1}\right) \quad ; \quad h(0; \text{adj}) = 1 - \frac{N}{N+k+1}$$
$$h(f; 0) = \frac{(N-1)}{2N}\left(1 + \frac{N+1}{N+k}\right) \quad ; \quad h(\text{adj}; 0) = 1 + \frac{N}{N+k}.$$

3) HOLOGRAPHY

The W_N minimal models have an interesting large N limit a la 't Hooft. Since $1/k$ (or more exactly $1/(k+N)$) plays the role of coupling in the WZW theory, can consider the limit $N, k \to \infty$ s.t. $\lambda \equiv \frac{N}{N+k}$ is held fixed. This limit appears to make sense.

$c \simeq N(1-\lambda^2)$. Operators get classified into single trace and multi-trace as one might expect in a large N limit. There is a spectrum of "light primaries" w/ $h \sim \frac{\lambda^2}{N}$. There are others $\sim O(1), O(N) \ldots O(N^3)$.

Due to a non-obvious symmetry of the $\frac{N}{N+k}$ "λ" grm. W_∞ algebra it turns out that $W_N \simeq W_\infty^{(q)}[\lambda]$. (w/ $c = c_{N,k}$).

Thus as $N \to \infty$ we get a limiting $W_\infty^{(q)}[\lambda]$ algebra.

This is crucial for the theory to match with the Vasiliev theory with $hs[\lambda]$ algebra. Indeed the proposal is for a duality between the large N 't Hooft limit (λ fixed) of the W_N minimal models and the Vasiliev theory w/ $hs[\lambda] \times hs[\lambda]$ algebra and one cplx. scalar field (w/ $m^2 = -1+\lambda^2$) having Δ_+ (usual bdy. condns.)

Thus we have λ and $N \sim c \sim 1/G_N$ as the two parameters of the two respective theories.

The Vasiliev theory w/ $hs[\lambda] \times hs[\lambda]$ algebra has $W_\infty^{(q)}[\lambda]$ as its asymptotic algebra.