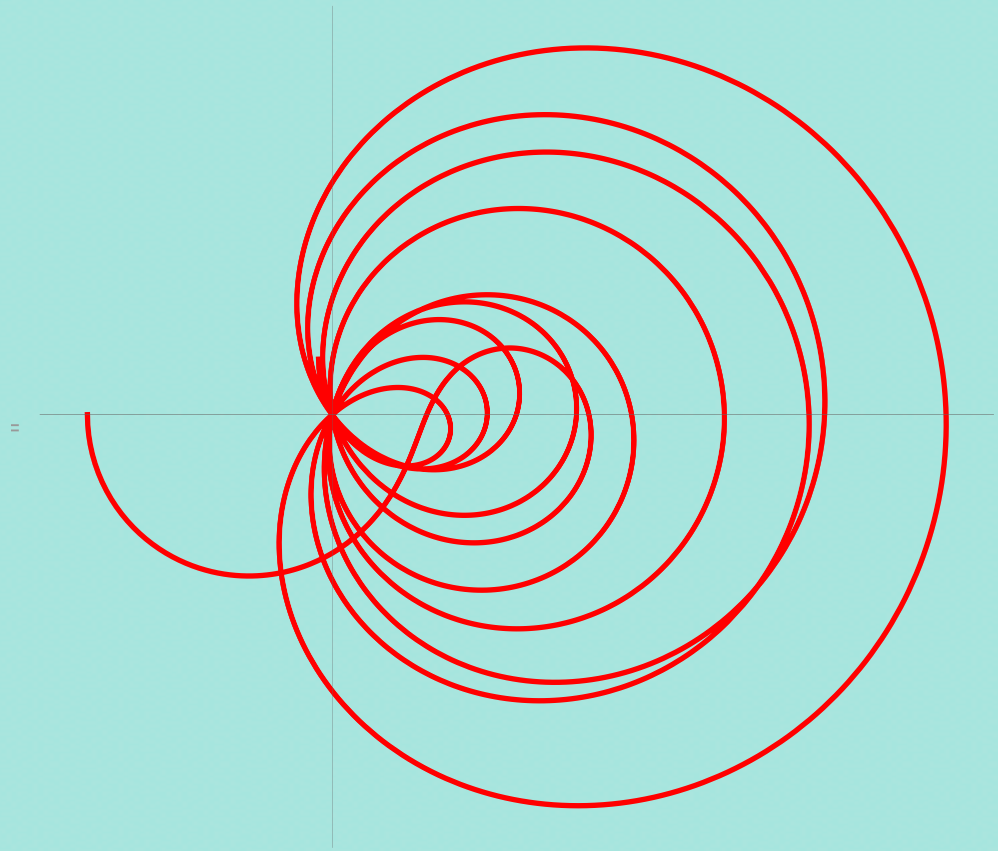
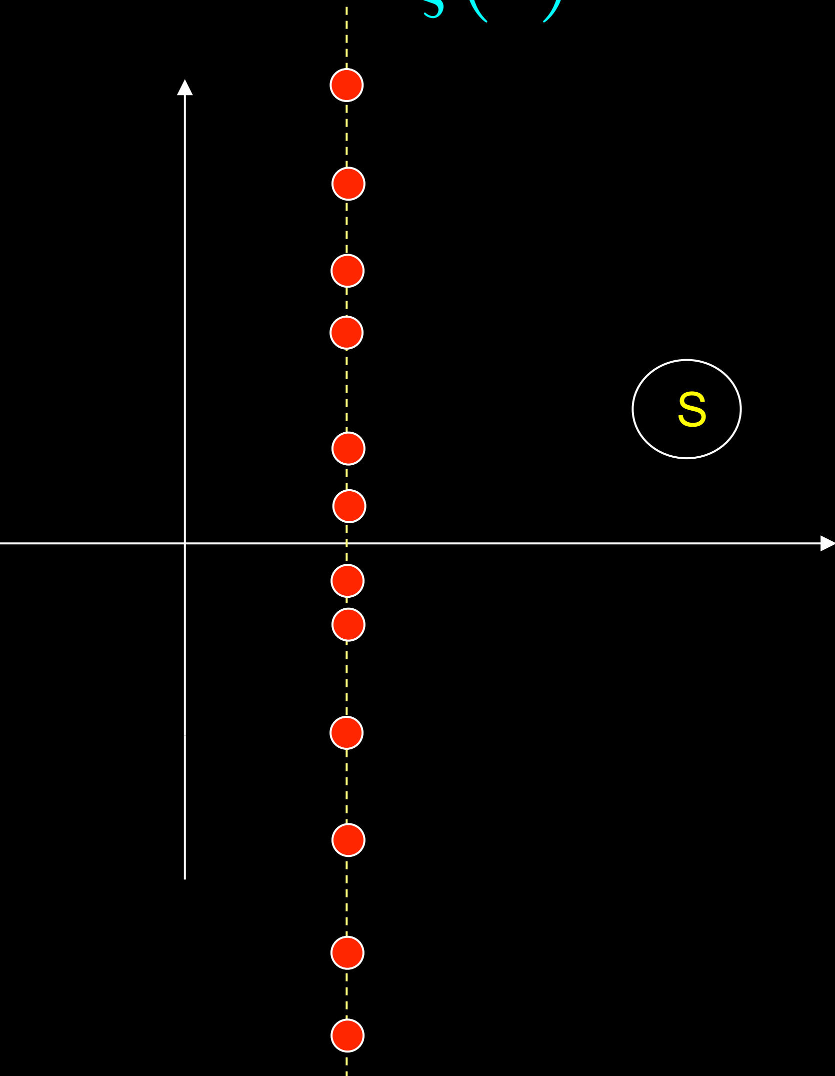
Generalised Riemann Hypothesis

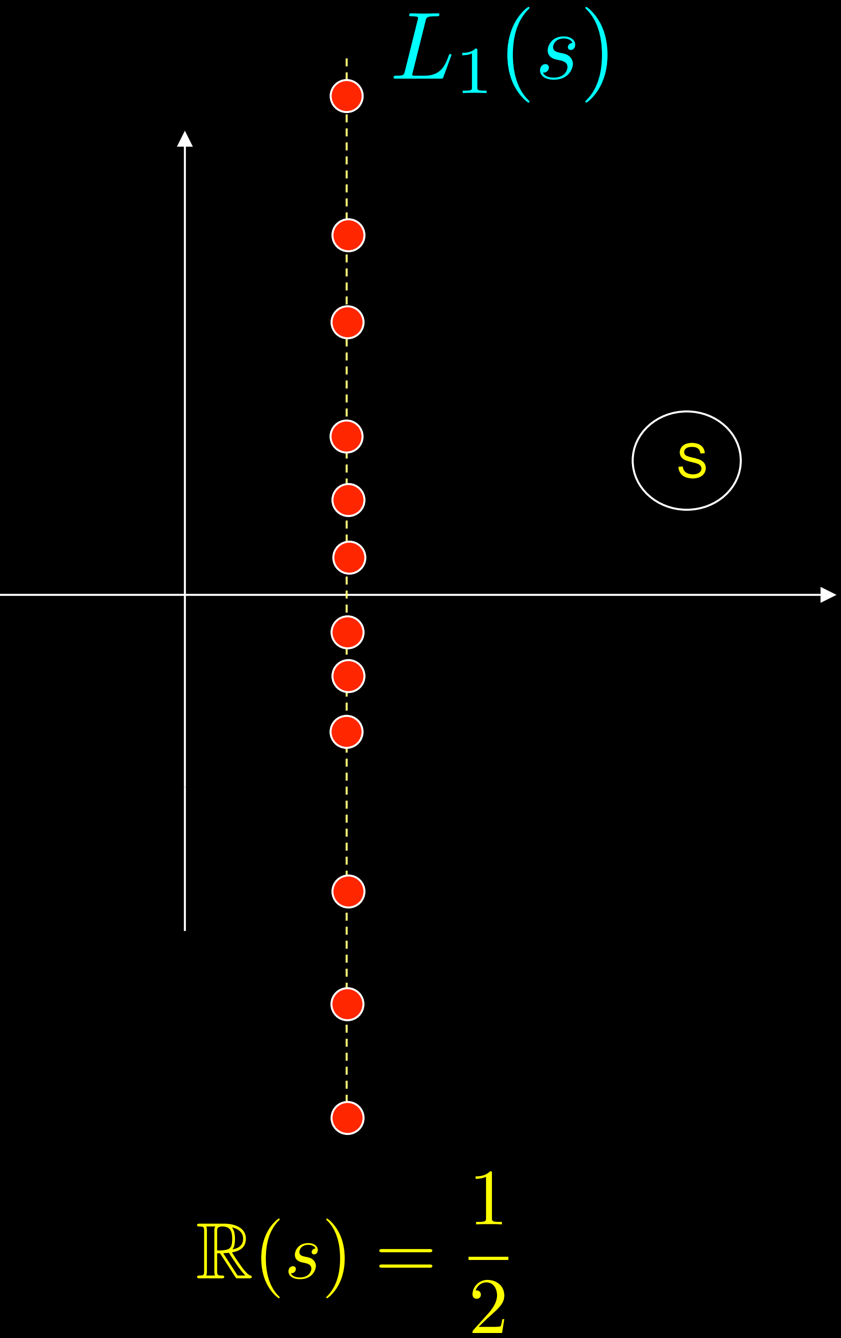


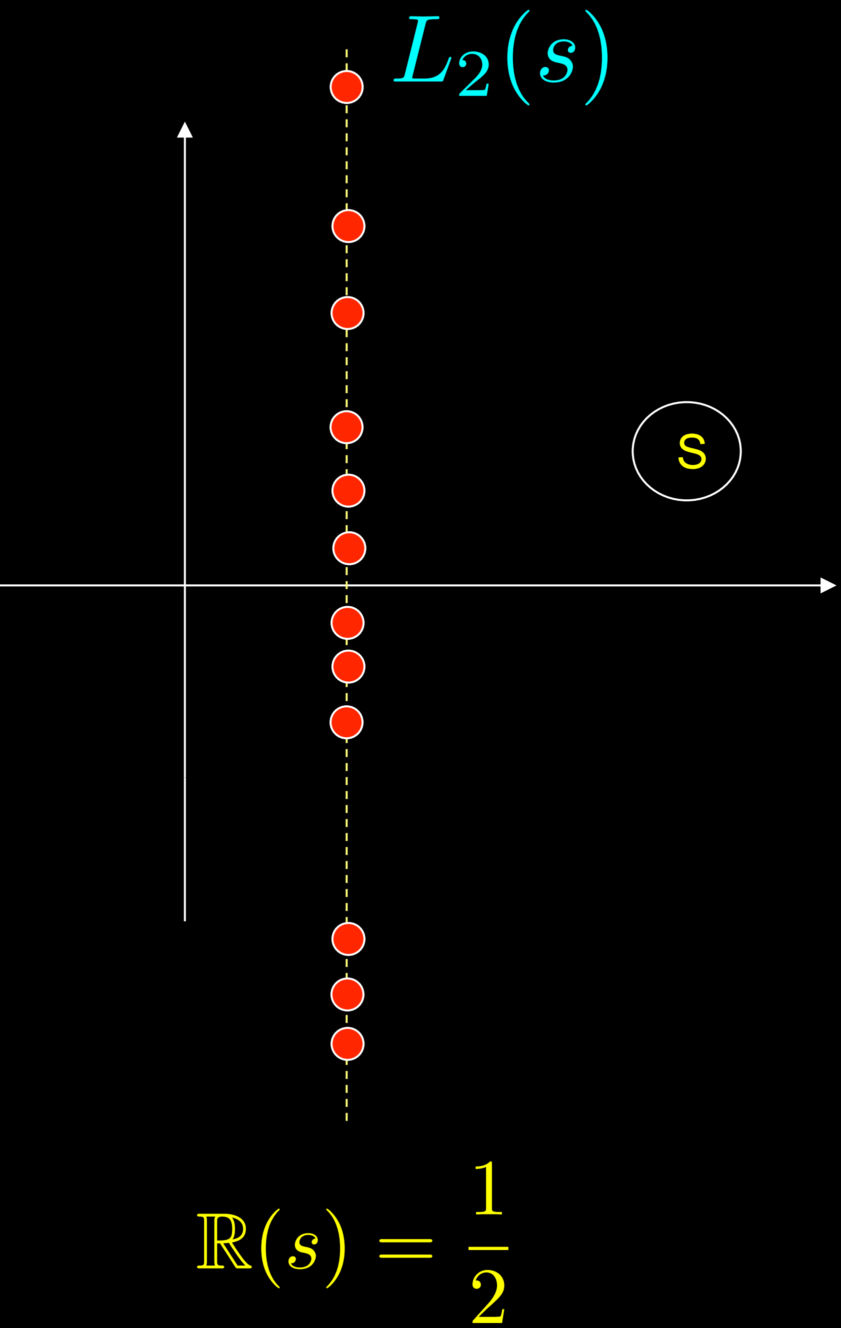
Giuseppe Mussardo

$\zeta(s)$

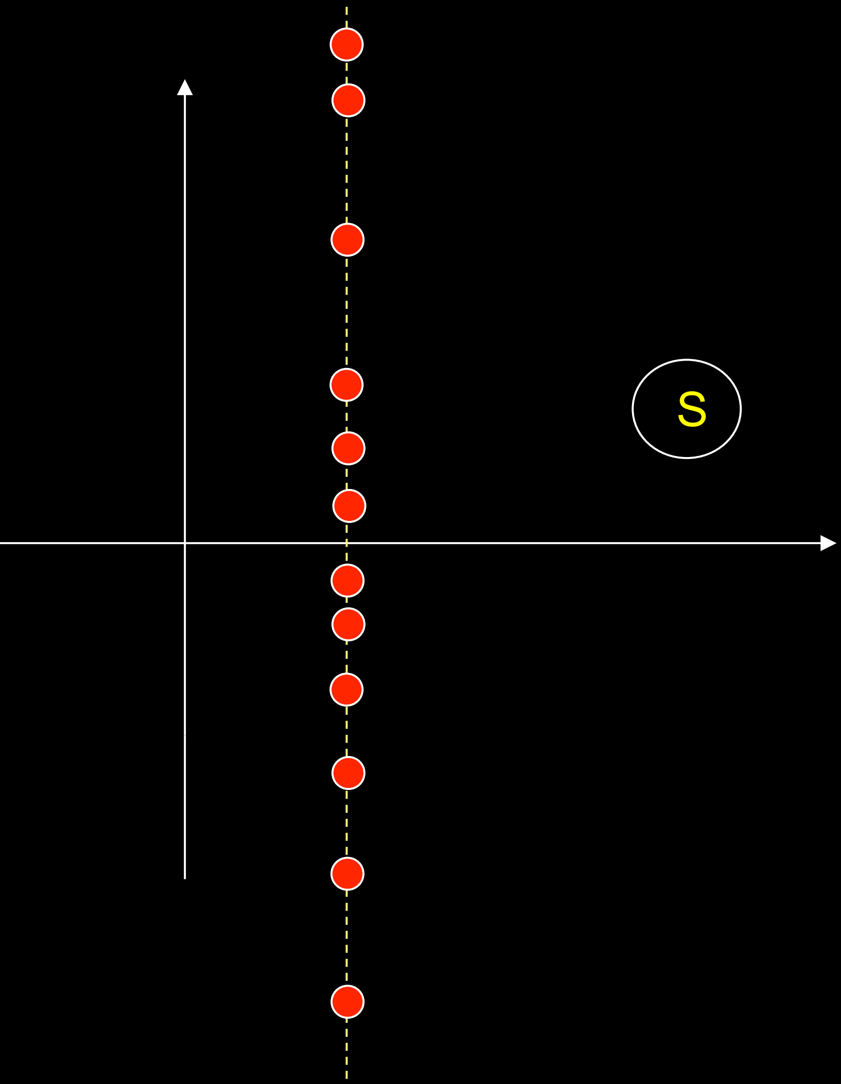


$$\mathbb{R}(s) = \frac{1}{2}$$

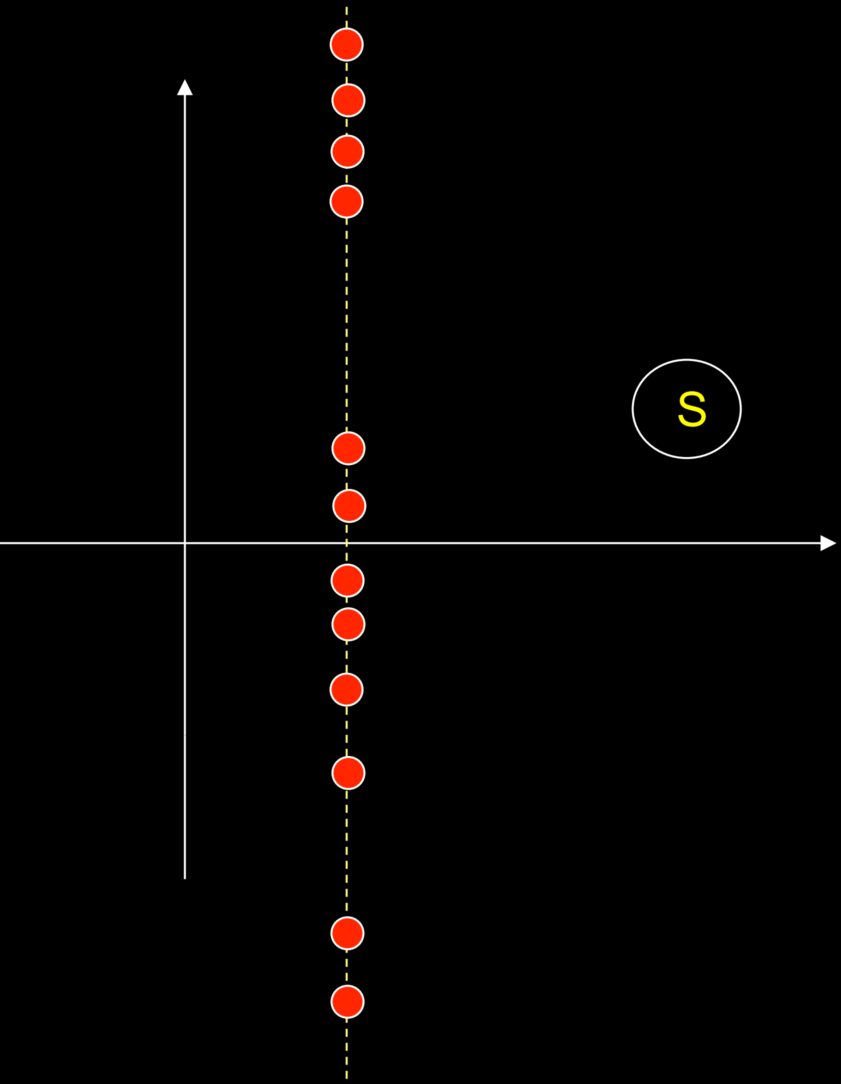

$L_1(s)$
s
$\mathbb{R}(s) = \frac{1}{2}$


$L_2(s)$
s
$\mathbb{R}(s) = \frac{1}{2}$

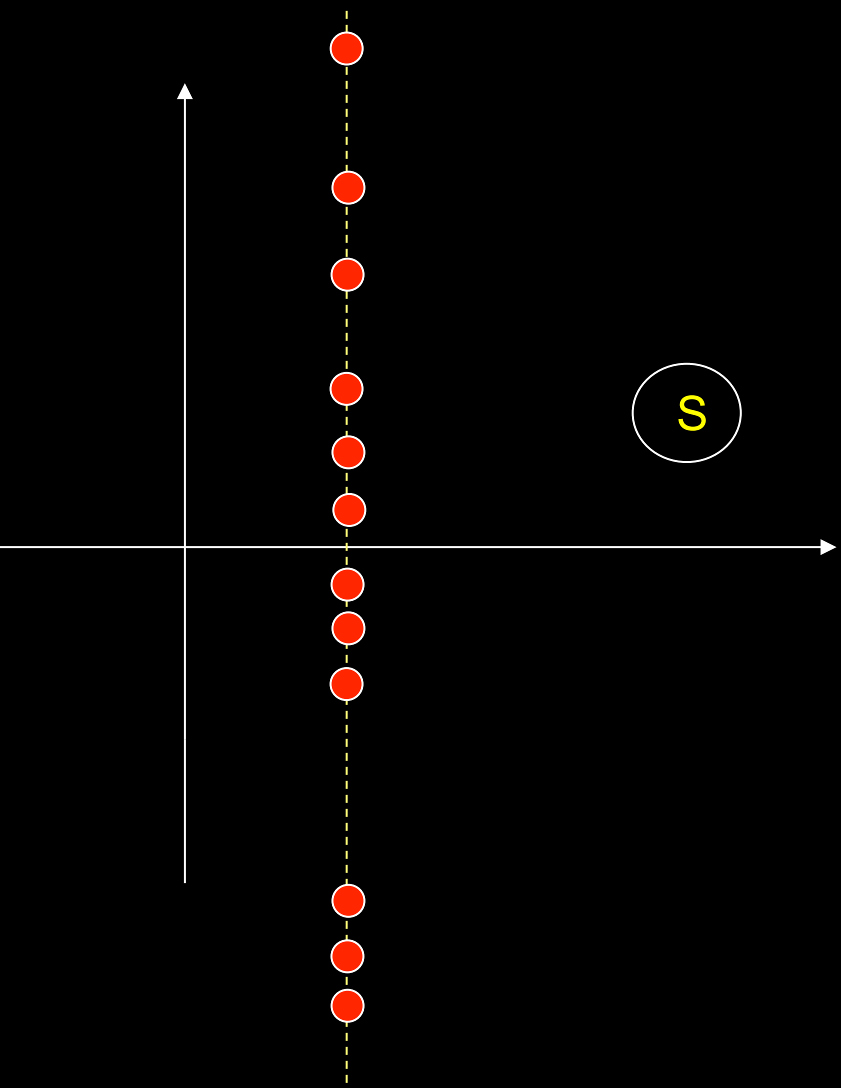
$$\mathbb{R}(s) = \frac{1}{2}$$



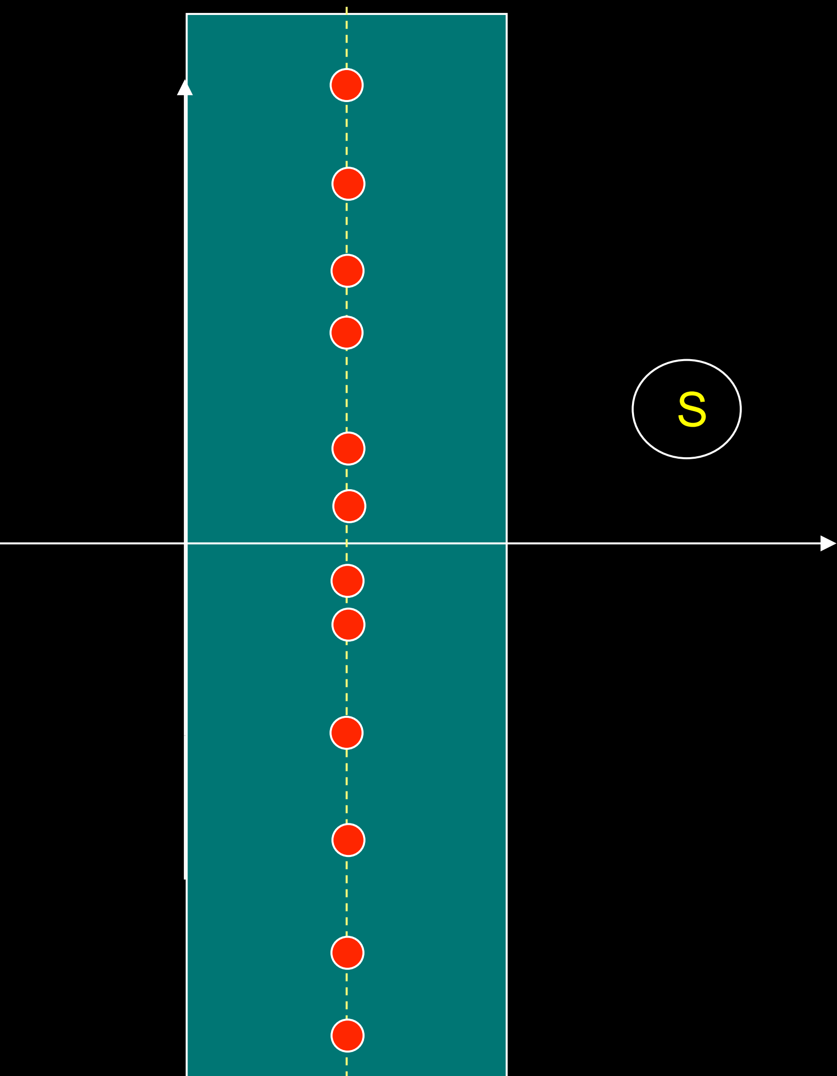
$$\mathbb{R}(s) = \frac{1}{2}$$



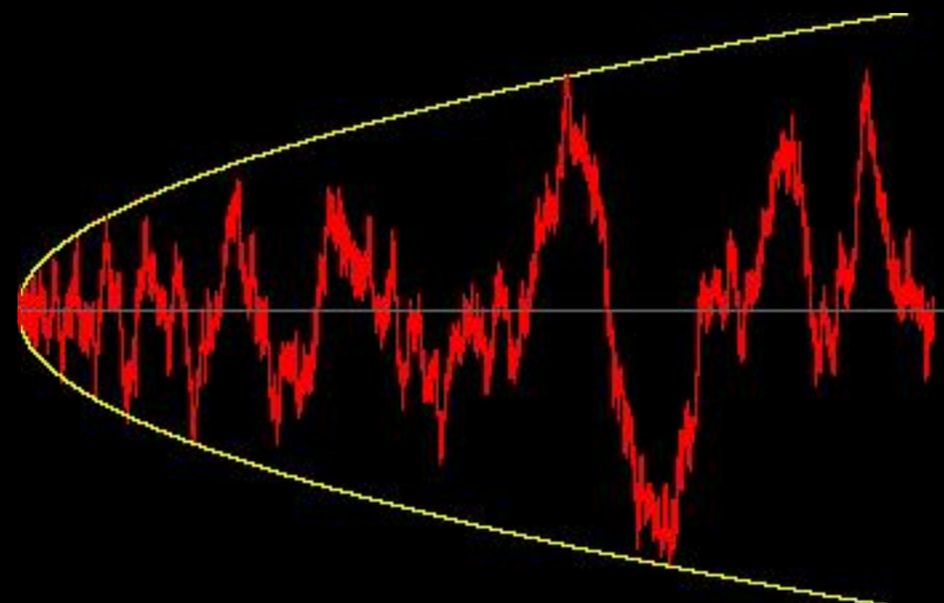
$$\mathbb{R}(s) = \frac{1}{2}$$



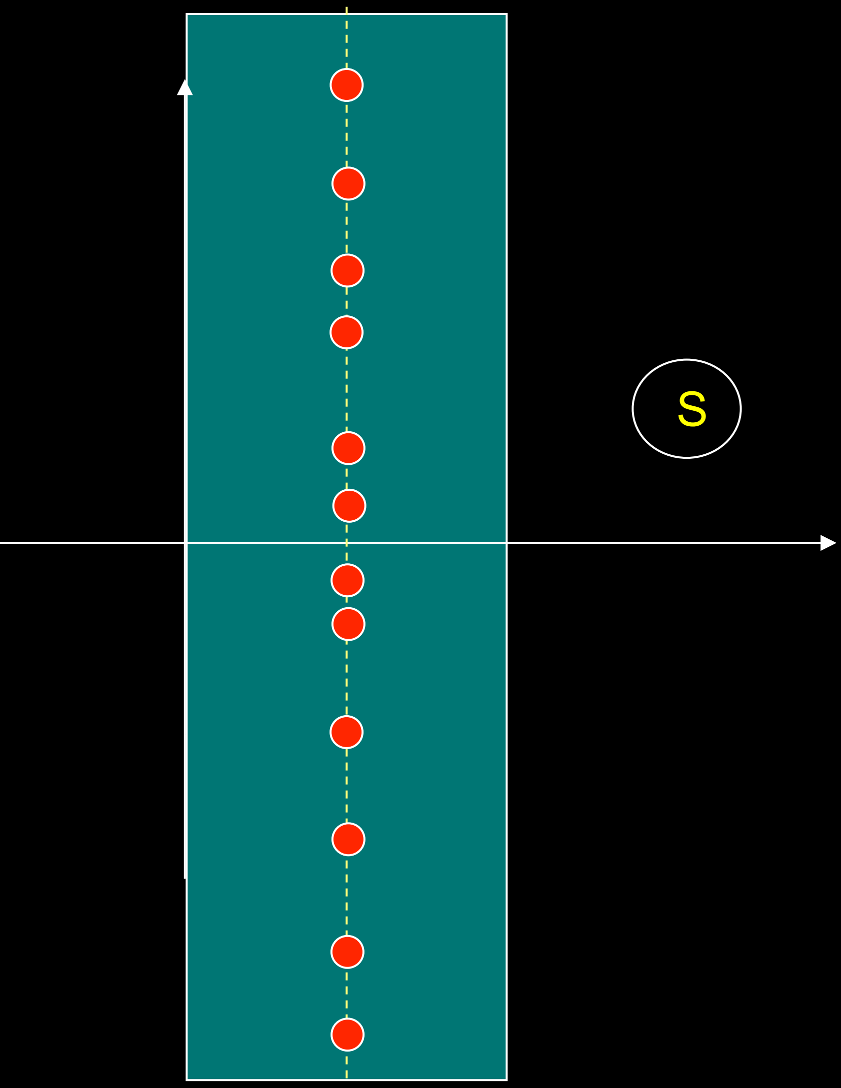
$$\mathbb{R}(s) = \frac{1}{2}$$



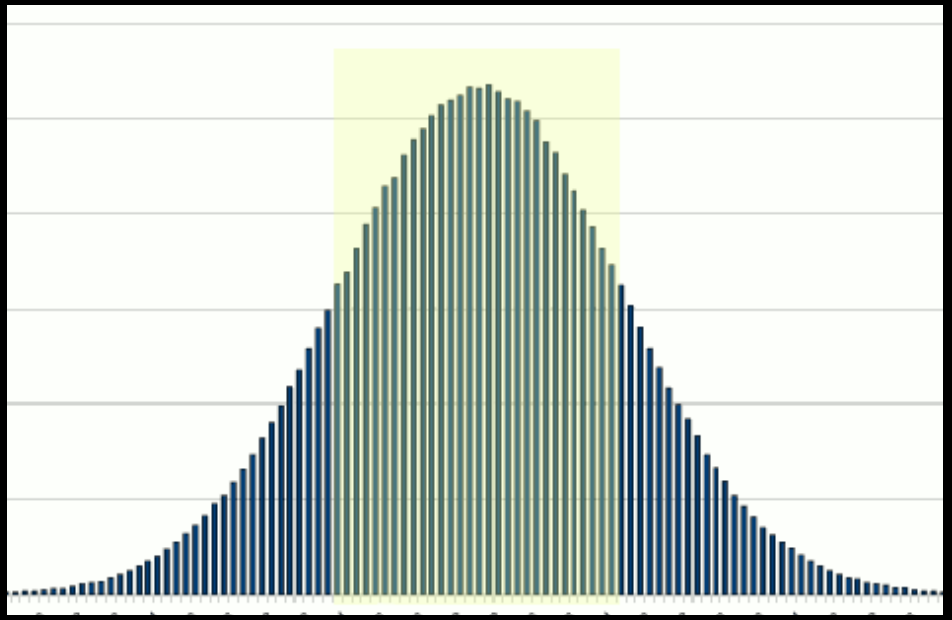
$$\mathbb{R}(s) = \frac{1}{2}$$



$$C_N \simeq N^{1/2}$$



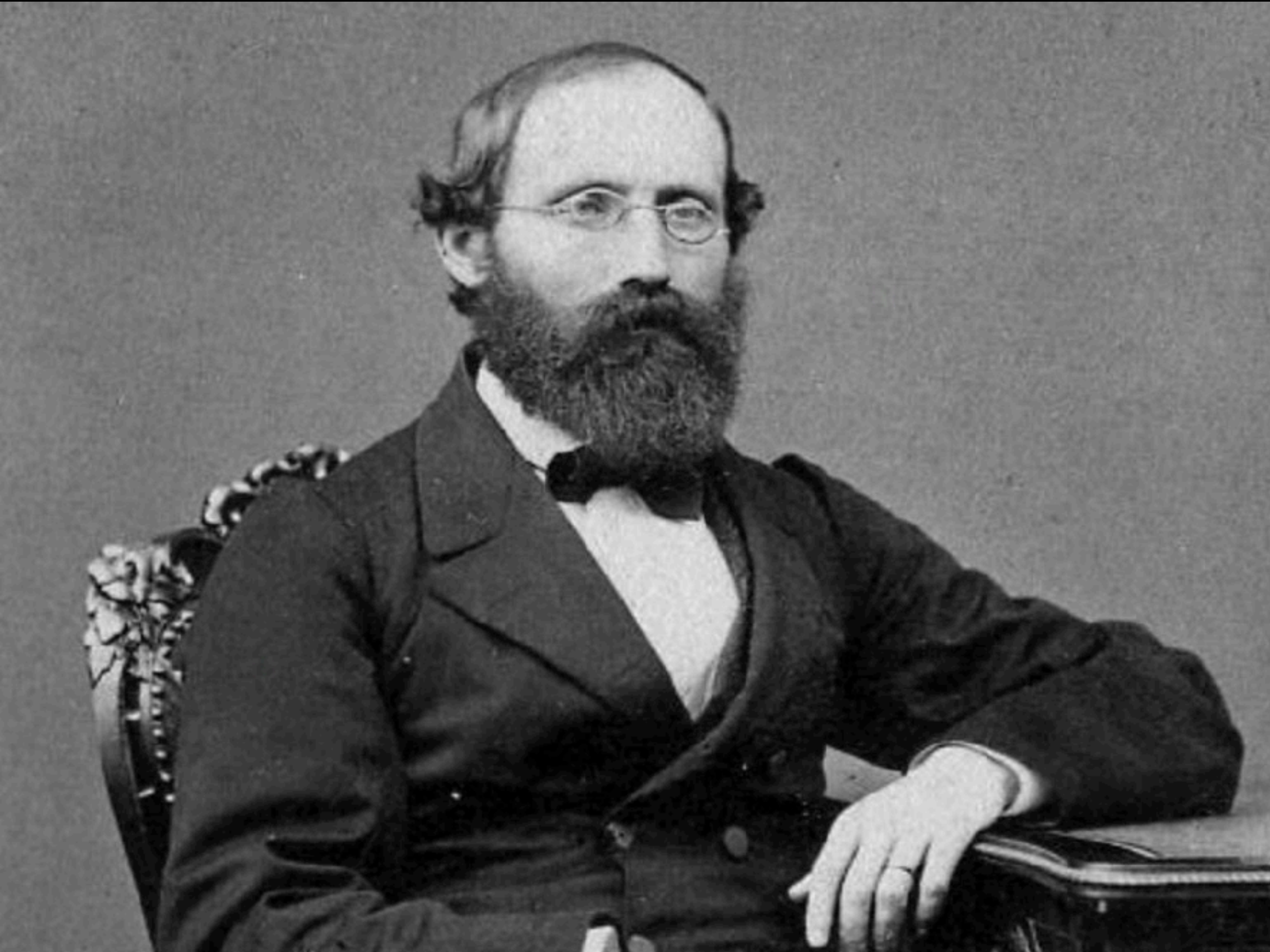
$$\mathbb{R}(s) = \frac{1}{2}$$



$$C_N \simeq N^{1/2}$$

Topics of the seminar

- The art of counting
- Seduced by the zeros
- Arithmetic Progressions
- Dirichlet characters, $\mathcal{L}$-functions and all that
- The grand picture: the Generalised Riemann Hypothesis
- Randomness vs Determinism



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \Sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

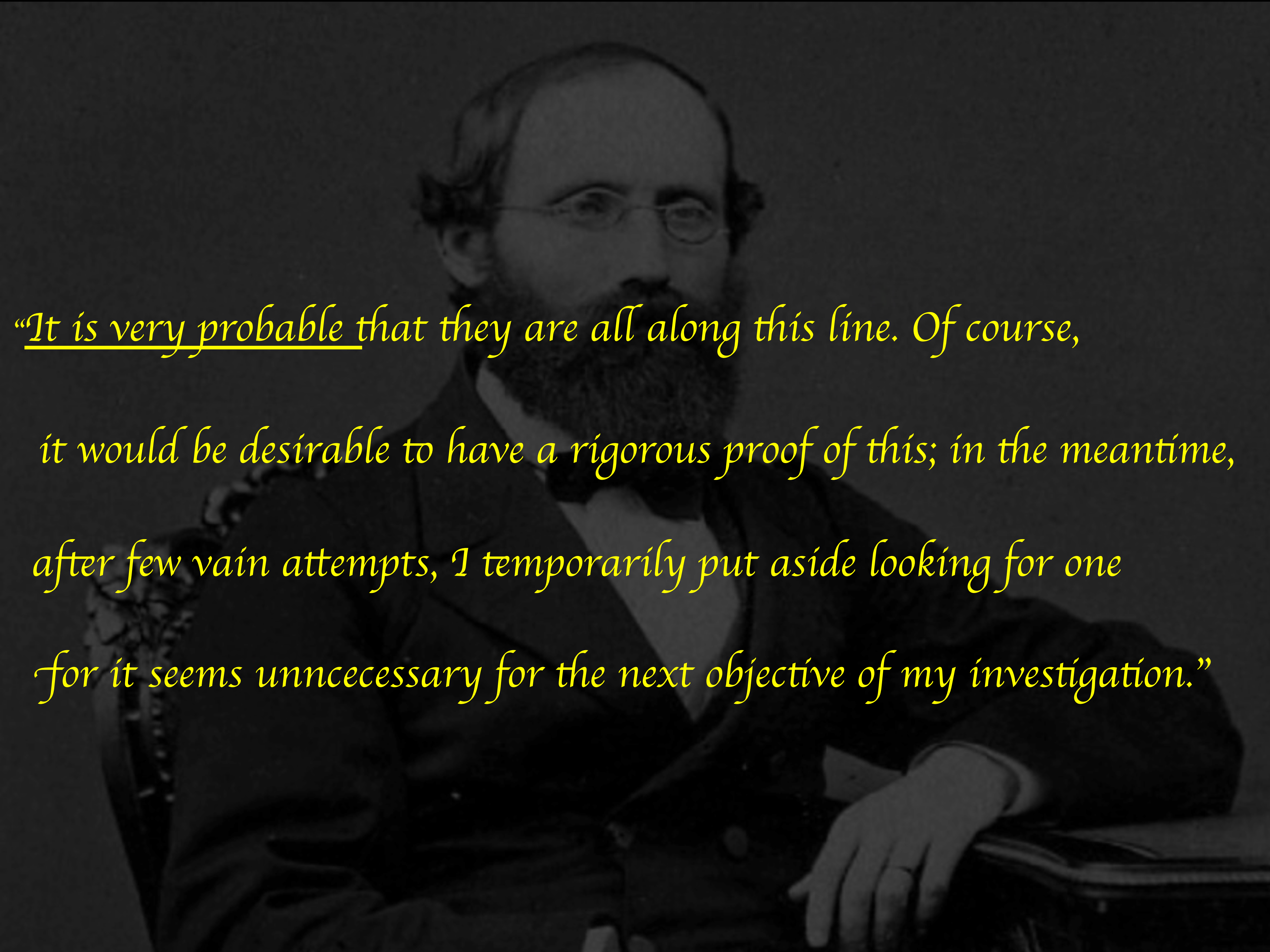
Betrachtet man nun das Integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

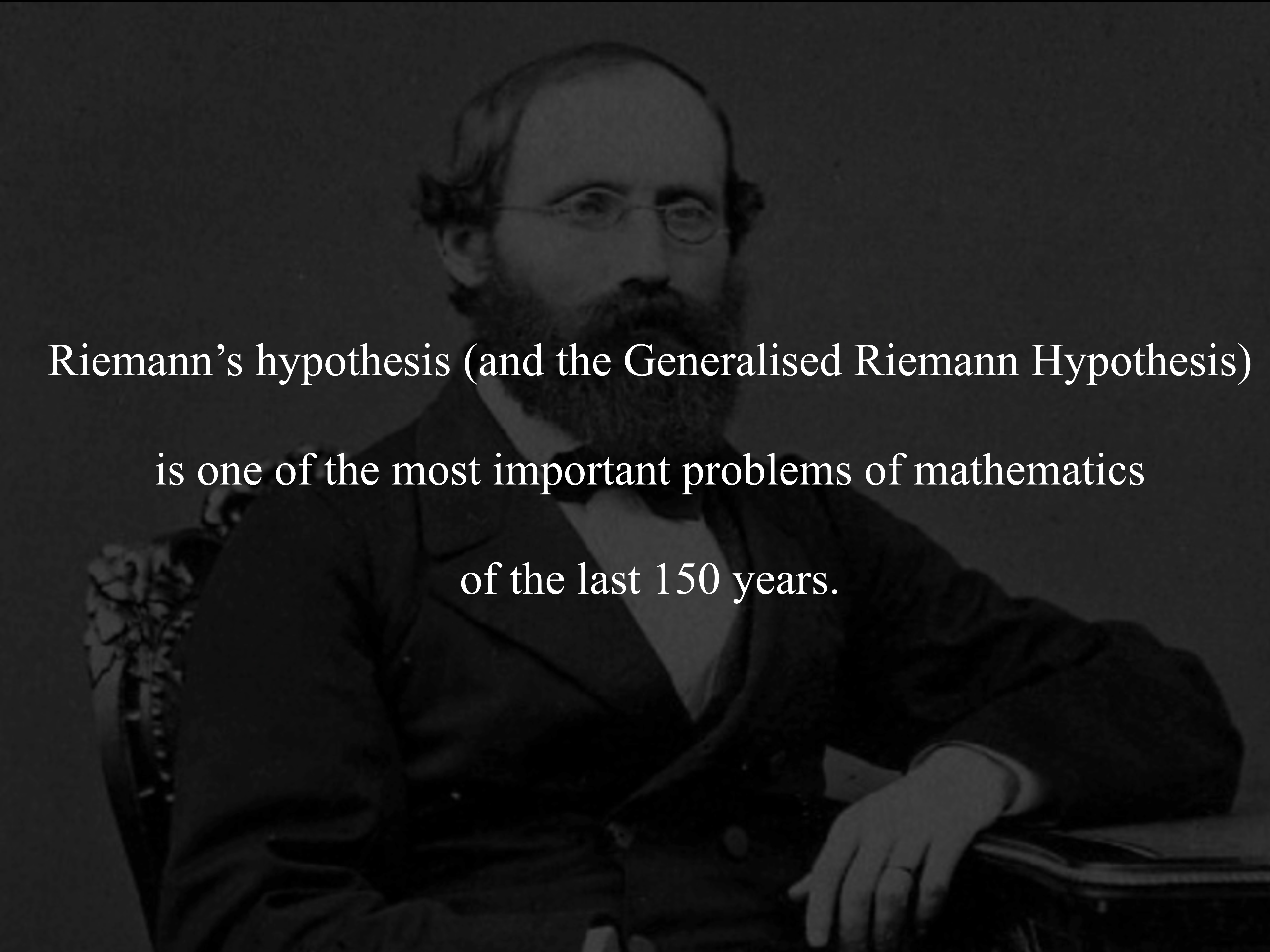
von $+\infty$ bis $+\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Unstetigkeitswerth der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich dieses leicht gleich

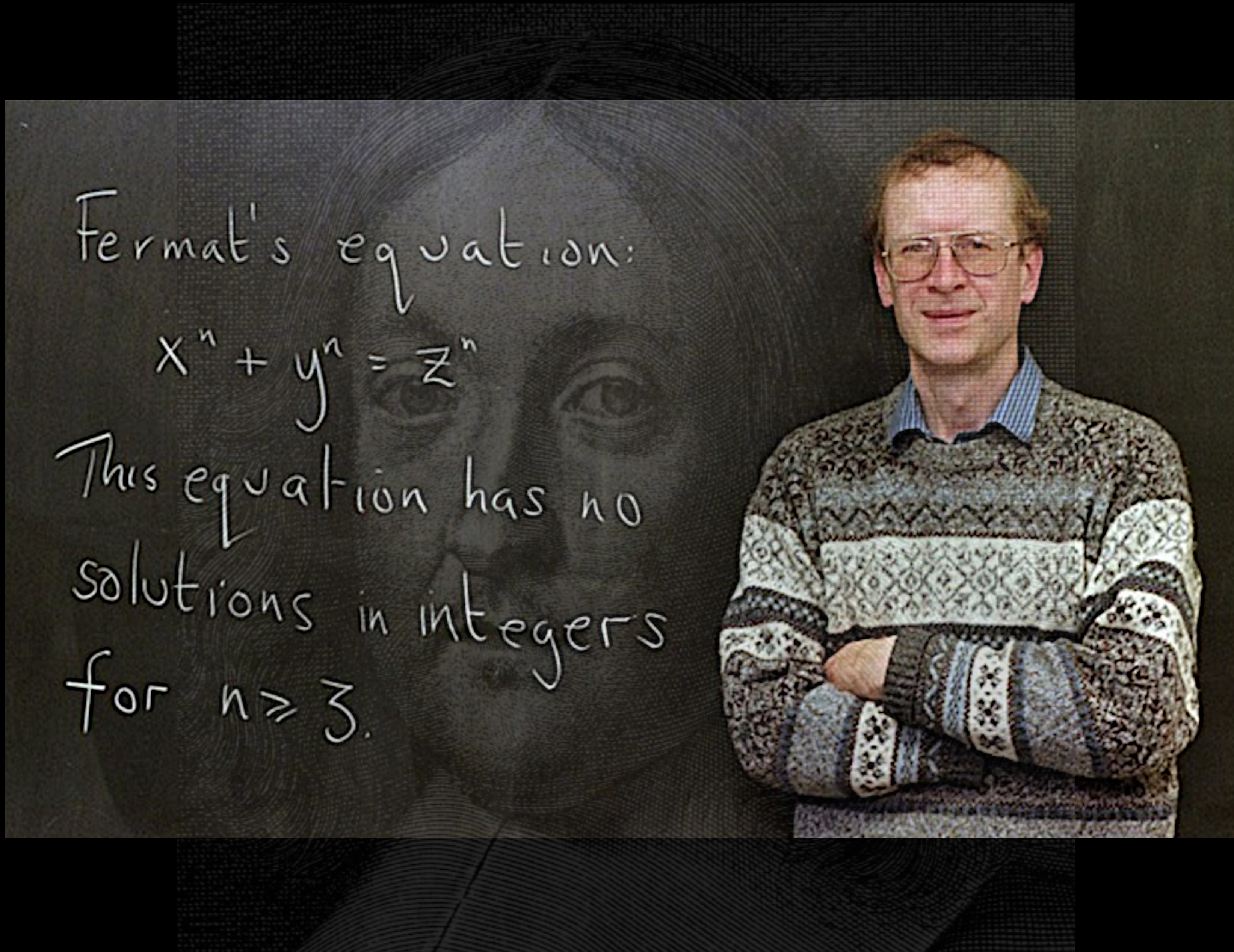
$$(e^{-\pi si} - e^{\pi si}) \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1},$$

vorausgesetzt, dass in der vieldeutigen Function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ der Logarithmus von $-x$ so bestimmt worden ist, dass er für ein negatives x reell wird. Man



"It is very probable that they are all along this line. Of course, it would be desirable to have a rigorous proof of this; in the meantime, after few vain attempts, I temporarily put aside looking for one for it seems unncecessary for the next objective of my investigation."


Riemann’s hypothesis (and the Generalised Riemann Hypothesis)
is one of the most important problems of mathematics
of the last 150 years.

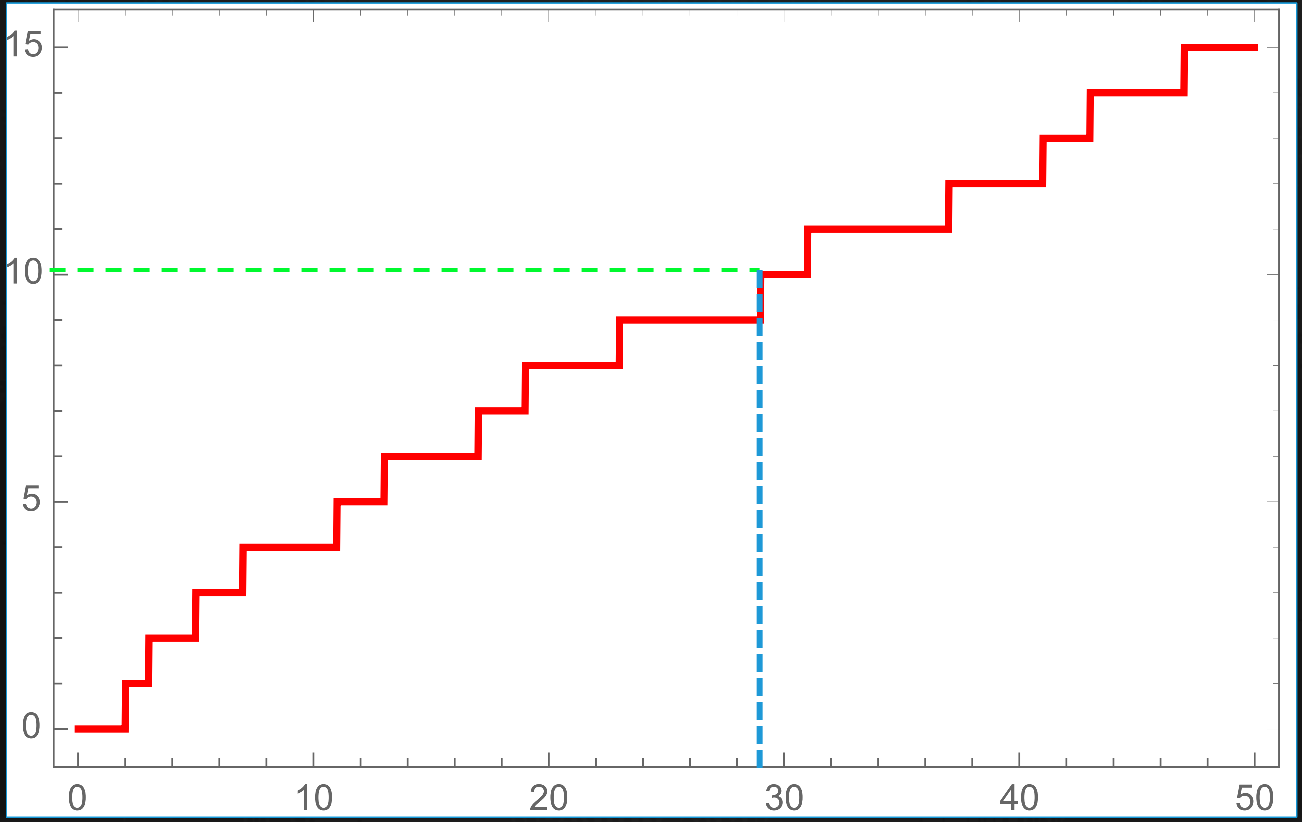

Fermat's equation:
$x^n + y^n = z^n$
This equation has no solutions in integers for $n \geq 3$.

Innumerable consequences of the Riemann Hypothesis

- Distribution of the primes (large gap, etc.)
- Growth of some arithmetic functions
- Rate of growth of $\zeta(z)$ along the critical line
- Zoo of zeta functions
- Quantum Mechanics, Primality Test
- Random Matrices and Quantum Chaos
- General Theory of Phase Transitions

Counting the Primes

$\pi(x)$: gives the number of primes less or equal to x



Counting the Primes

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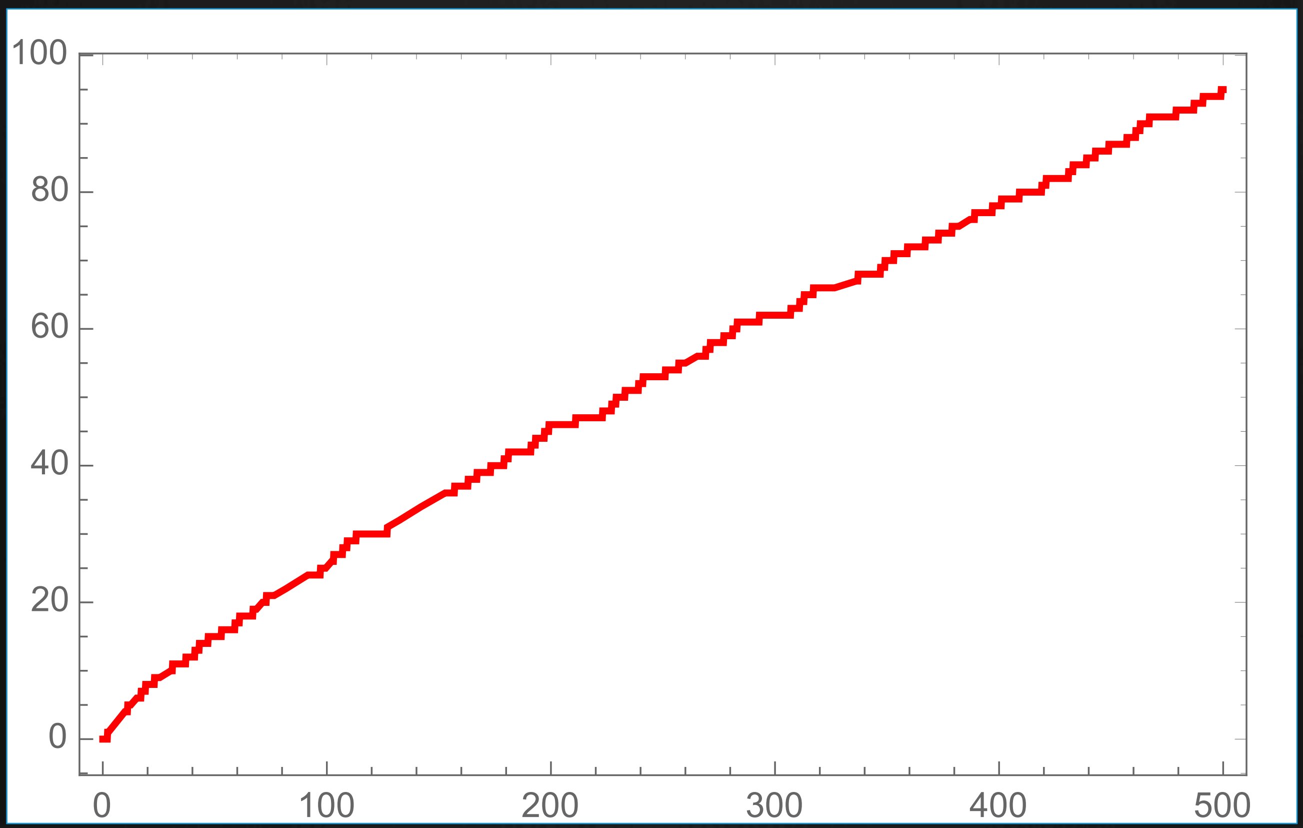

Counting the Primes

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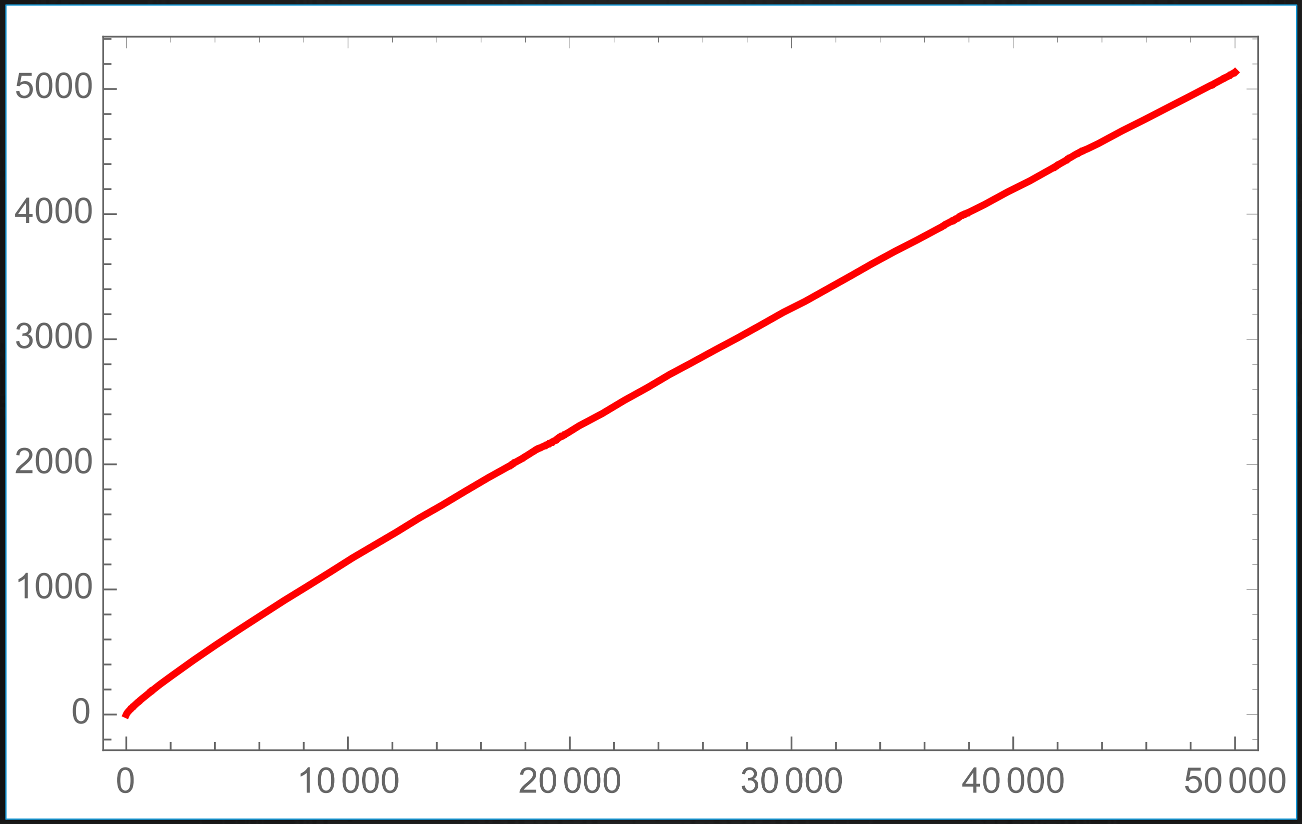


Prime Number Theorem: Gauss

$$\pi(x) \simeq Li(x) \equiv \int_1^x \frac{dt}{\log t} \simeq \frac{x}{\log x}$$

A simple proof

- assuming no correlation between numbers, 1/p is the probability that a number x is divisible by the prime p
- $$W(x) \simeq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots = \prod_{p<x}\left(1-\frac{1}{p}\right)$$
- $$\log W(x) \simeq \sum_{p<x} \log\left(1-\frac{1}{p}\right) \simeq -\sum_p \frac{1}{p} \simeq -\int_1^x dn \frac{W(n)}{n}$$
- $$\frac{W'(x)}{W(x)} = \frac{W(x)}{x} \longrightarrow W(x) = \frac{1}{\log x}$$

Primes are rare but not so rare,
their number is larger, say,
of the number of squares n^2

$$\pi_p(x) \simeq \frac{x}{\log x}$$

$$\pi_{n^2}(x) \simeq \sqrt{x}$$

x	$\pi(x) = \#\{\text{primes } \leq x\}$	Overcount: $\mathrm{Li}(x) - \pi(x)$
10^8	5761455	753
10^9	50847534	1700
10^{10}	455052511	3103
10^{11}	4118054813	11587
10^{12}	37607912018	38262
10^{13}	346065536839	108970
10^{14}	3204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	279238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2220819602560918840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207

The Riemann zeta Function

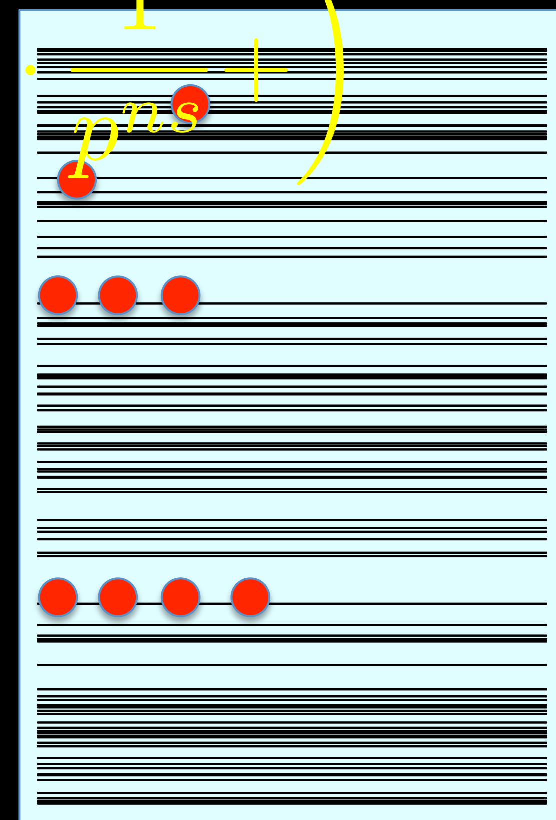
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \frac{1}{p^{ns}} + \right)$$

Free bosonic system

$$E_p = \log p$$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$



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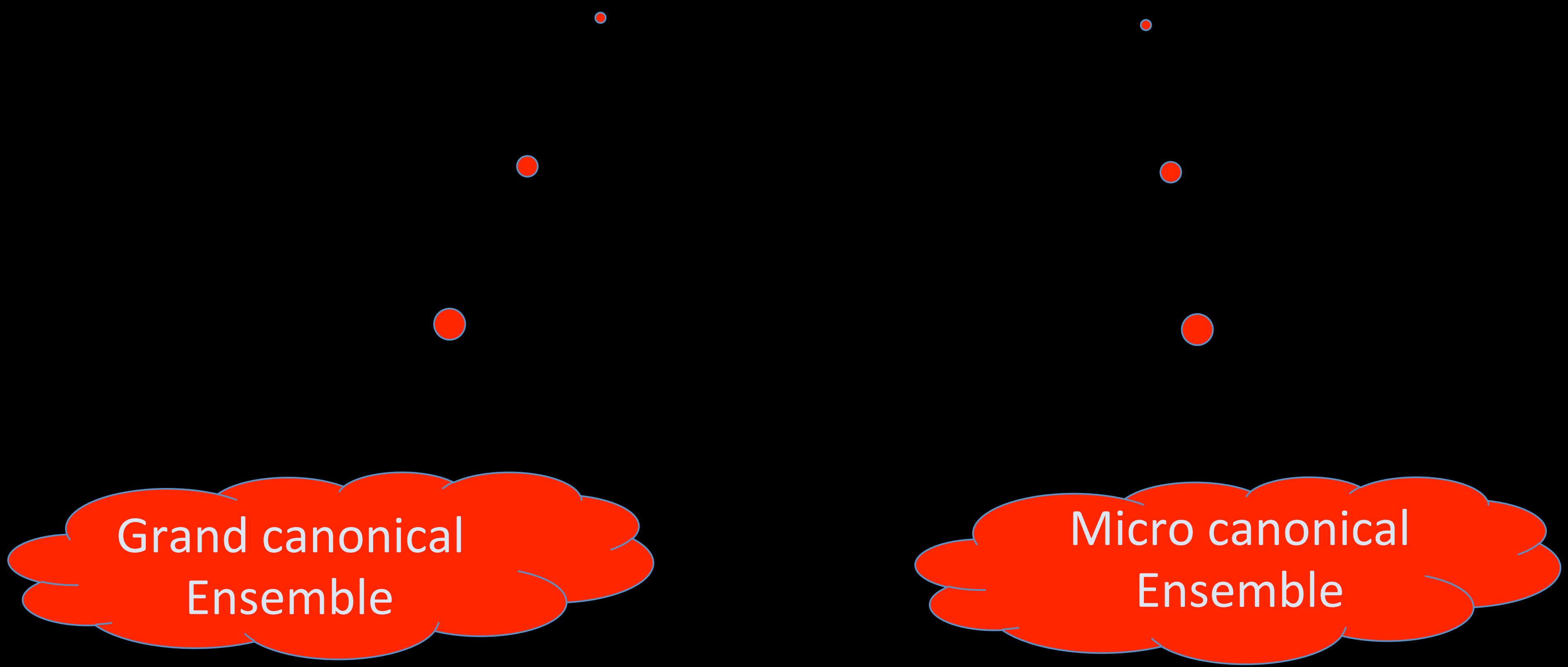

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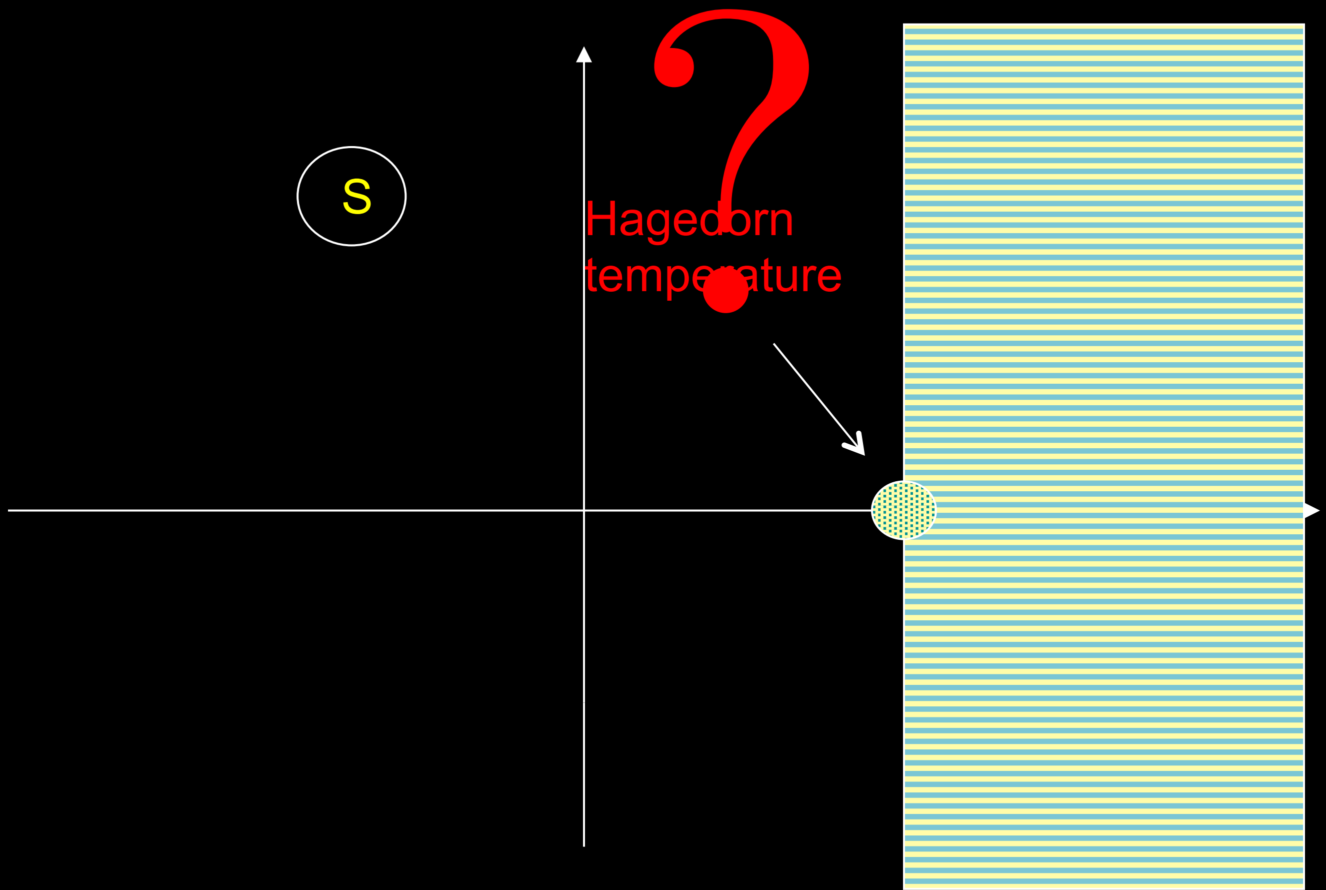


The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Analytic continuation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint_C \frac{(-x)^{s-1}}{e^x - 1}$$

The Riemann zeta Function

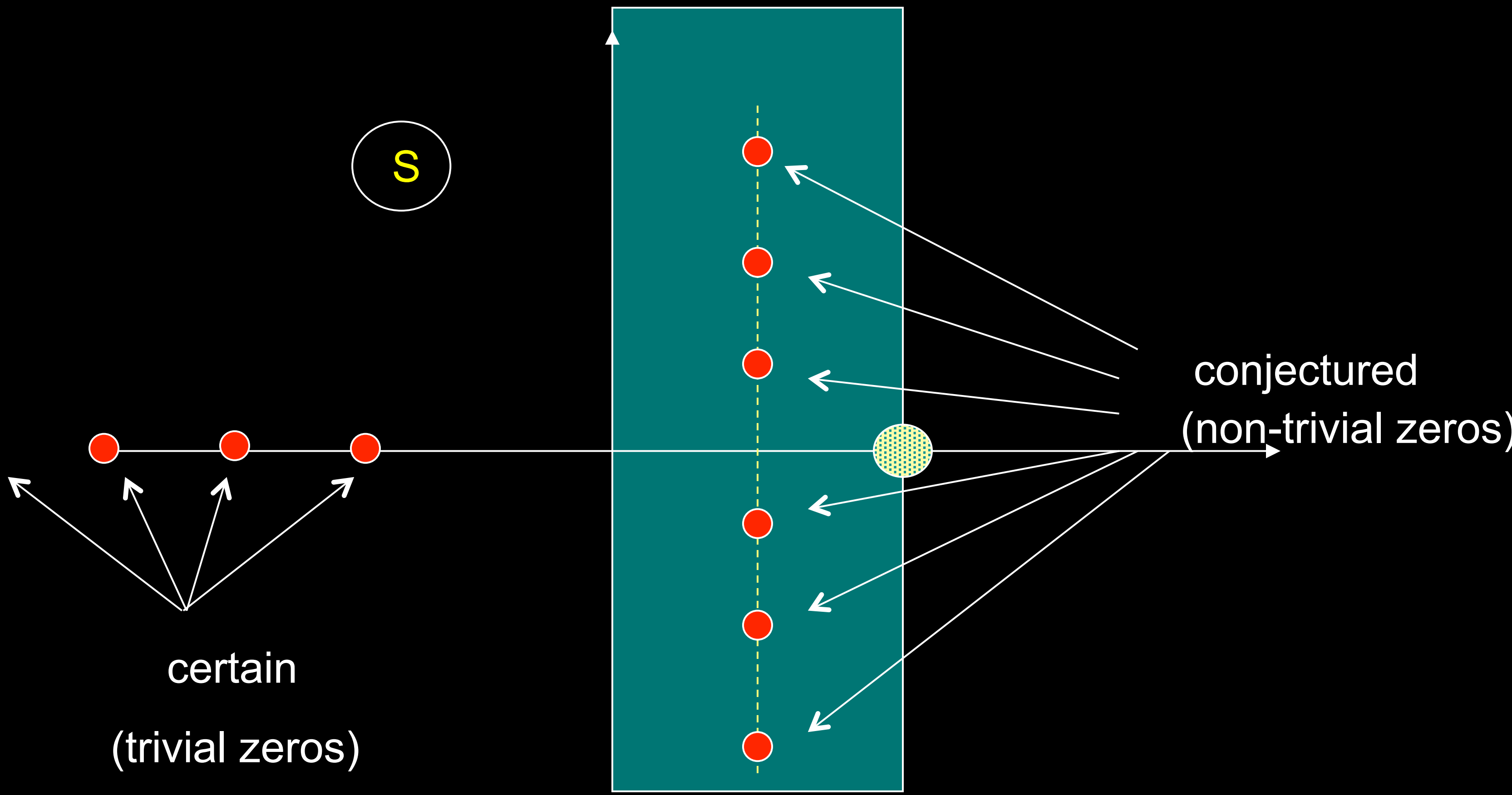
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Functional equation

$$\boxed{\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)}$$

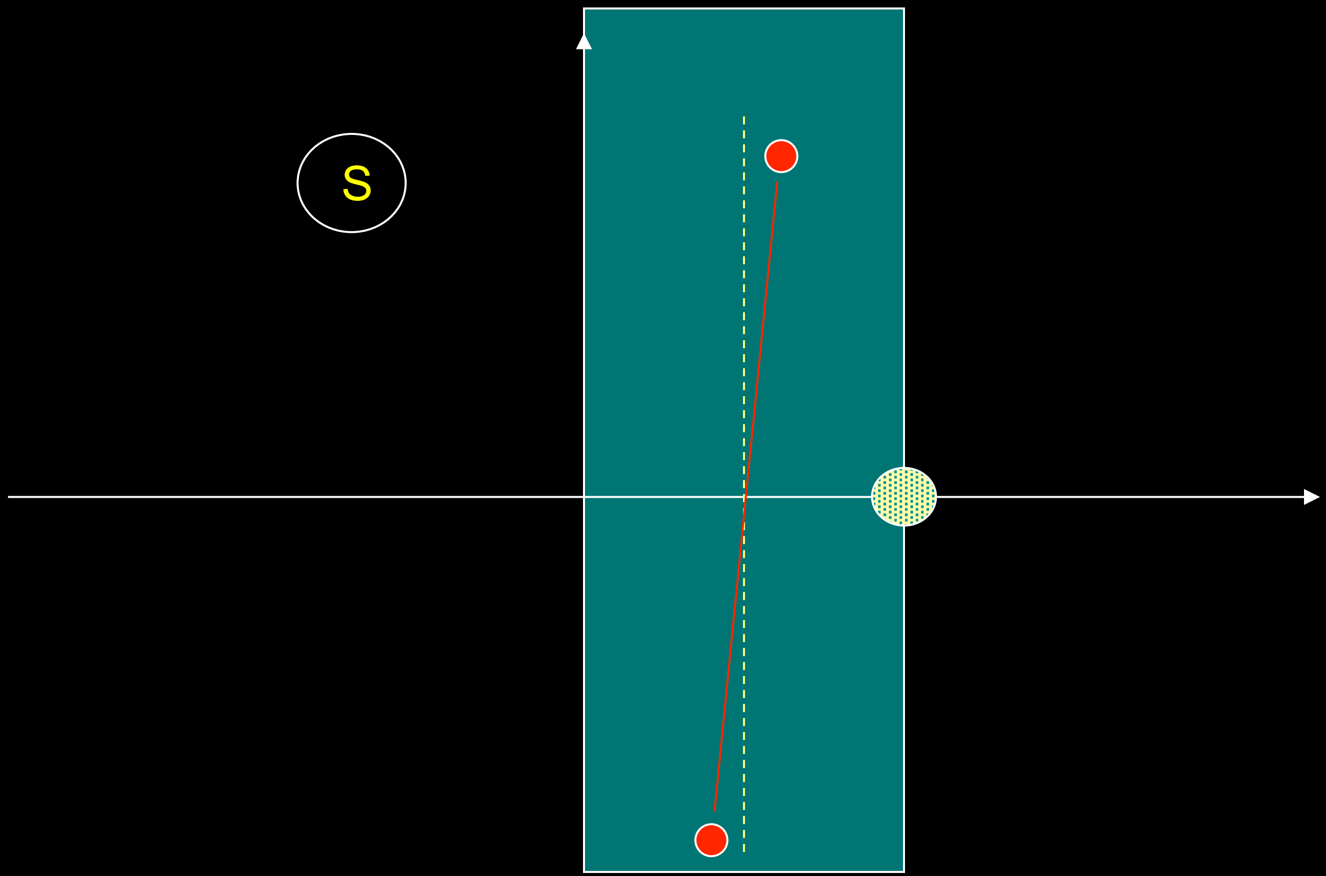
The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$



The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$



Connection between the primes and the Riemann ζ(s) function

$$\log \zeta(s) = -\sum_{p} \log\left(1 - \frac{1}{p^s}\right)$$

$$= -\sum_{n=1}^{\infty} [\pi(n) - \pi(n-1)] \log\left(1 - \frac{1}{n^s}\right)$$

$$= -\sum_{n=1}^{\infty} \pi(n) \left[\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right]$$

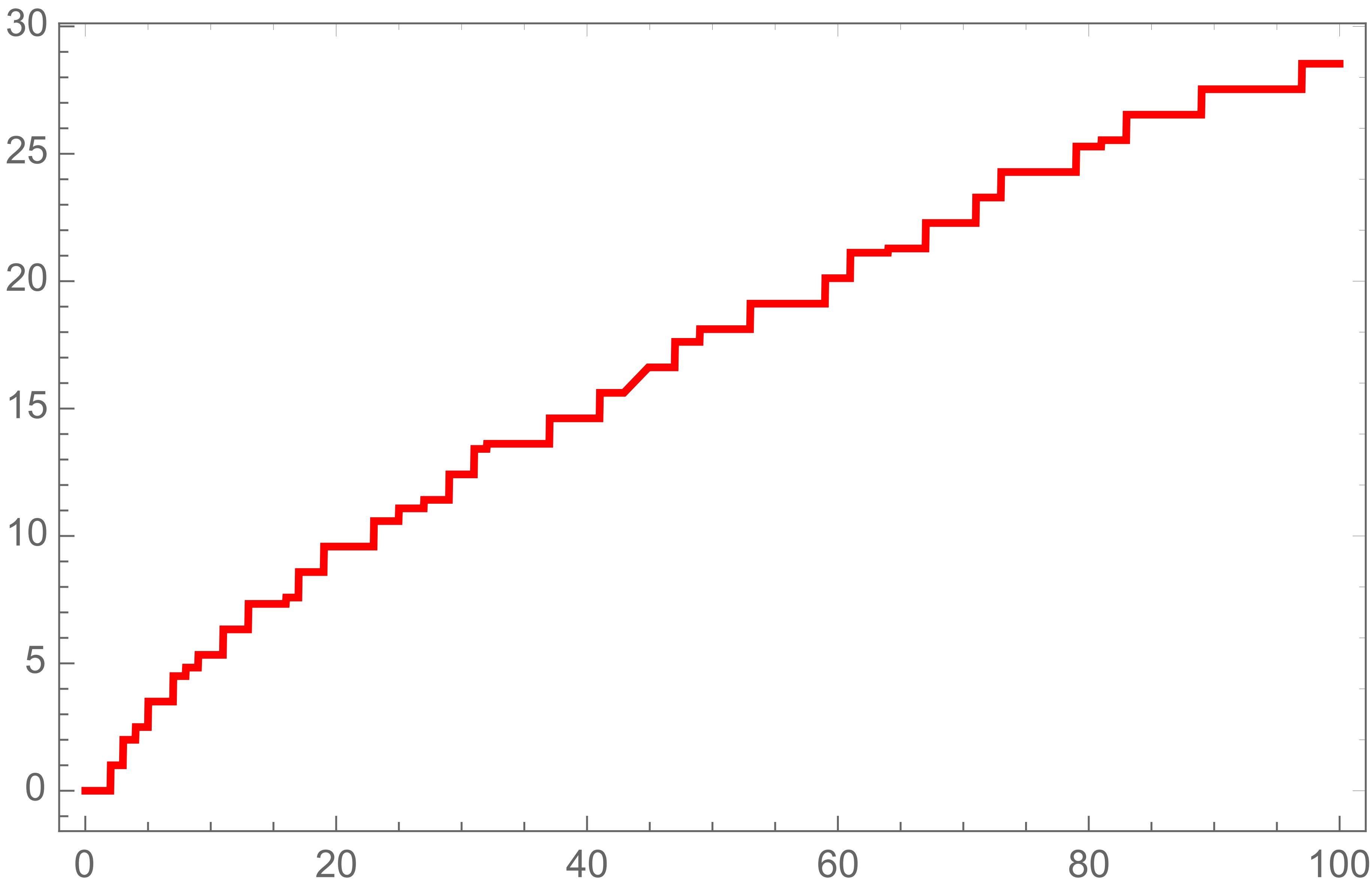
$$= \sum_{n=1}^{\infty} \pi(n) \int_{n}^{n+1} \frac{s}{x(x^s - 1)} dx \quad = s \int_{1}^{\infty} \frac{\pi(x)}{x(x^s - 1)} dx$$

Connection between the primes and the Riemann ζ(s) function

$$\log \boxed{\zeta(s)} = s \int_1^\infty \frac{\boxed{\pi(x)}}{x(x^s - 1)}\,dx$$

$$= s \int_0^\infty \frac{J(x)}{x^{s+1}}\,dx$$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$



30
25
20
15
10
5
0
0
20
40
60
80
100

Connection between the primes and the Riemann ζ(s) function

$$\log \zeta(s) = s \int_1^\infty \frac{\pi(x)}{x(x^s - 1)} dx$$

$$= s \int_0^\infty \frac{J(x)}{x^{s+1}} dx$$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$

Connection between the primes and the Riemann ζ(s) function

$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n})$$

$$\mu(n) = \begin{cases} 1 & \text{if n is squarefree with an even number of prime factors} \\ -1 & \text{if n is squarefeee with an odd number of prime factors} \\ 0 & \text{if n has a squared prime factor} \end{cases}$$

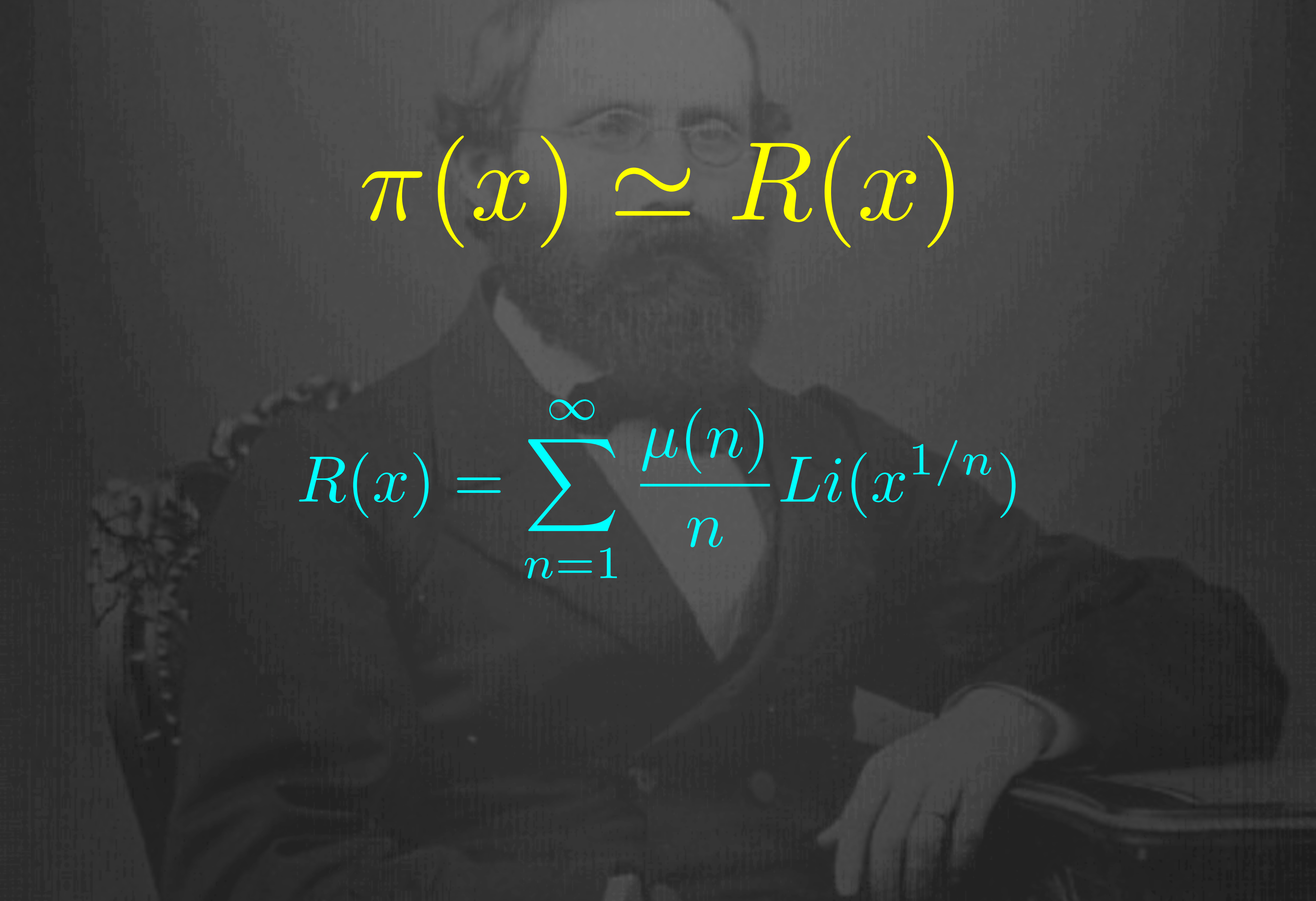
Connection between the primes and the Riemann ζ(s) function

$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n})$$

$$\boxed{\begin{aligned} J(x) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log\zeta(s)\, x^s \,\frac{ds}{s} \\ &= Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \end{aligned}}$$

Prime Number Theorem: Riemann

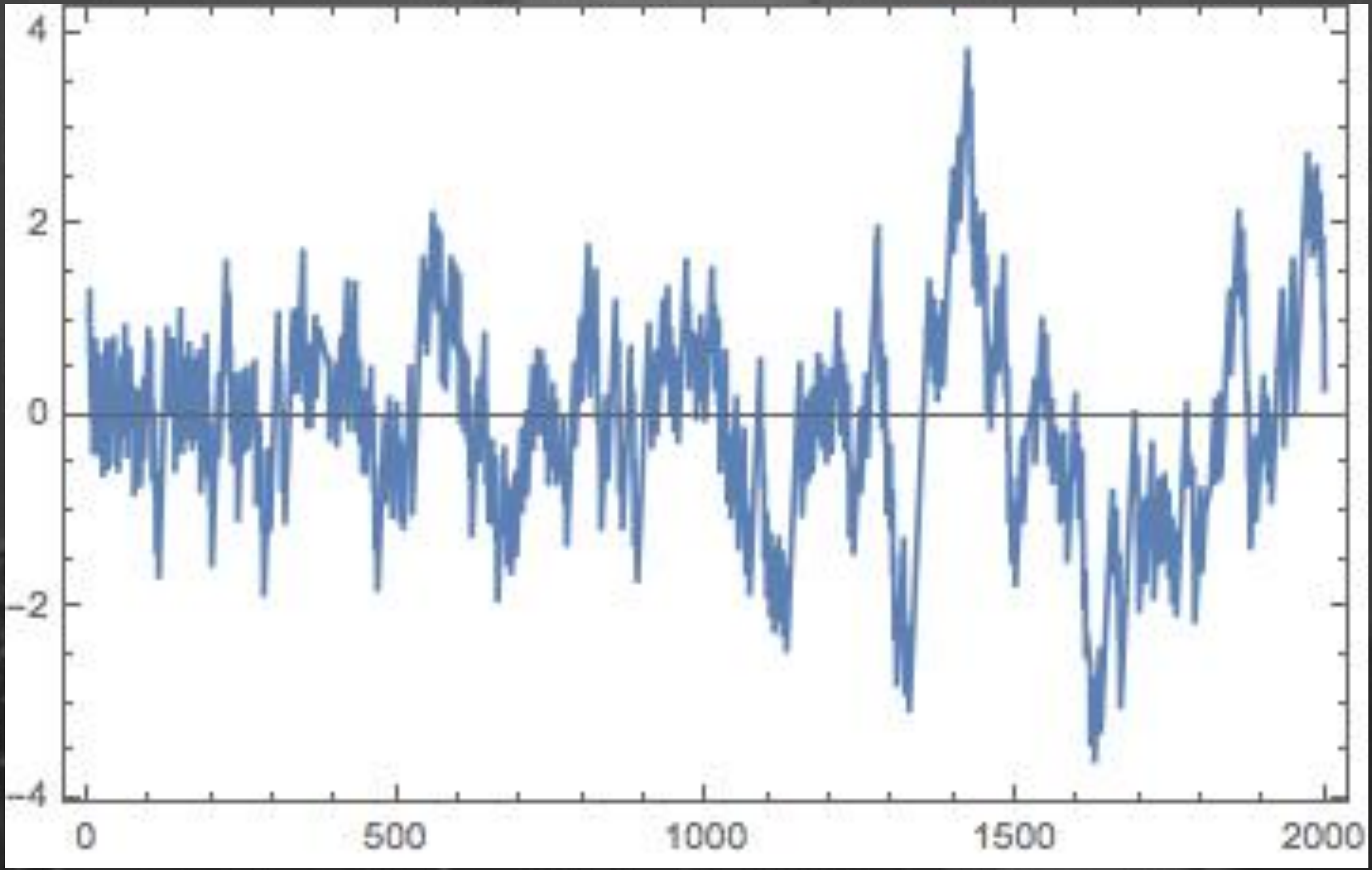


$$\pi(x) \simeq R(x)$$

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n})$$

x	#{primes $\leq x$}	Gauss's overcount	Riemann's overcount
10^8	5761455	753	131
10^9	50847534	1700	−15
10^{10}	455052511	3103	−1711
10^{11}	4118054813	11587	−2097
10^{12}	37607912018	38262	−1050
10^{13}	346065536839	108970	−4944
10^{14}	3204941750802	314889	−17569
10^{15}	29844570422669	1052618	76456
10^{16}	279238341033925	3214631	333527
10^{17}	2623557157654233	7956588	−585236
10^{18}	24739954287740860	21949554	−3475062
10^{19}	234057667276344607	99877774	23937697
10^{20}	2220819602560918840	222744643	−4783163
10^{21}	21127269486018731928	597394253	−86210244
10^{22}	201467286689315906290	1932355207	−126677992

"The music of the primes"



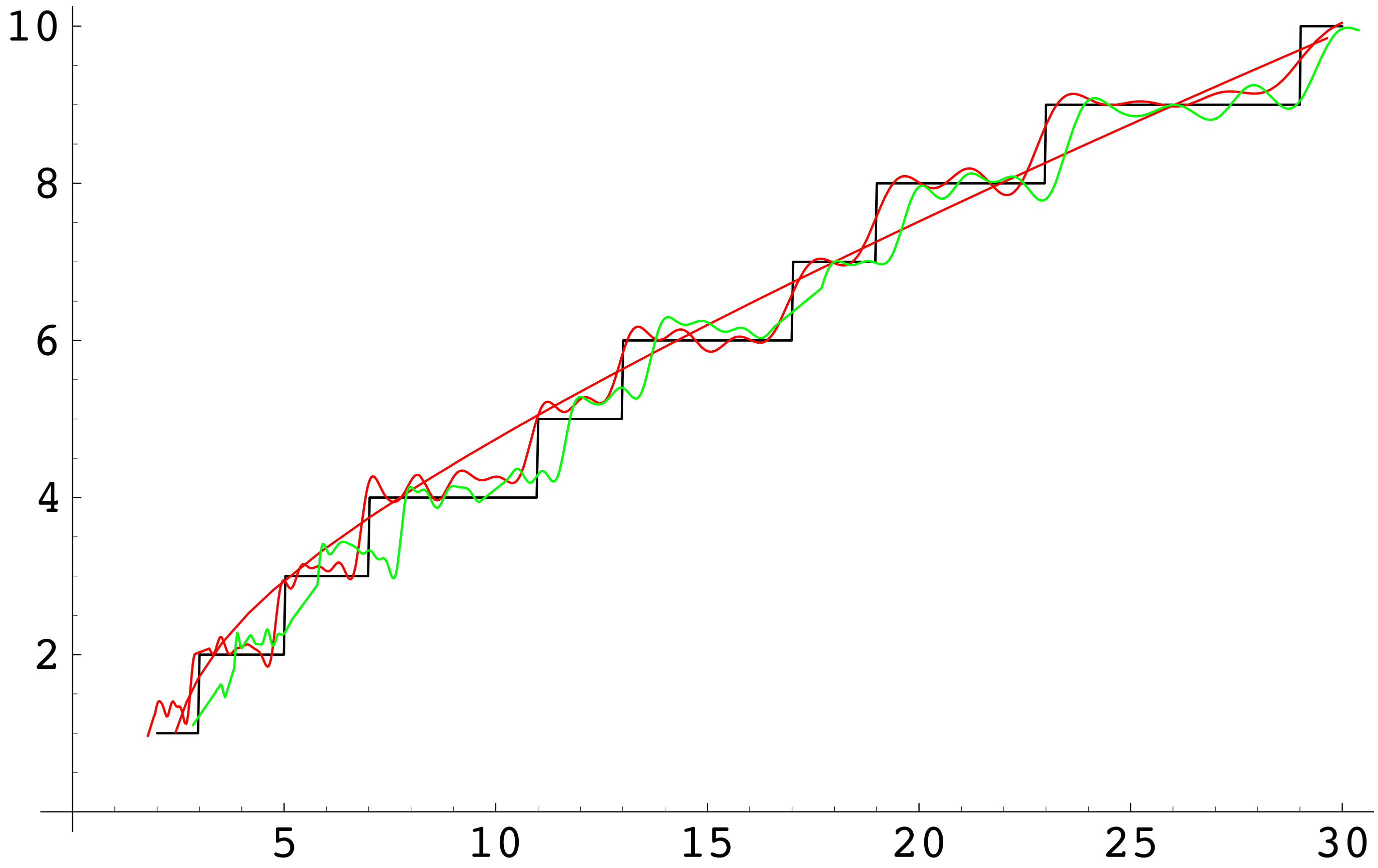
Prime Number Theorem: Riemann

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n})$$

$$\boxed{\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})}$$

ρ : non-trivial zeros of the ζ(z) Riemann function in the critical strip

Climbing the staircase



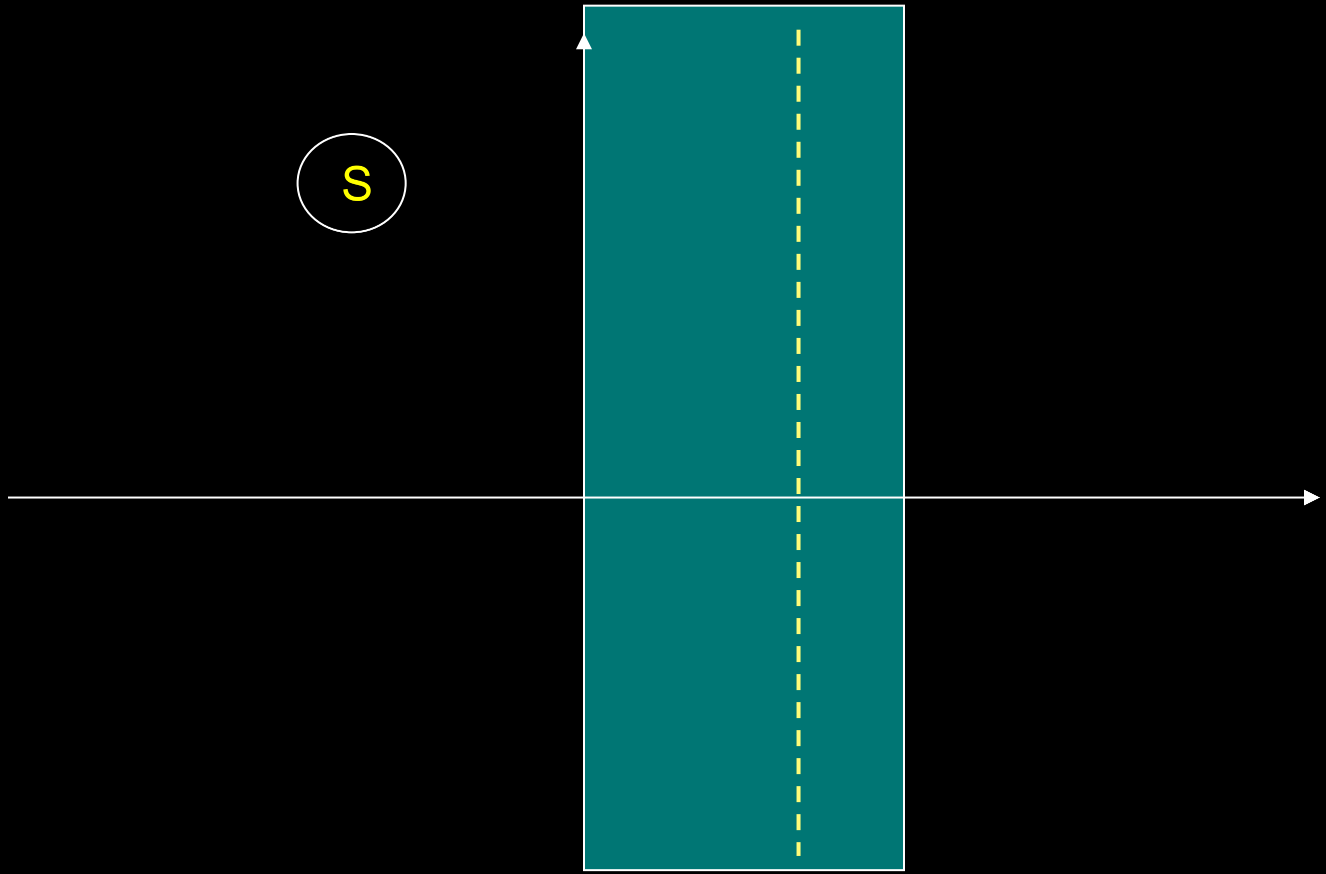
Summary

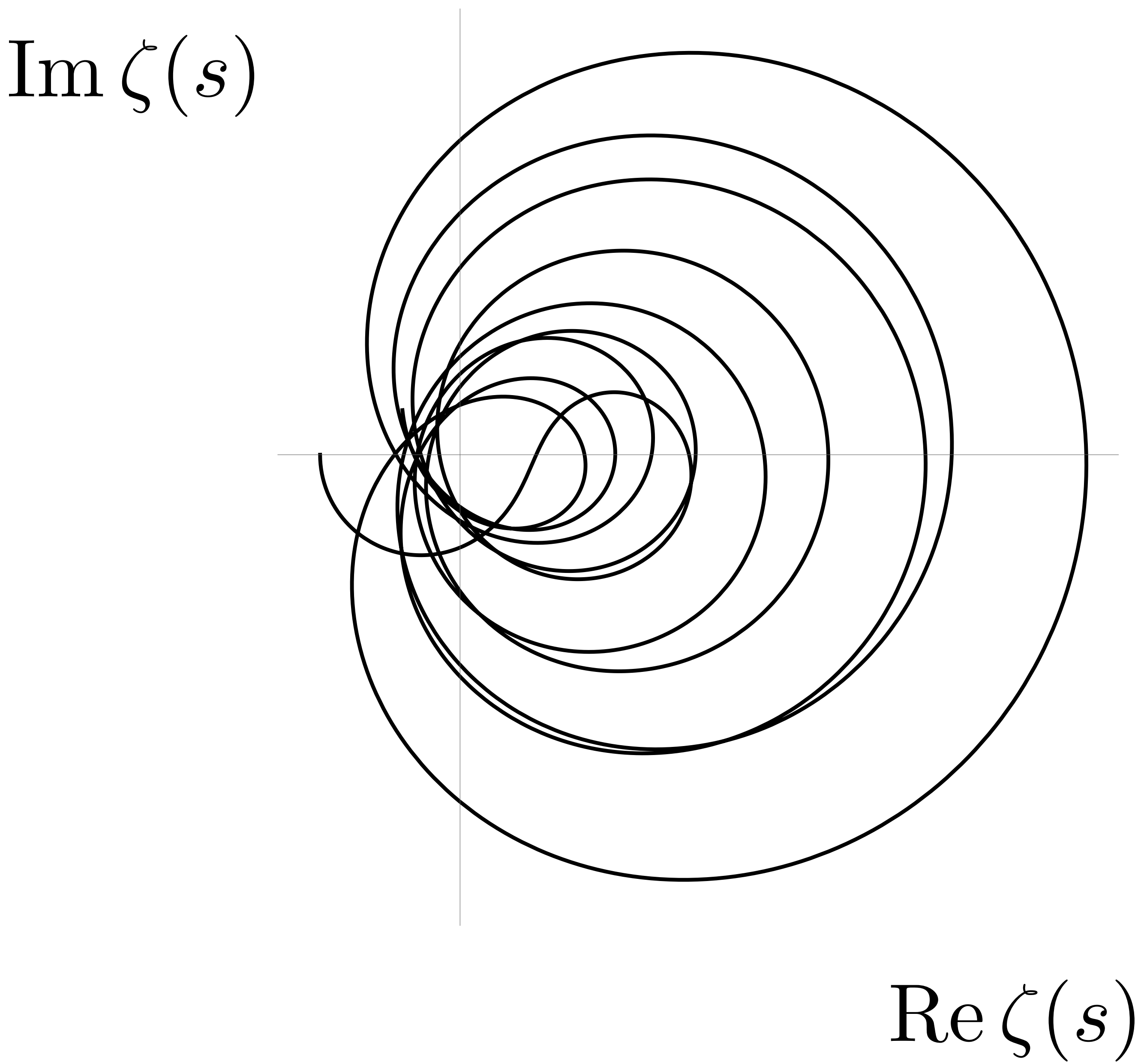
- The Riemann function has to do with the counting of the primes

- If the Riemann hypothesis is true, we have a clear estimate of the error

$$|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\log x$$

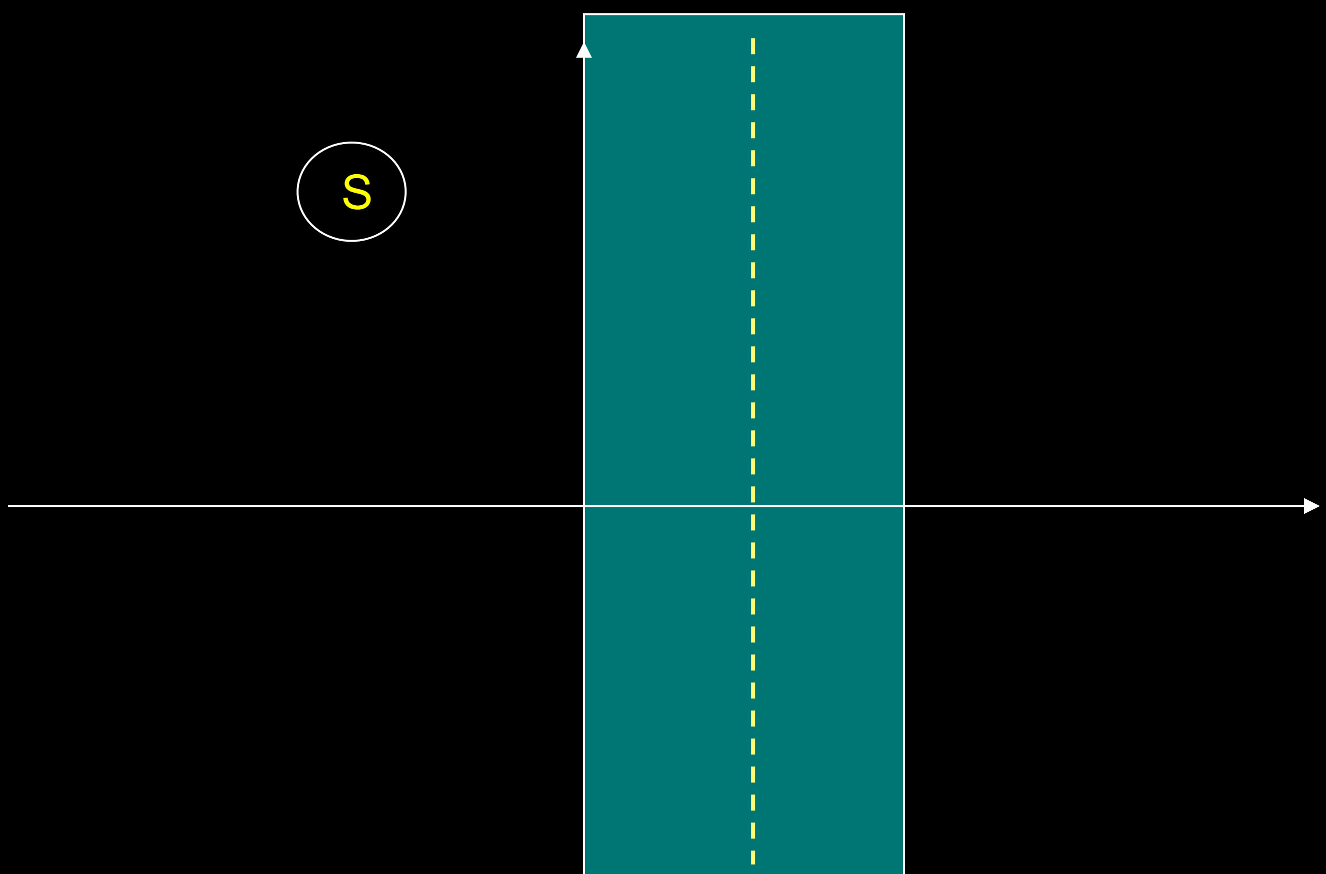
Strolling in the critical strip

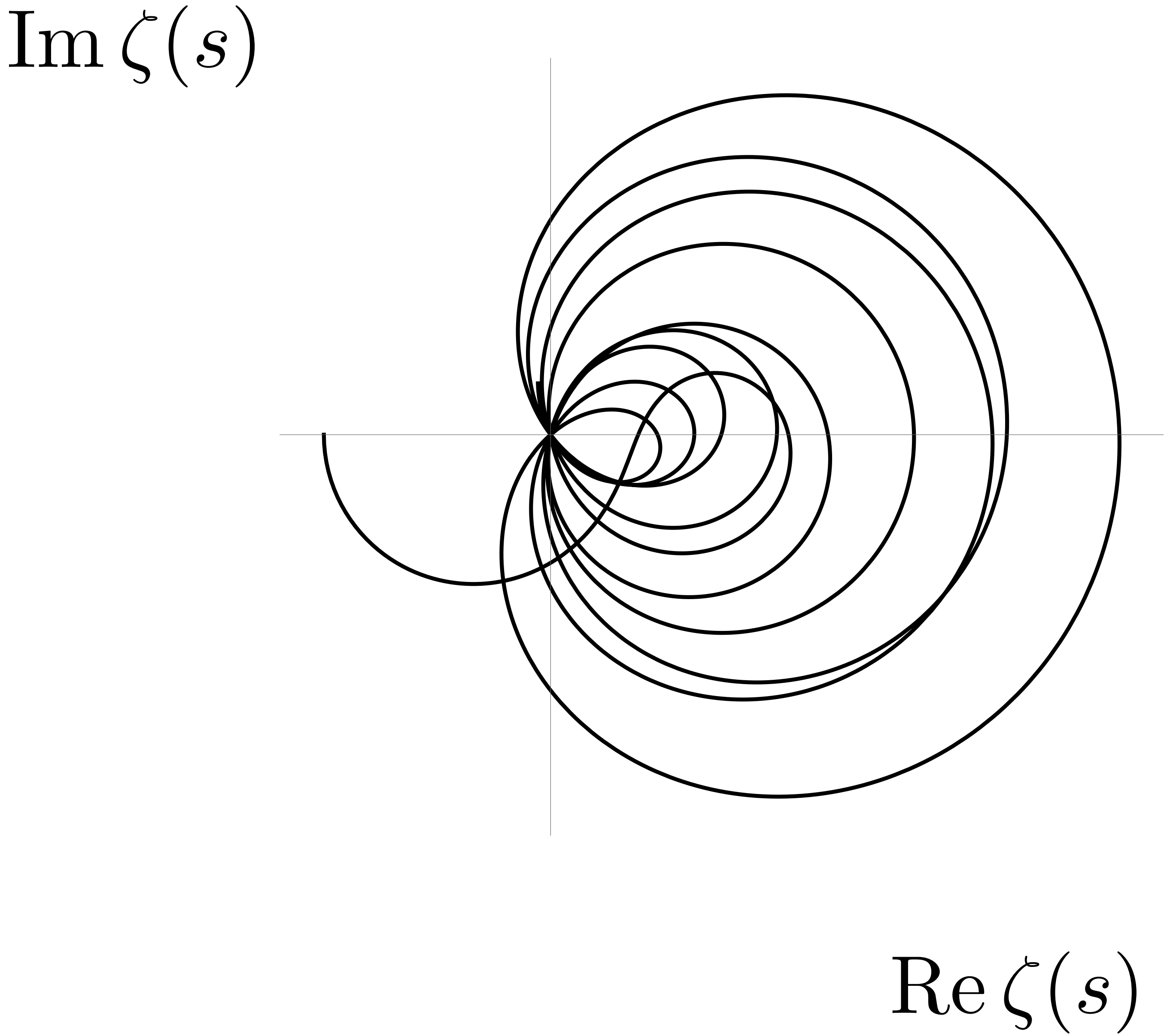




$\mathrm{Im}\,\zeta(s)$
$\mathrm{Re}\,\zeta(s)$

Along the critical line

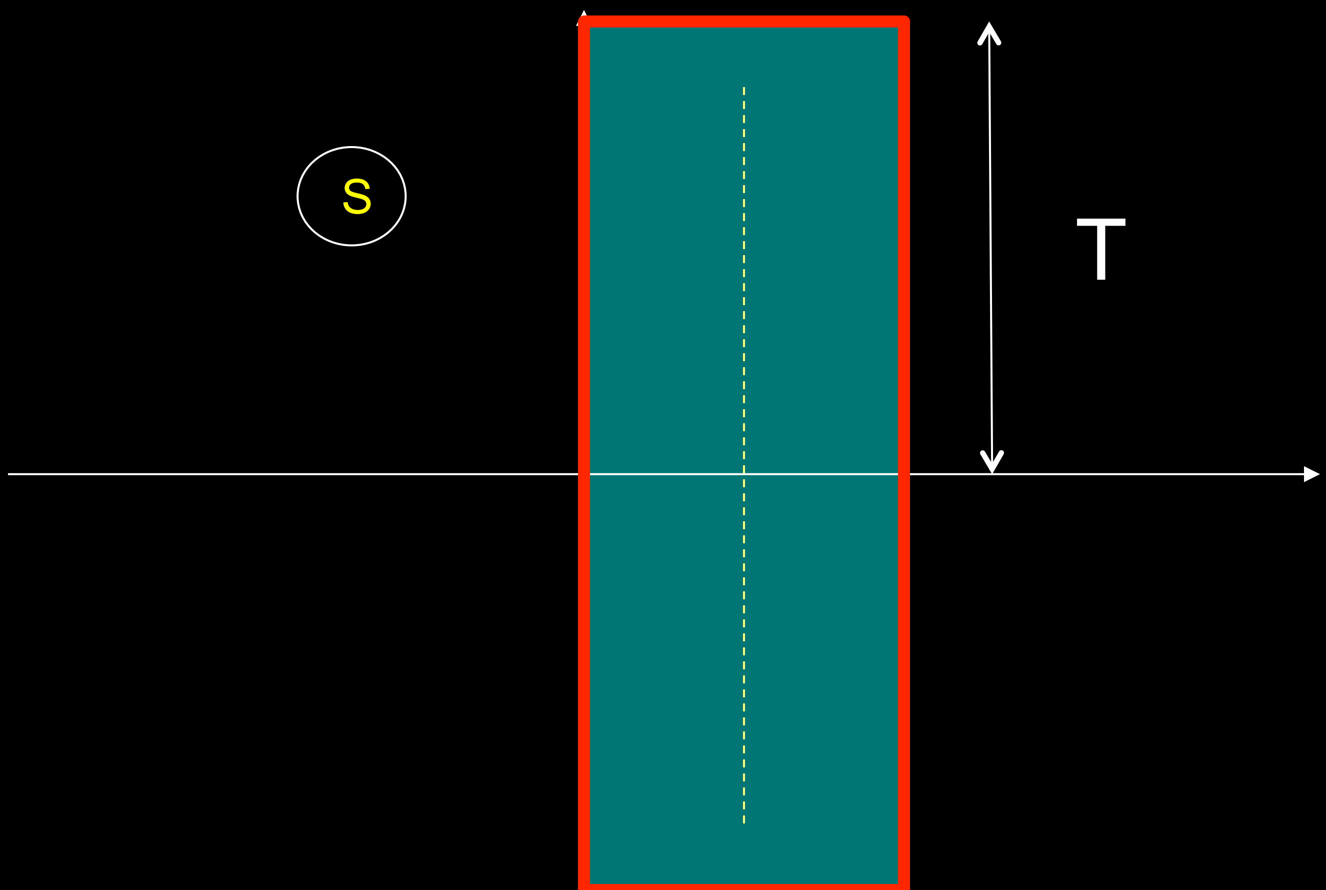



$\mathrm{Im}\,\zeta(s)$
$\mathrm{Re}\,\zeta(s)$

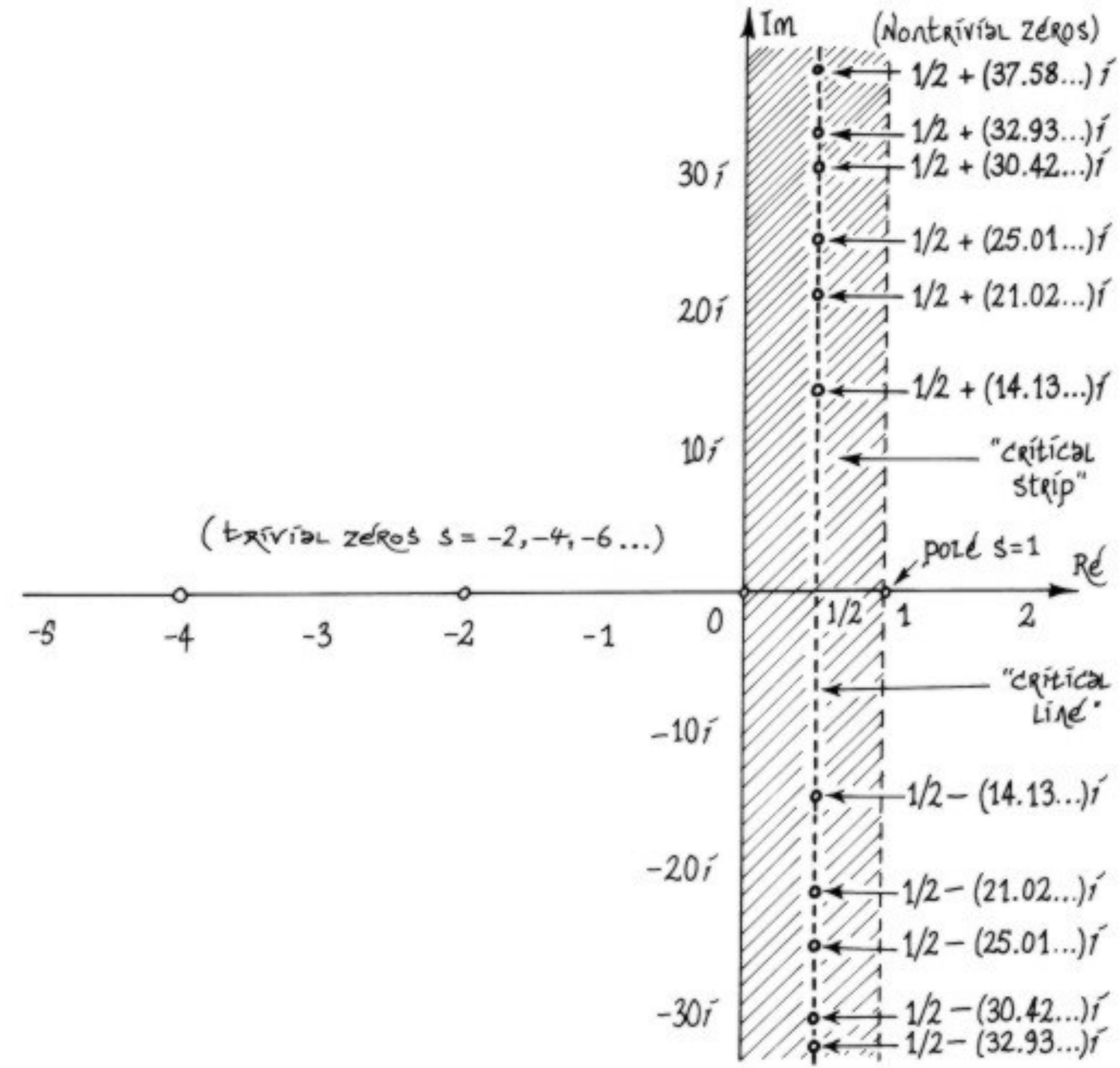
Number of zeros in the critical strip

(Riemann, von Mangoldt)

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + \mathcal{O}\left(\frac{1}{T}\right)$$



Year	Number of zeros	Computed by
1859 (approx.)	1	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1,041	E. C. Titchmarsh
1953	1,104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,500,000	J. B. Rosser, et al.
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, et al.
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, et al.
2001	10,000,000,000	J. van de Lune (unpublished)
2004	900,000,000,000	S. Wedeniwski
2004	10,000,000,000,000	X. Gourdon


Im
(Nontrivial zeros)
1/2 + (37.58...) i
1/2 + (32.93...) i
1/2 + (30.42...) i
30 i
1/2 + (25.01...) i
1/2 + (21.02...) i
20 i
1/2 + (14.13...) i
10 i
"critical strip"
(trivial zeros s = −2, −4, −6 ...)
pole s=1
Re
-5
-4
-3
-2
-1
0
1/2
1
2
"critical line"
−10 i
1/2 − (14.13...) i
−20 i
1/2 − (21.02...) i
1/2 − (25.01...) i
−30 i
1/2 − (30.42...) i
1/2 − (32.93...) i

Riemann’s Zeros and Random Matrices

$$\begin{pmatrix} 6 & 2 & 3 & 4 & 7 & 8 & 1 & 2 \\ 5 & 4 & 9 & 1 & 4 & 2 & 6 & 5 \\ 4 & 5 & 6 & 3 & 7 & 5 & 0 & 1 \\ 1 & 0 & 2 & 7 & 4 & 6 & 3 & 9 \\ 5 & 1 & 9 & 8 & 7 & 0 & 0 & 3 \end{pmatrix}$$

Riemann's Zeros and Random Matrices

Riemann’s Zeros and Random Matrices

Montgomery-Odlyzko Law

the nontrivial zeros of the Riemann zeta function and the eigenvalues of GUE matrices share a particular 'statistical fingerprint'.

$$\rho_k = \frac{1}{2} + it_k \qquad\qquad s_k = t_k - t_{k-1}$$

$$s_k \to \frac{s_k}{< s >}$$

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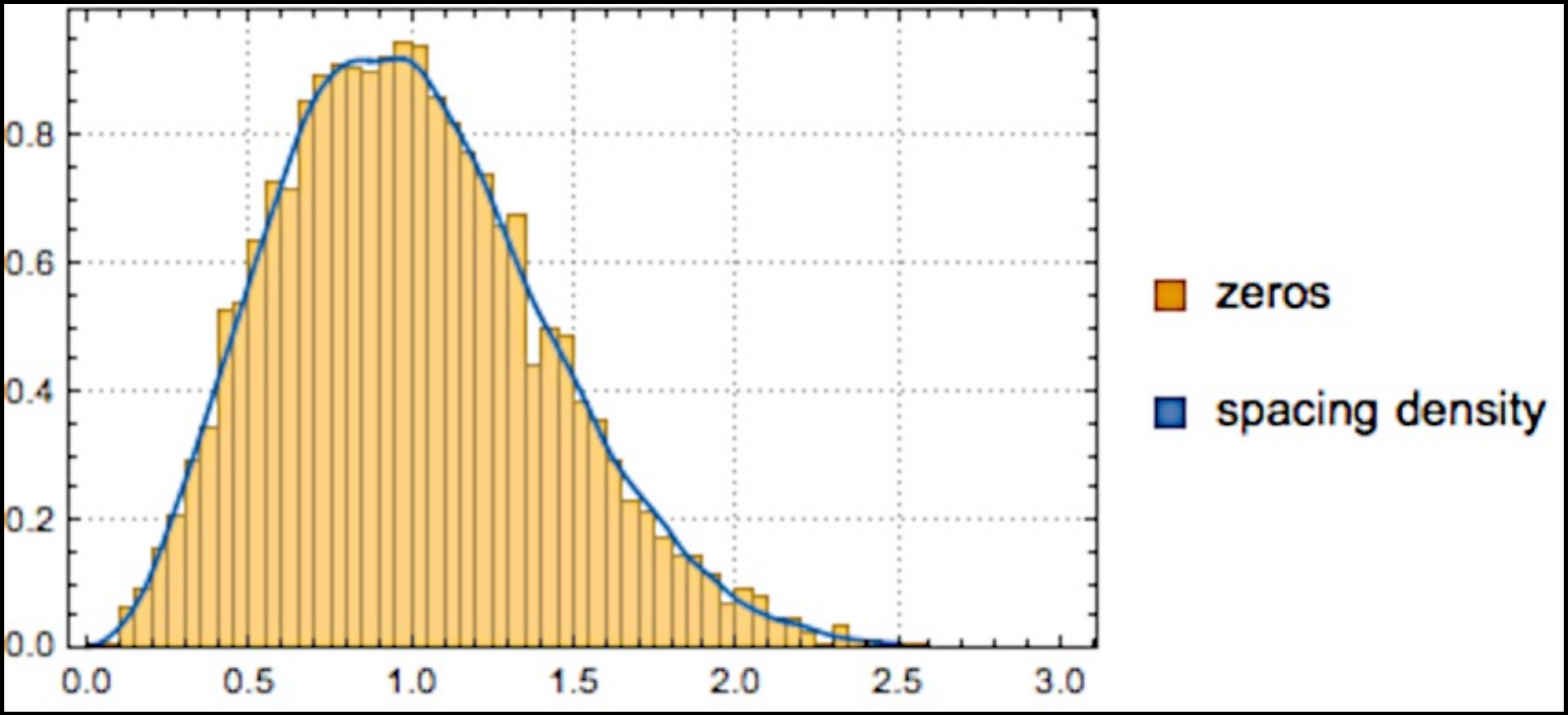

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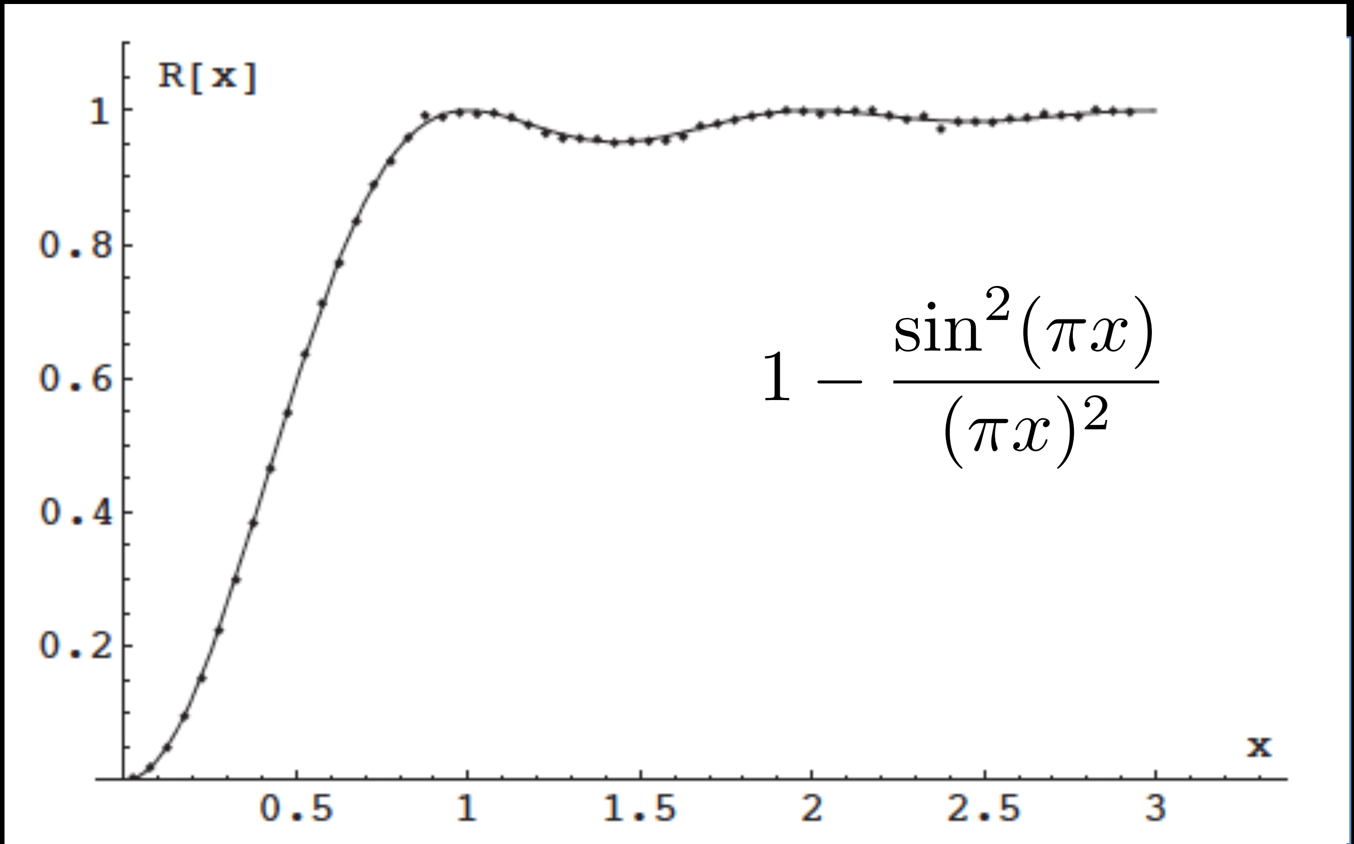


<u>Voronin Universality Theorem</u>

Voronin (1975) proved the remarkable analytical property of the Riemann zeta function that, roughly speaking,

<u>**ANY**</u> non-vanishing analytic function

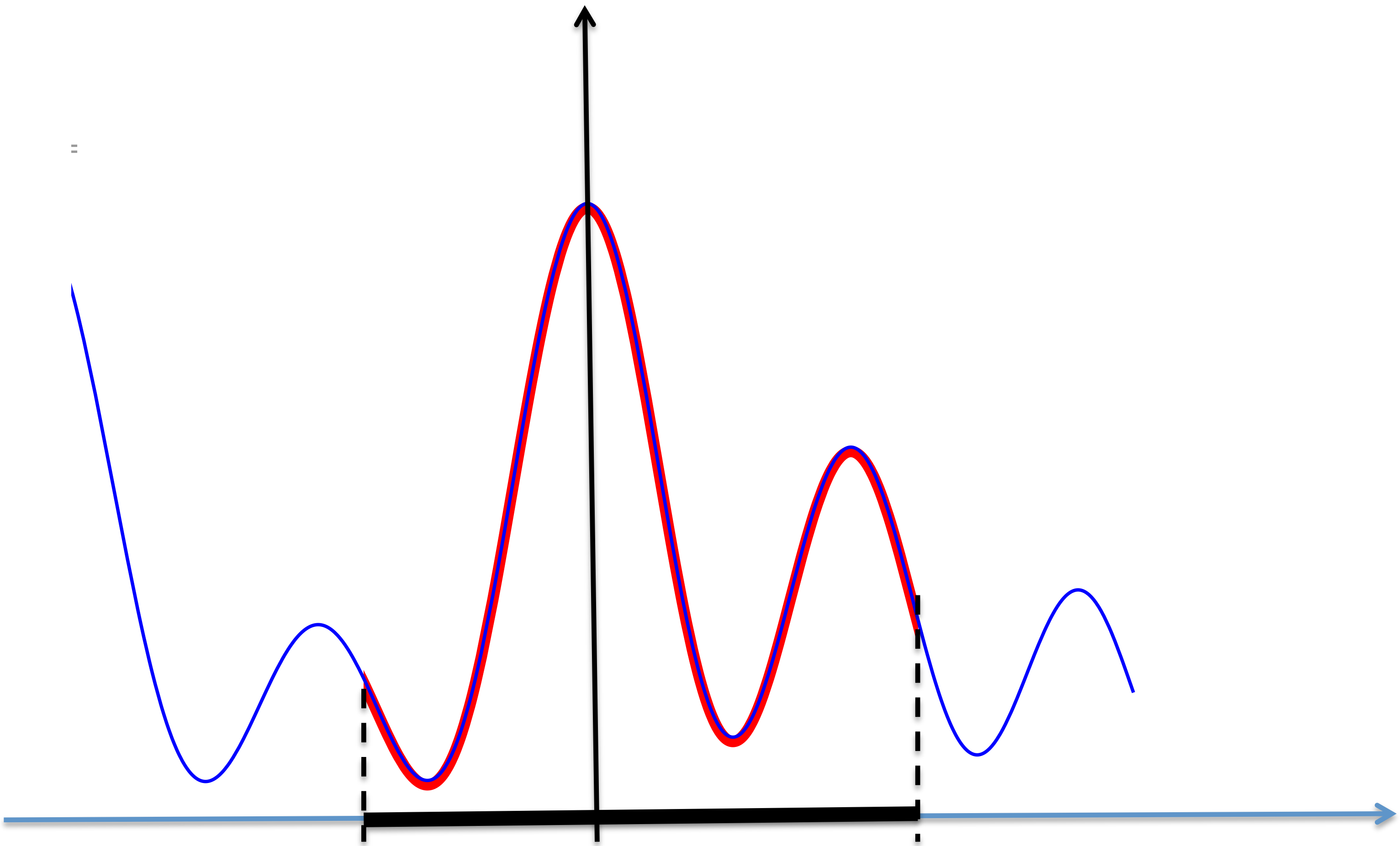
can be approximated uniformly by certain purely imaginary shifts of the zeta function in the critical strip.

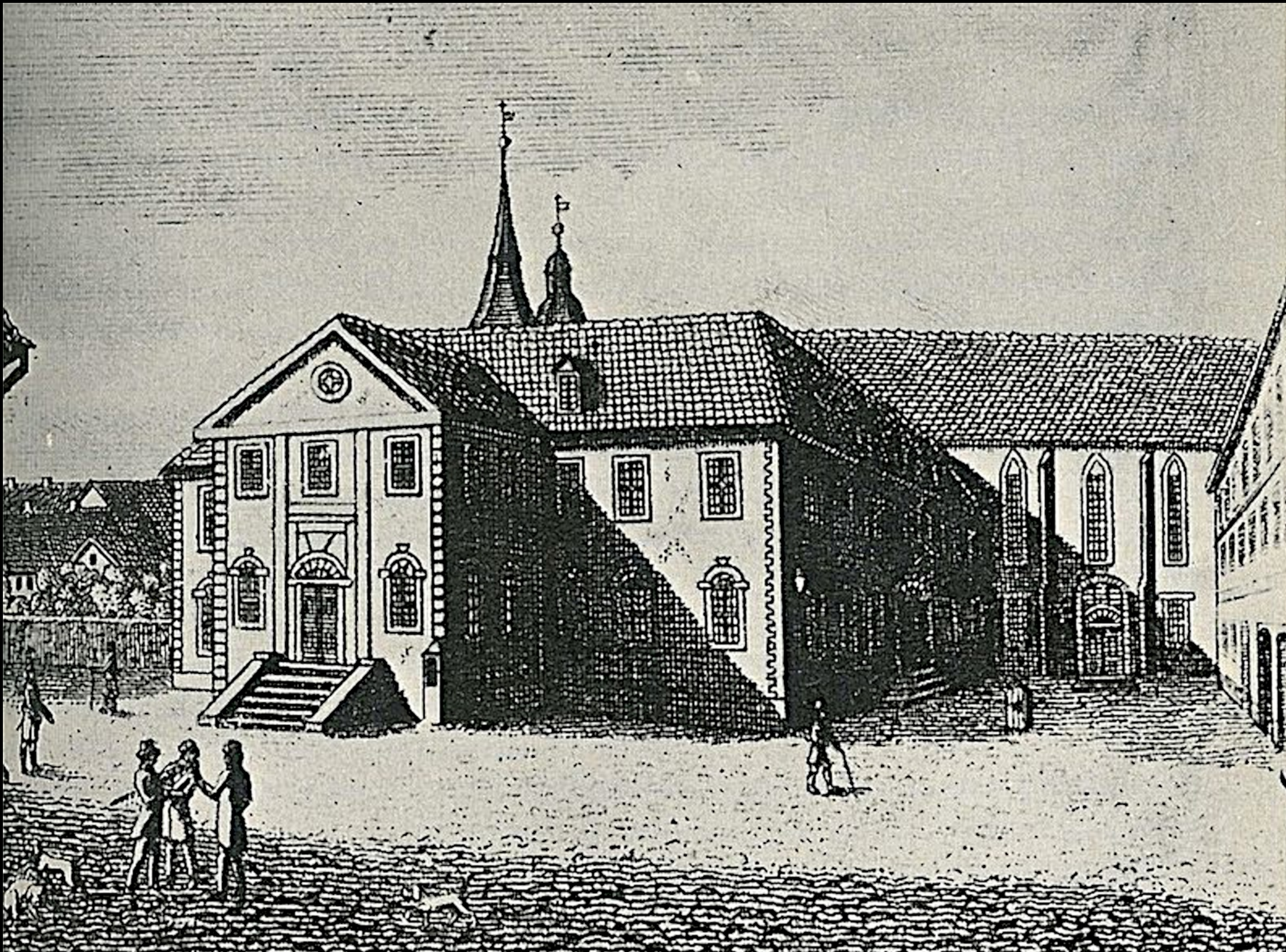
Voronin Universality Theorem

g(x) a non-vanishing continuous function in the disk | x| < r, where 0 < r < 1/4

$$\underset{|x|<r}{Max}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-g(s)\right|<\epsilon$$

Voronin Universality Theorem





Arithmetic Progressions

$$S_n = q\,n + h$$

$$q, h \in \mathbb{N}$$

Dirichlet question: Under which conditions on q and h, the sequence contains infinite number of primes?

Dirichlet theorem: q and h must be coprime! Necessary but also sufficient condition

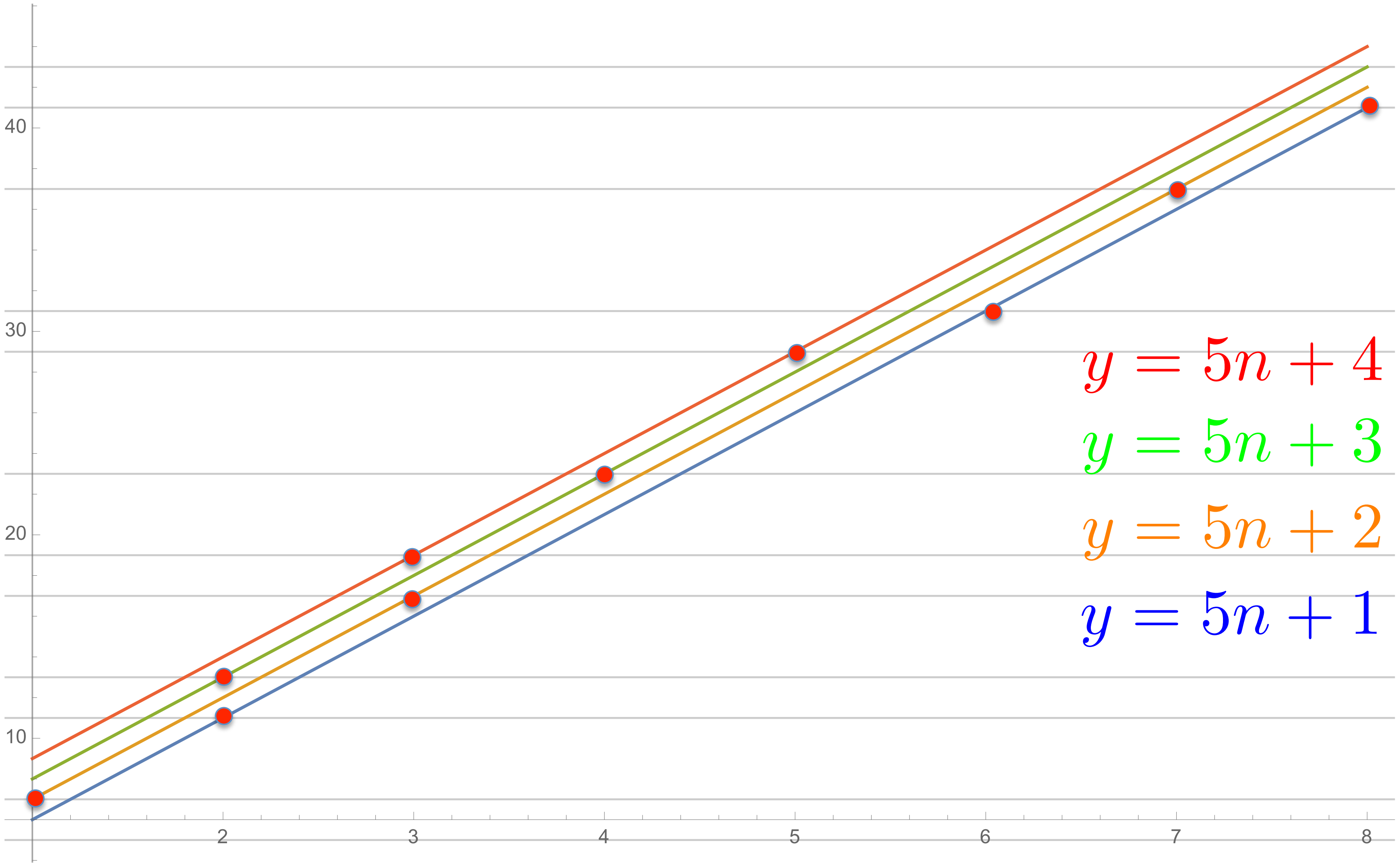
Examples

$$S_n = 5n + 3$$

S_n = {8, 13, 18, 23, 28, 33, 38, 43, 48, 53, 58, 63, 68, 73, 78, 83, 88, 93, 98, 103,…}

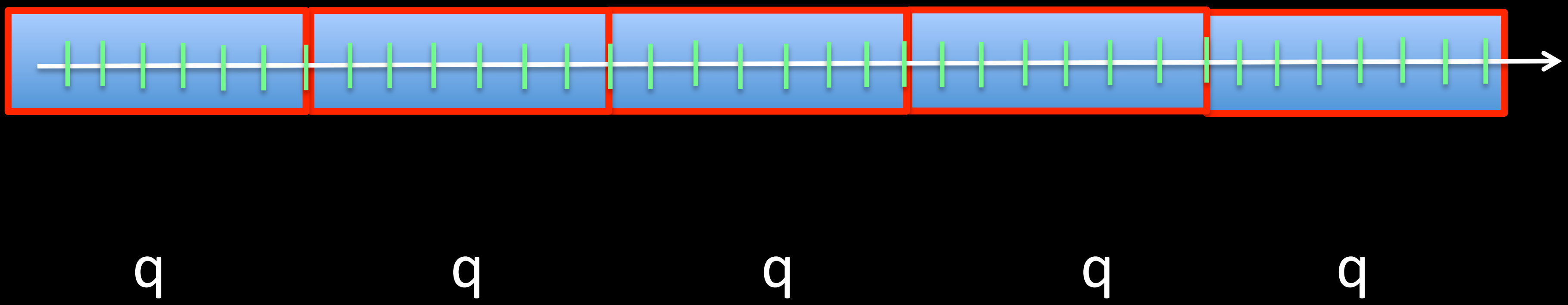
$$S_n = 5n + 2$$

S_n = {7, 12, 17, 22, 27, 32, 37, 42, 47,52, 57, 62, 67, 72, 77, 82, 87, 92, 97, 102, 107, 117, 122, 127, 132,…}



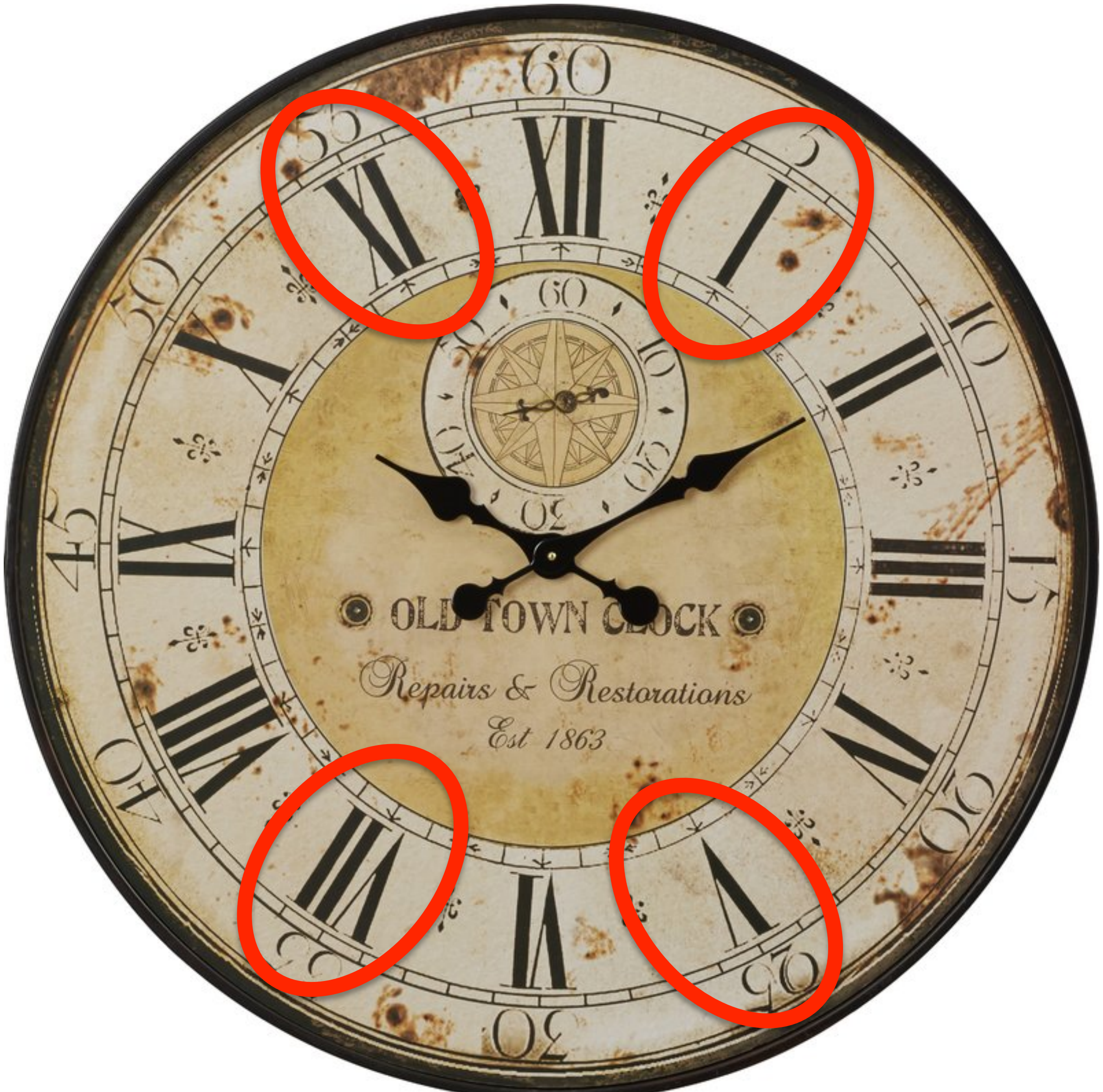
$y = 5n + 4$
$y = 5n + 3$
$y = 5n + 2$
$y = 5n + 1$
40
30
20
10
2
3
4
5
6
7
8

Lattices on integers



Modular Arithmetic

Modular Arithmetic



Group Multiplication Table

1	5	7	11
5	1	11	7
7	11	1	5
11	7	5	1

$$5 \times 5 = 25 = \cancel{24} + 1$$

$$5 \times 7 = 35 = \cancel{24} + 11$$

$$5 \times 11 = 55 = \cancel{48} + 7$$

$$2 \times 2 = 4$$

$$2 \times 8 = 16 = \cancel{12} + 4$$

The abelian group of prime residue classes modulo q

$(\mathbb{Z}/q\mathbb{Z})$: = { a mod q : gcd(a,q) = 1 }

$$\#\,\text{elements} = \varphi(q) = q \prod_{p \mid q} \left(1 - \frac{1}{p}\right)$$

Examples

q=7 $\quad (\mathbb{Z}/7\mathbb{Z}) = \{1, 2, 3, 4, 5, 6\}$ $\quad$ (q-1)

q=20 $\quad (\mathbb{Z}/20\mathbb{Z}) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

Characters

- These are 1-d representation of the residue class group
- There are as many characters as many elements, i.e. φ(q)

$$\chi_a(n) = e^{i\theta_a(n)} \qquad a = 1, 2, \ldots \varphi(q)$$

- These arithmetic functions satisfy a series of properties

Characters

1. $\chi(n+q) = \chi(n).$

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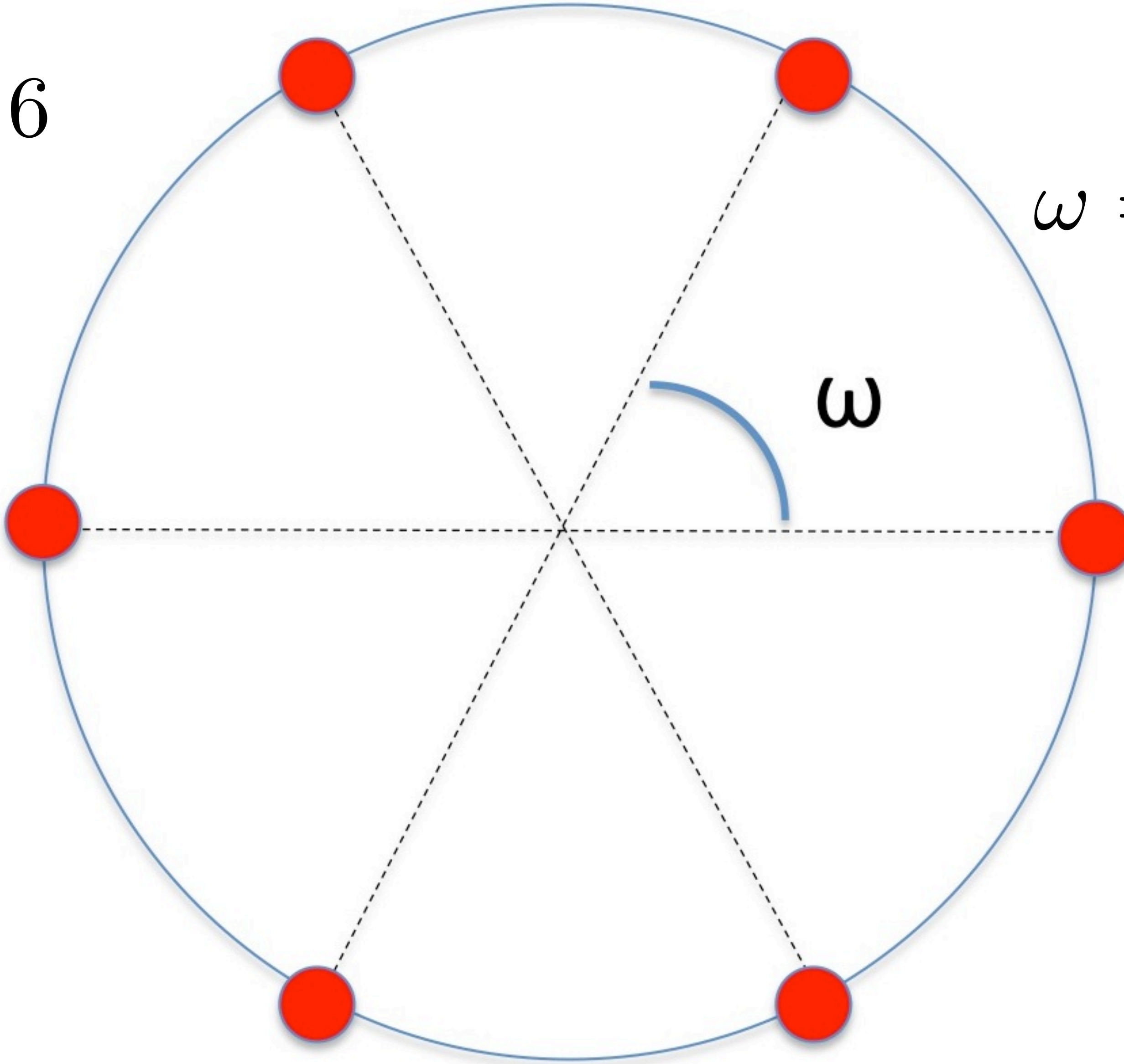
5. If $(n,q) = 1$ then $(\chi(n))^{\varphi(q)} = 1$,
   namely $\chi(n)$ have to be $\varphi(q)$-roots of unity.
6. If χ is a Dirichlet character so
   is the complex conjugate $\overline{\chi}$.

q=7

$$\varphi(7) = 6$$



$$\omega = e^{i2\pi/6}$$
$$= e^{i\pi/3}$$

Principal character

n	1	2	3	4	5	6	7
$\chi_1(n)$	1	1	1	1	1	1	0
$\chi_2(n)$	1	ω^2	ω	$-\omega$	$-\omega^2$	-1	0
$\chi_3(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1	0
$\chi_4(n)$	1	1	-1	1	-1	-1	0
$\chi_5(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1	0
$\chi_6(n)$	1	$-\omega$	$-\omega^2$	ω^2	ω	-1	0

Dirichlet L-functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

Functional equation

$$L(1-s,\chi) = i^{-a} \frac{q^{s-1}\,\Gamma(s)}{(2\pi)^s} \cos(\pi s/2)\, L(s,\overline{\chi})$$

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

Residue at the pole s = 1

$$\operatorname{Res} L(s,\chi) = \frac{1}{q} \sum_{r=0}^{q} \chi(r) = \begin{cases} \frac{\varphi(q)}{q} & \text{if } \chi = \chi_1 \\ 0 & \text{if } \chi \neq \chi_1 \end{cases} .$$


Urbi et Orbi

Generalised Riemann Hypothesis

(Piltz 1884)

The non-trivial zeros of ALL Dirichlet functions

(for any modulus q and any character)

are ALL along the critical line placed at

$$\mathbb{R}e\, s = \frac{1}{2}$$

For what the zeros are concerned,
we have to deal with

Two different cases

1. L-functions of principal characters

2. L-functions of non-principal characters

1. L-functions of principal characters

All these functions are equivalent to the Riemann function !

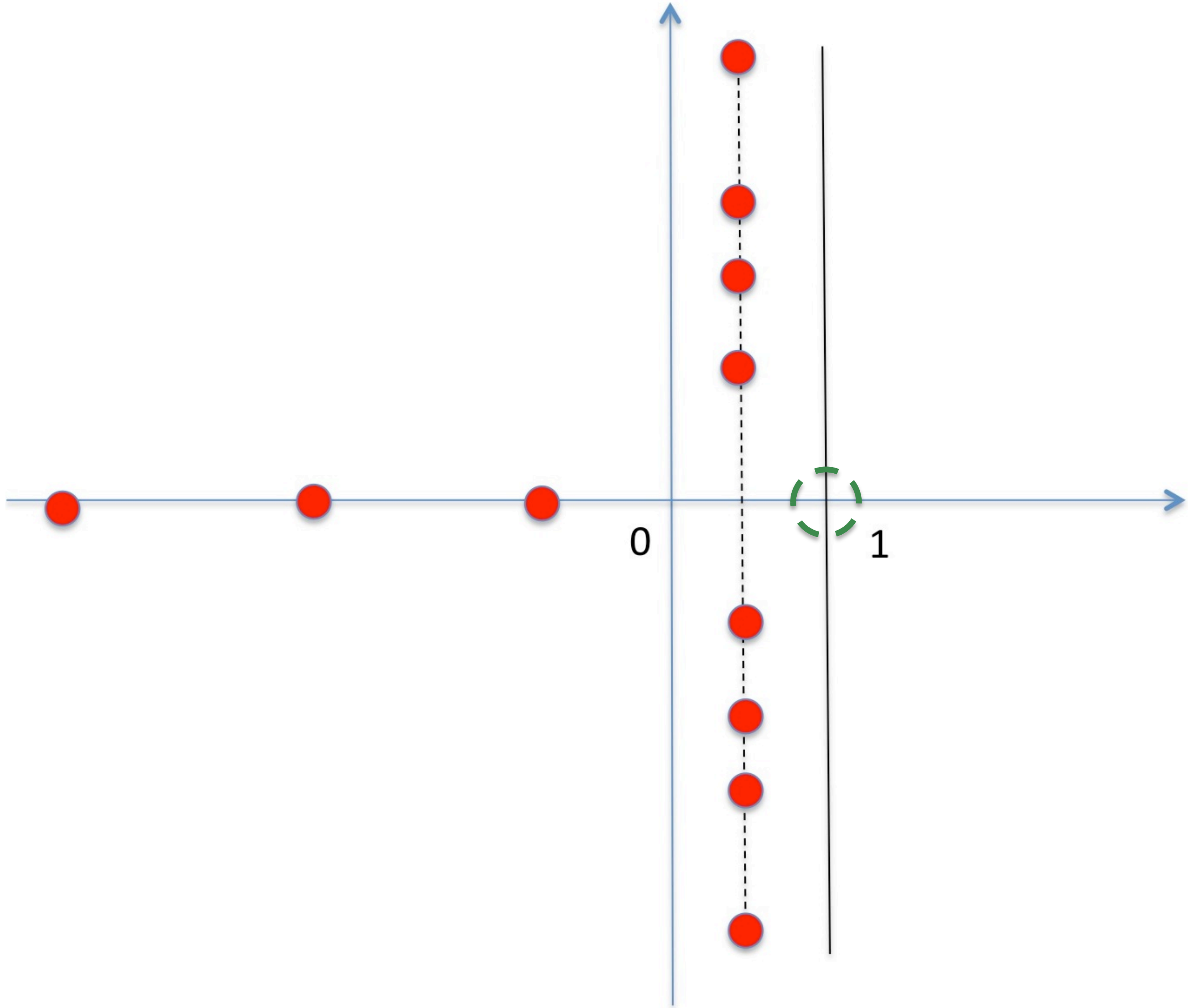
$$L(s,\chi_1) = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p | q} \left(1 - \frac{1}{p^s}\right)^{-1}$$

The same zeros

2. L-functions of non-principal characters

All these functions are essentially on the same footing,
i.e. they share the same general properties

L-function of non-principal characters are ENTIRE FUNCTIONS


0
1

This is an enormous simplification for approaching the proof of the

Generalised Riemann Hypothesis

Generalized Riemann Hypothesis and Stochastic Time Series

Giuseppe Mussardo

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André LeClair

Cornell University, Physics Department, Ithaca, NY 14850

We show that the domain of convergence of the infinite product of Dirichlet L-functions of non-principal characters can be extended from $\Re(s) > 1$ down to $\Re(s) > \frac{1}{2}$, without encountering any zeros before reaching this critical line. The possibility of doing so can be traced back to a universal diffusive random walk behavior $C_N = \mathcal{O}(N^{1/2})$ of a series C_N over the primes which underlies the convergence of the infinite product of the Dirichlet functions. The series C_N presents several aspects in common with stochastic time series and its control requires to address a problem similar to the Single Brownian Trajectory Problem in statistical mechanics. In the case of the Dirichlet functions of non principal characters, we show that this problem can be solved in terms of a self-averaging procedure based on an ensemble $\mathcal{E}$ of block variables computed on extended intervals of primes. Those intervals, called *inertial intervals*, ensure the ergodicity and stationarity of the time series underlying the quantity C_N. The ensemble $\mathcal{E}$ is severely constrained by the Dirichlet theorem on the equidistribution of reduced residue classes modulo q and the Lemke Oliver-Soundararajan conjecture on the distribution of pairs of residues on consecutive primes. The infinity of primes also ensures the absence of rare events which would have been responsible for a different scaling behavior than the universal law $C_N = \mathcal{O}(N^{1/2})$ of the random walks.

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I. INTRODUCTION

The Generalised Riemann Hypothesis (GRH) states that the non-trivial zeros of *all* the Dirichlet L-functions $L(s,\chi)$ of the complex variable $s = \sigma + it$ lie along the critical line $\Re(s) = \frac{1}{2}$. In this paper we focus our attention on L-functions associated to non-principal characters (see Appendix A for their definitions). As we are going to show below, the fact that the non-trivial zeros of these functions align themselves along such a critical line finds a natural and universal explanation in terms of a random walk, in the sense that $\sigma = \frac{1}{2}$ is the universal critical exponent of a random walk process $C_N = \mathcal{O}(N^{1/2})$ which exists for any Dirichlet L-function of a non principal character. Such a scaling behavior of the quantity C_N is a consequence of the Dirichlet theorem on the equi-distribution of reduced residue classes modulo q [1, 2] and the Lemke Oliver-Soundararajan conjecture on the distribution of pairs of residues on consecutive primes [3]. Moreover, there are very intriguing analogies between the deterministic series C_N coming from the Dirichlet L-functions and the stochastic time series. The close relation between these two entities will play a crucial role in the analysis that follows. From a statistical mechanics point of view, there is also a very close and interesting parallel of the series C_N with the Single Brownian Trajectory Problem (see, for instance [5–8] and references therein), problem which consists in extracting the statistical properties of a sequence of events which happen only once!

This paper is devoted to a short presentation of our findings, whereas in [4] we provide a thorough discussion of many other aspects of L-functions and a detailed derivation of the results discussed here. For general definitions and properties of the Dirichlet L functions, we refer the reader to standard references, for instance [9–13].

***L*-functions.** For $\Re(s) > 1$ there are two equivalent representations of these functions, one given in terms of an infinite series on the natural numbers m, the other in terms of an infinite product over the sequence of primes p_n (hereafter labelled in ascending order)

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1} . \tag{1}$$

This equation expresses the so-called *Euler identity*. The quantities $\chi(m)$ which enter the definition of $L(s,\chi)$ are the so-called Dirichlet characters. When non-zero, they are pure phases and therefore

…lised Riemann Hypothesis, Time Series and Normal Law Distribution

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…titute for Theoretical Physics, Utrecht University
… University, Physics Department, Ithaca, NY 14850

… series B_N involving Dirichlet characters of modulo q on primes and
…ts that the B_N are ruled by normal law probability distributions.
… these series B_N present a universal diffusive random walk behav-
…e Dirichlet theorem on the equidistribution of reduced residue classes
…r-Soundararajan conjecture on the distribution of pairs of residues
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I. INTRODUCTION

…the normal law probability distribution – the same law which
…d plays a dominant role in all fields of statistical mechanics –
…y of the so-called Generalised Riemann Hypothesis (GRH).
…f the problem, we present the central idea of our approach,
…ious gaps required to develop this idea.

…ial zeros of the Dirichlet L-functions $L(s,\chi)$ of the complex
… two equivalent representations, one given in terms of an
…her in terms of an infinite product over the sequence of

$$= \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1} . \tag{1}$$

The quantities $\chi(m)$ which enter the definition of
…-zero, they are pure phases and therefore expressed

$$\chi(m) \neq 0 . \tag{2}$$

…etic functions, a property which is at the origin
…composition of any integer in terms of primes.
…ion [1–3], where all $\chi(m) = 1$. For a thorough
…m some standard books, reviews or lecture

…the non-trivial zeros of *all* the L- functions
…k has been devoted to this problem and
…buted to the development of the subject.
…ns which are important for our purposes,
…ormula $N(T,\chi)$ for the number of zeros
… result later refined by Fujii [10], who
… the ordinate between $[T+H,T]$. The
…al line was analyzed by Iwaniec, Luo
…g the original Montgomery-Odlyzko
… relation to random matrix theory
…ctions were studied by Conrey and, in a
…over, it was also estimated by Conrey, Iwaniec,
… non-trivial zeros are on the critical line [18].

A very surprising result

(Grosswald and Schnitzer)

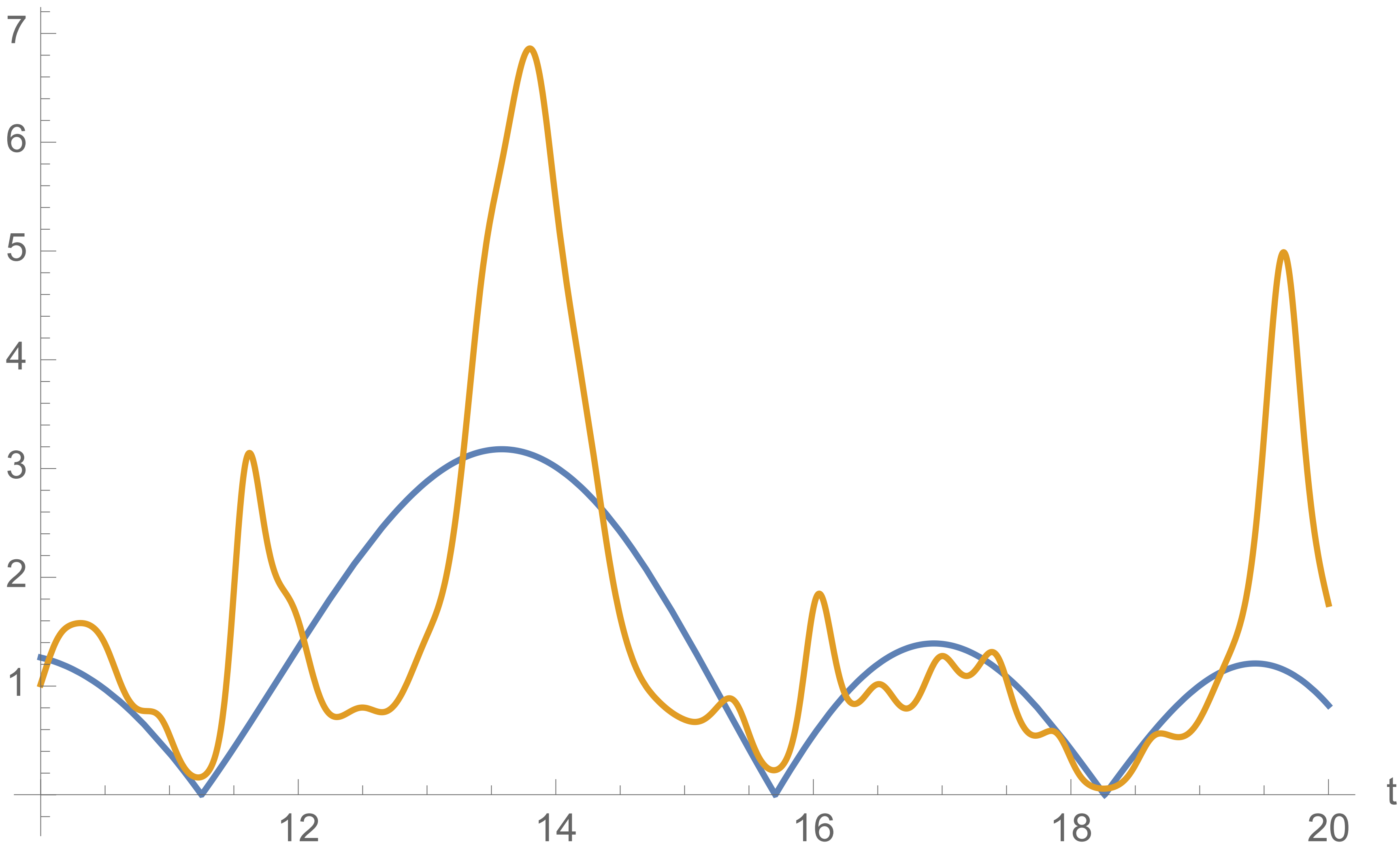
$$\mathcal{P} = \{p'_1, p'_2, \ldots\}, \text{ randomly chosen integers } p'_n$$

$$p_n \le p'_n < p_n + K, \qquad p'_n = p_n \pmod{q}$$

where $K \ge q$ is an arbitrary integer

$$L'(s,\chi) = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p'_n)}{(p'_n)^s}\right)^{-1}$$

L'(s) has exactly the same zeros as L(s) in the critical strip !!



7
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1
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14
16
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20
t

$$L'(s,\chi) = \prod_{n=1}^{\infty}\left(1 - \frac{\chi(p'_n)}{(p'_n)^s}\right)^{-1}$$

$$L(s,\chi) = \prod_{n=1}^{\infty}\left(1 - \frac{\chi(p_n)}{(p_n)^s}\right)^{-1}$$

$$\boxed{L(s,\chi) = R(s)\, L'(s,\chi)}$$

$$R(s) = \prod_{n=1}^{\infty}\left(\frac{1 - \frac{\chi(p_n)}{(p_n)^s}}{1 - \frac{\chi(p'_n)}{(p'_n)^s}}\right)^{-1} \neq 0$$

Our Main Idea

- We will look at enlarging the domain of convergence of the infinite product P(s) of the L-functions for non-principal characters.

$$P(s,\chi) = \prod_{n=1}^{\infty}\left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

- To do this, we take the log(P(s)). The singularities of this quantity depend ONLY on the zeros of the L-functions

$$s = \sigma + it$$

Our Main Idea

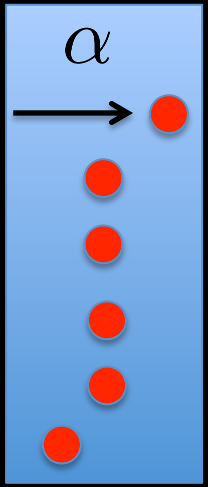
$$\log P(s) = \sum_{n=1}^{\infty} \frac{\cos(t\log p_n - \theta(p_n))}{p_n^{\sigma}} = \sigma \int_2^{\infty} \frac{B(u)}{u^{\sigma+1}}\,du$$

$$B(N) = \sum_{n=1}^{N} \cos(t\log p_n - \theta(p_n))$$

$$B(N) \leq N^{\alpha}$$

Then the integral above diverges at σ = α.

So α identifies the real part of the first zero in the critical strip



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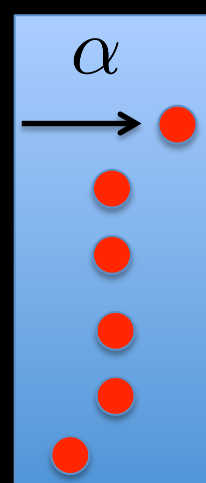
$$B(N) = \sum_{n=1}^{N} \cos(\omega_n t - \theta(p_n))$$

$$B(N) = N^{\alpha}$$

Then the integral above diverges at σ = α.

$$\omega_n = \log p_n$$

So α identifies the real part of the first zero in the critical strip



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$$B(N) \simeq N^{\alpha}$$

If α=1/2, i.e.

$$B \simeq \sqrt{N}$$

then all zeros are on the critical line!

Corrections by logarithms do not spoil the conclusion

$$\boxed{B(N) \simeq \sqrt{N}\,\log^a N}$$

$$\log P(s) = \int_2^\infty \frac{B(u)}{u^{\sigma+1}} = \int_2^\infty \frac{\sqrt{u}\,\log^a u}{u^{\sigma+1}}$$

$$= \frac{\Gamma(a+1)}{(\sigma - 1/2)^{a+1}}$$

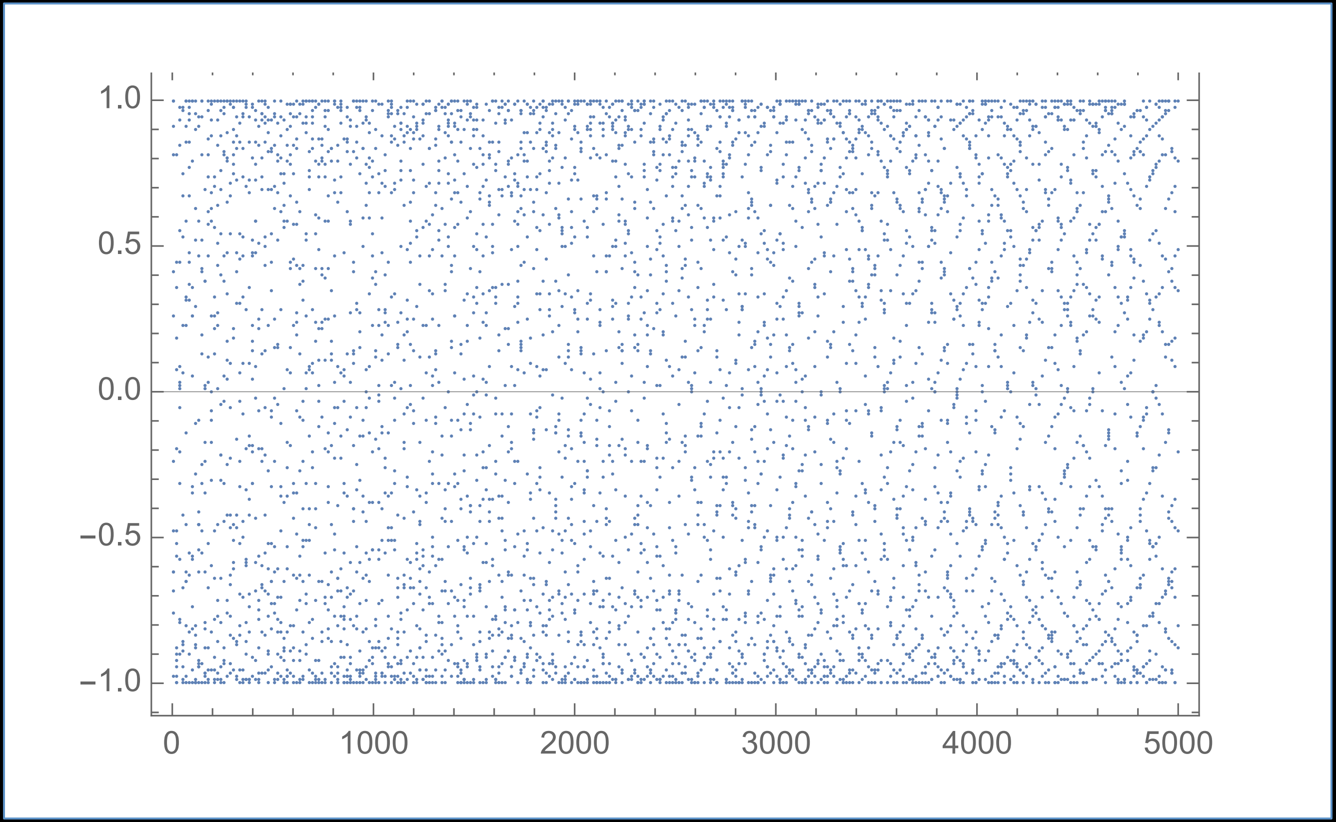
Properties of the series B(N)

$$B(N) = \sum_{n=1}^{N} b_n(t)$$

$$b_n(t) = \cos(t \log p_n - \theta(p_n))$$

$b_n(t)$

$t = 10$



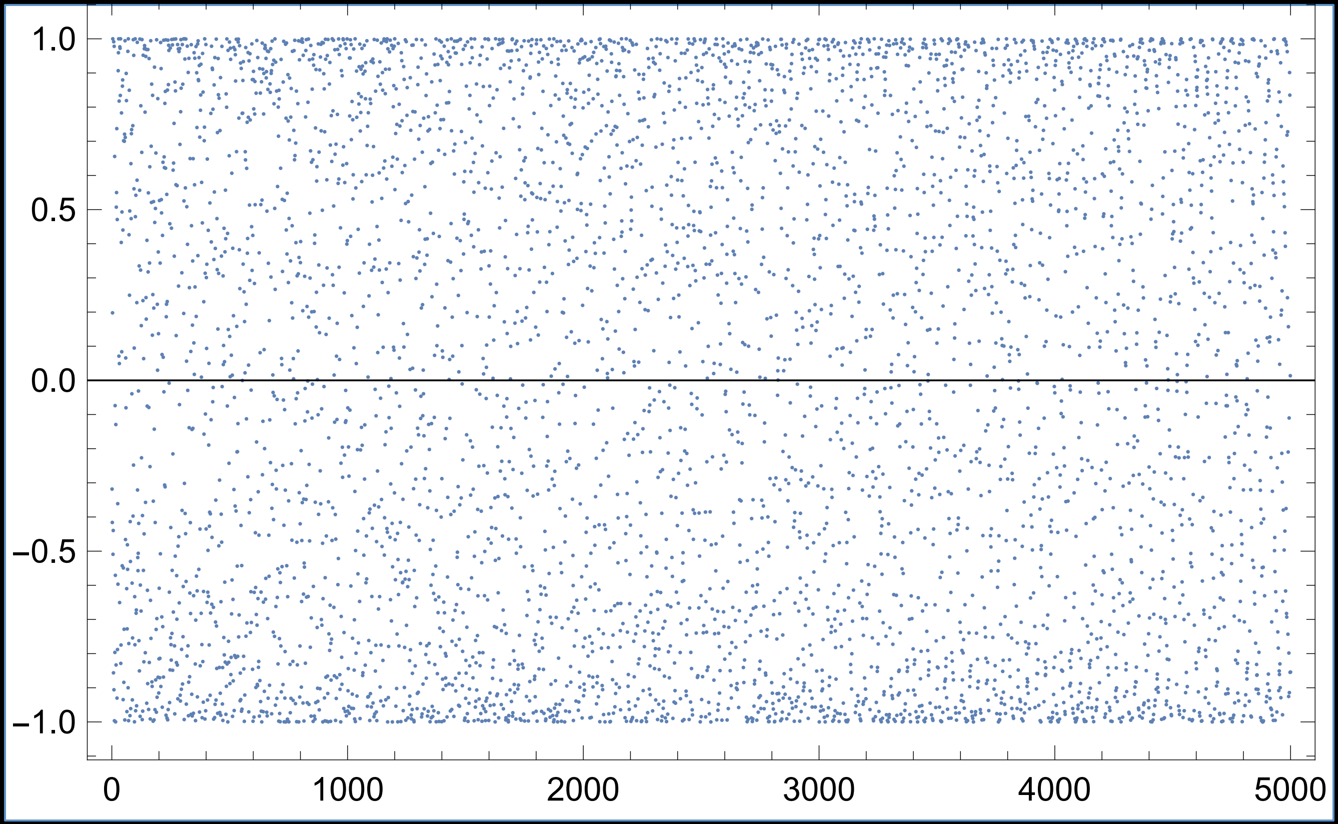
Properties of the series B(N)

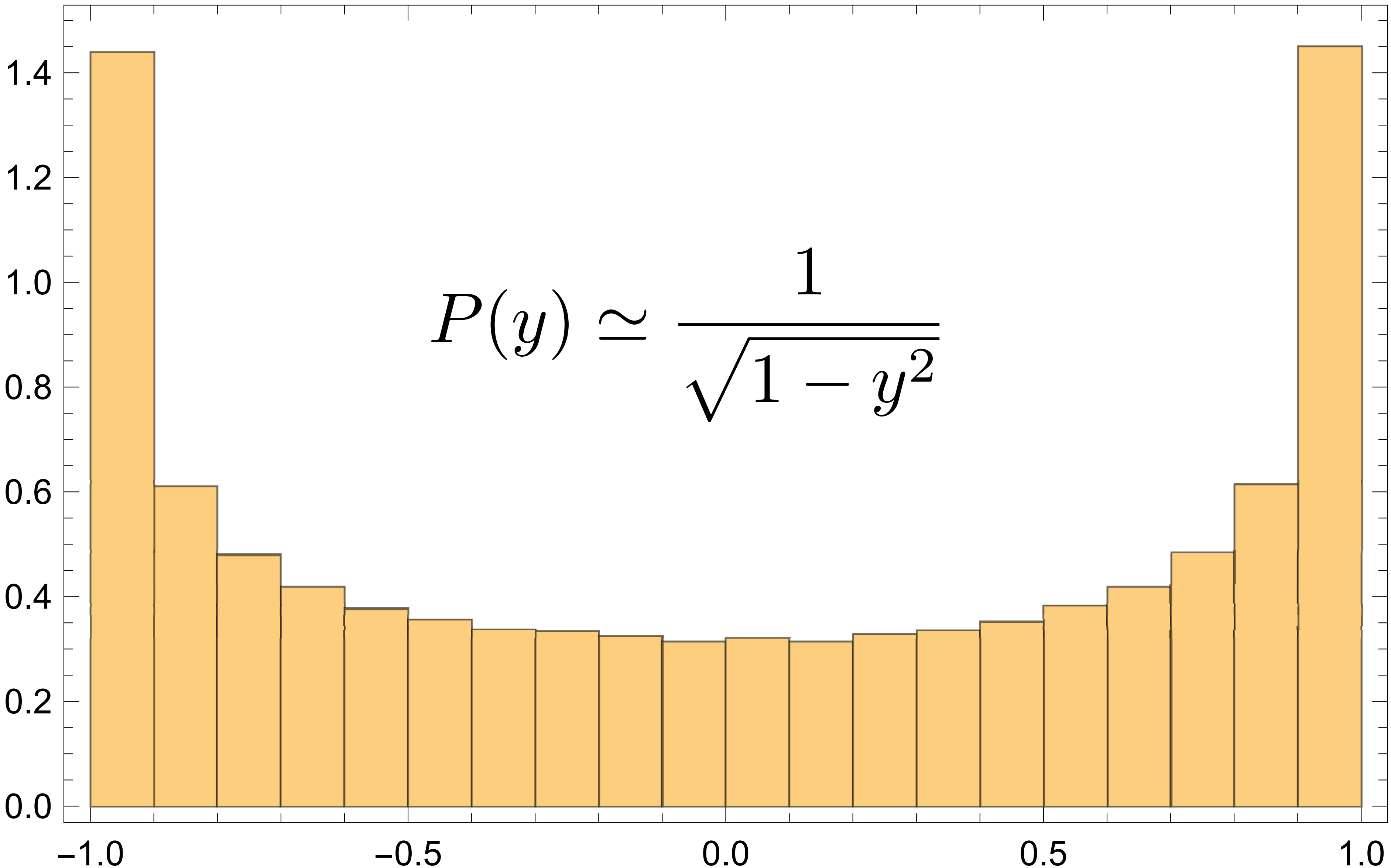
$$B(N) = \sum_{n=1}^{N} b_n(t)$$

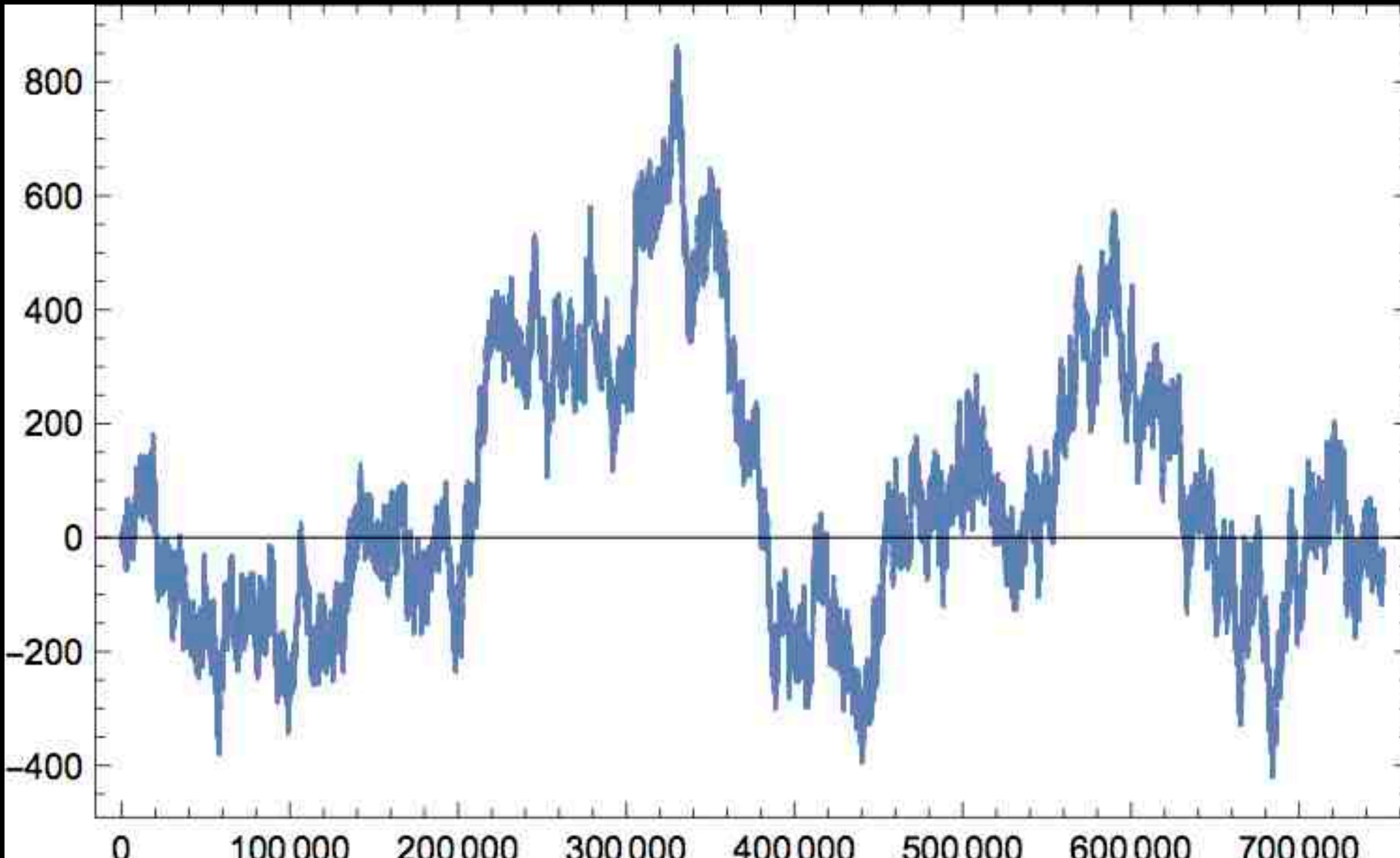
$$b_n(t) = \cos(t \log p_n - \theta(p_n))$$

$b_n(t)$

$t = 10^3$



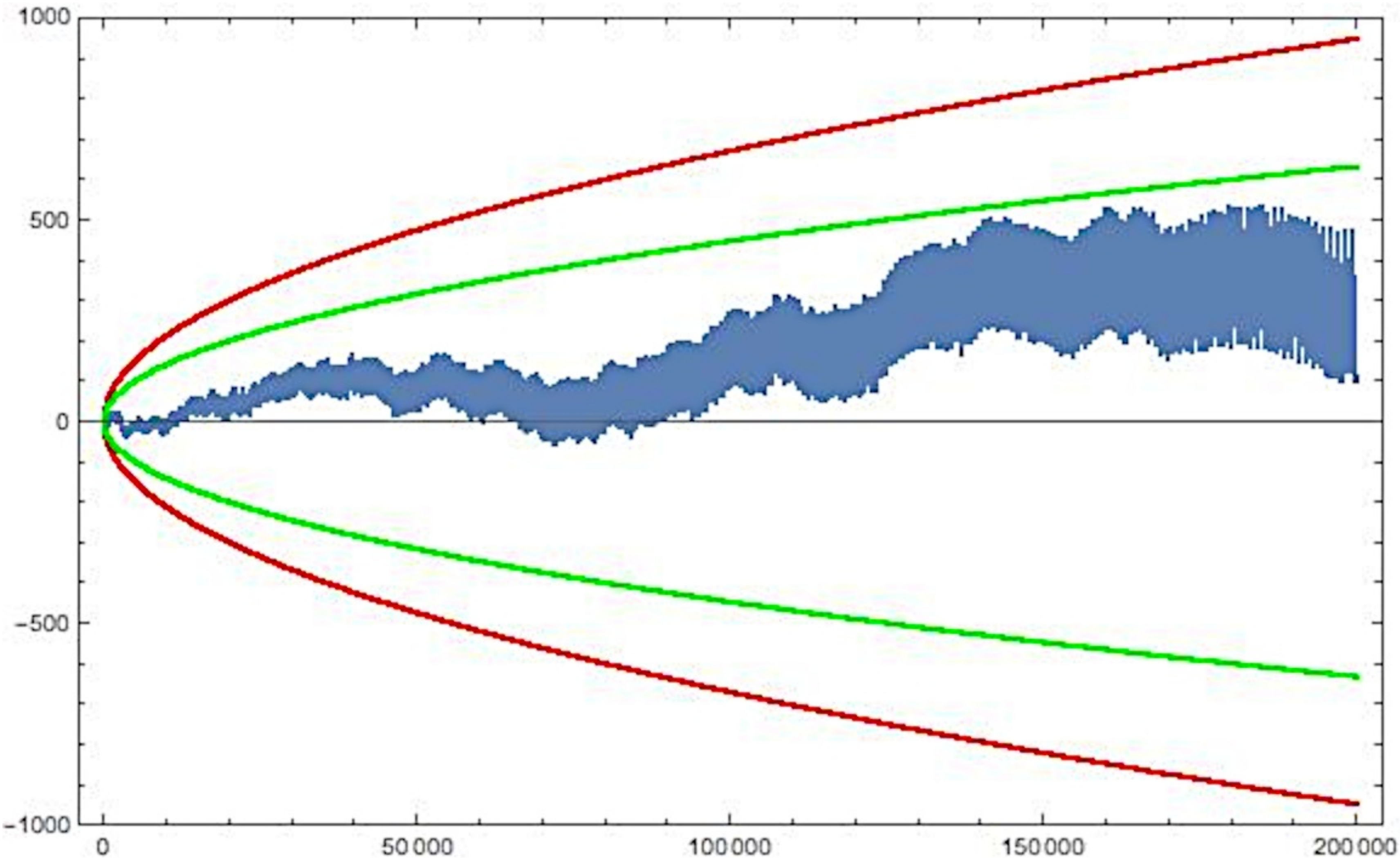

$P(y) \simeq \frac{1}{\sqrt{1-y^2}}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0


800
600
400
200
0
-200
-400
0
100 000
200 000
300 000
400 000
500 000
600 000
700 000

Kac's theorem

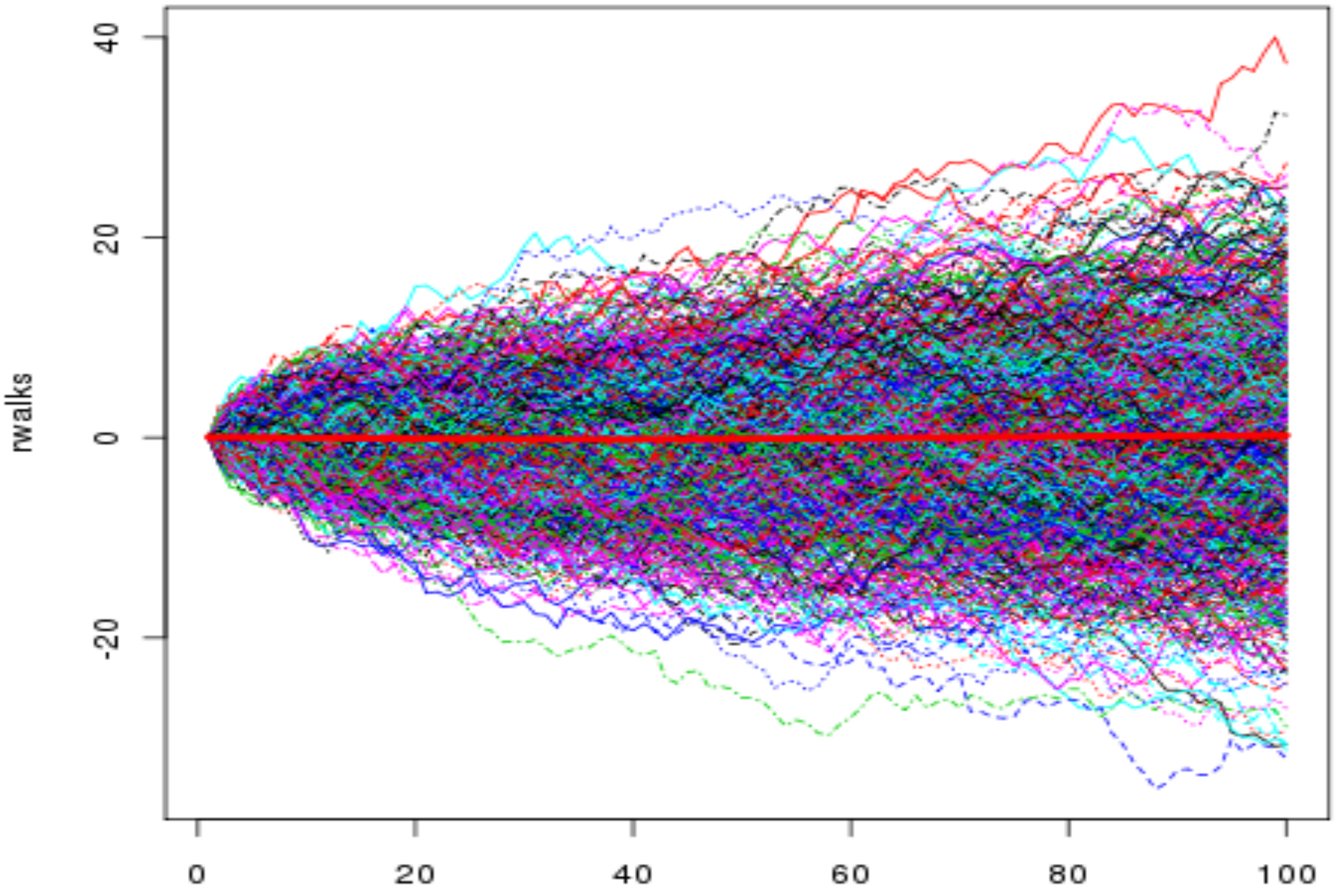
$$P\left[a < \sqrt{2}\,\frac{\cos\omega_1 t + \cos\omega_2 t + \cdots \cos\omega_N t}{\sqrt{N}} < b\right]$$

$$= \frac{1}{\sqrt{\pi}}\int_a^b e^{-x^2/2}\,dx$$

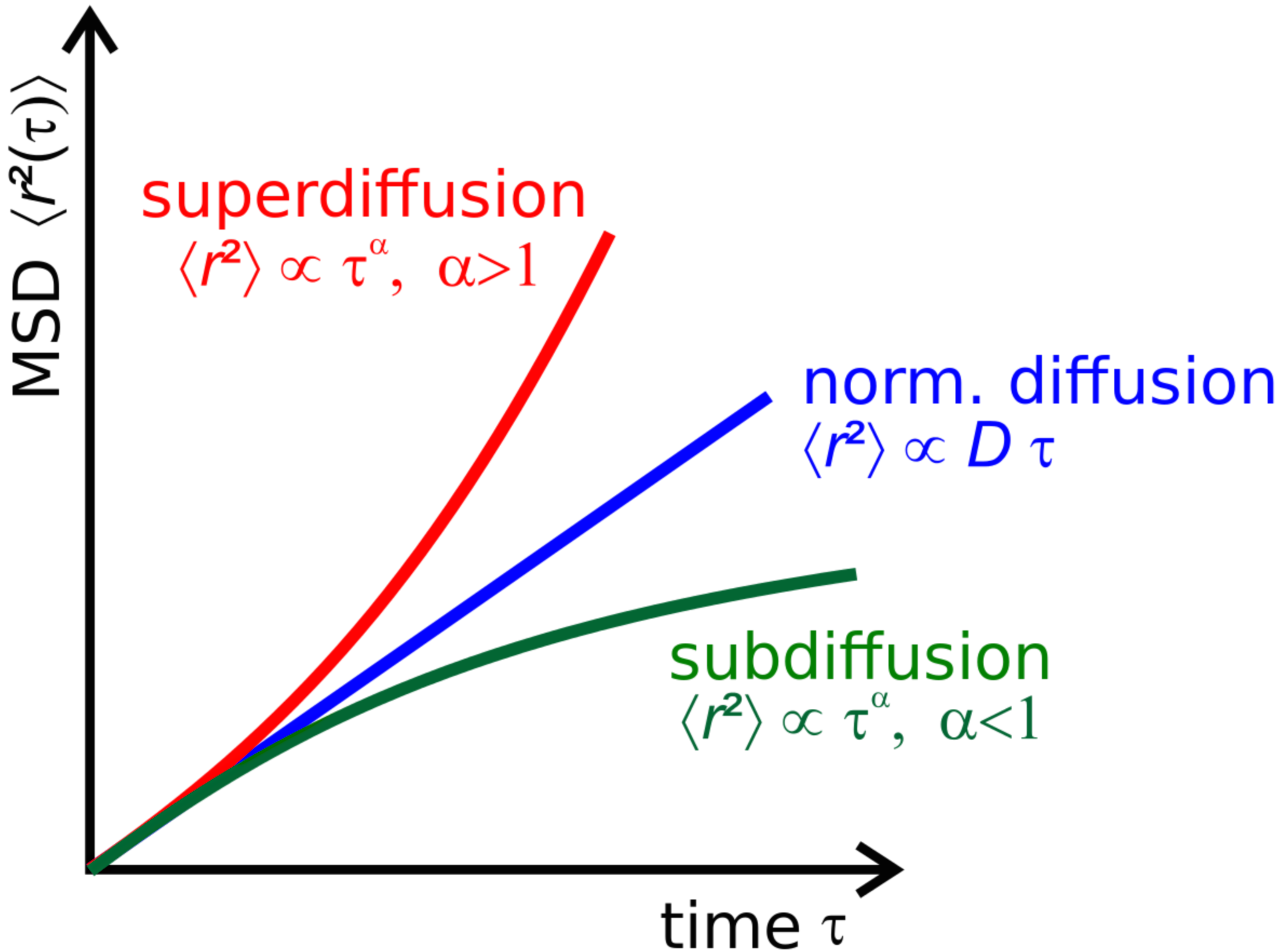


1000
500
0
-500
-1000
0
50000
100000
150000
200000

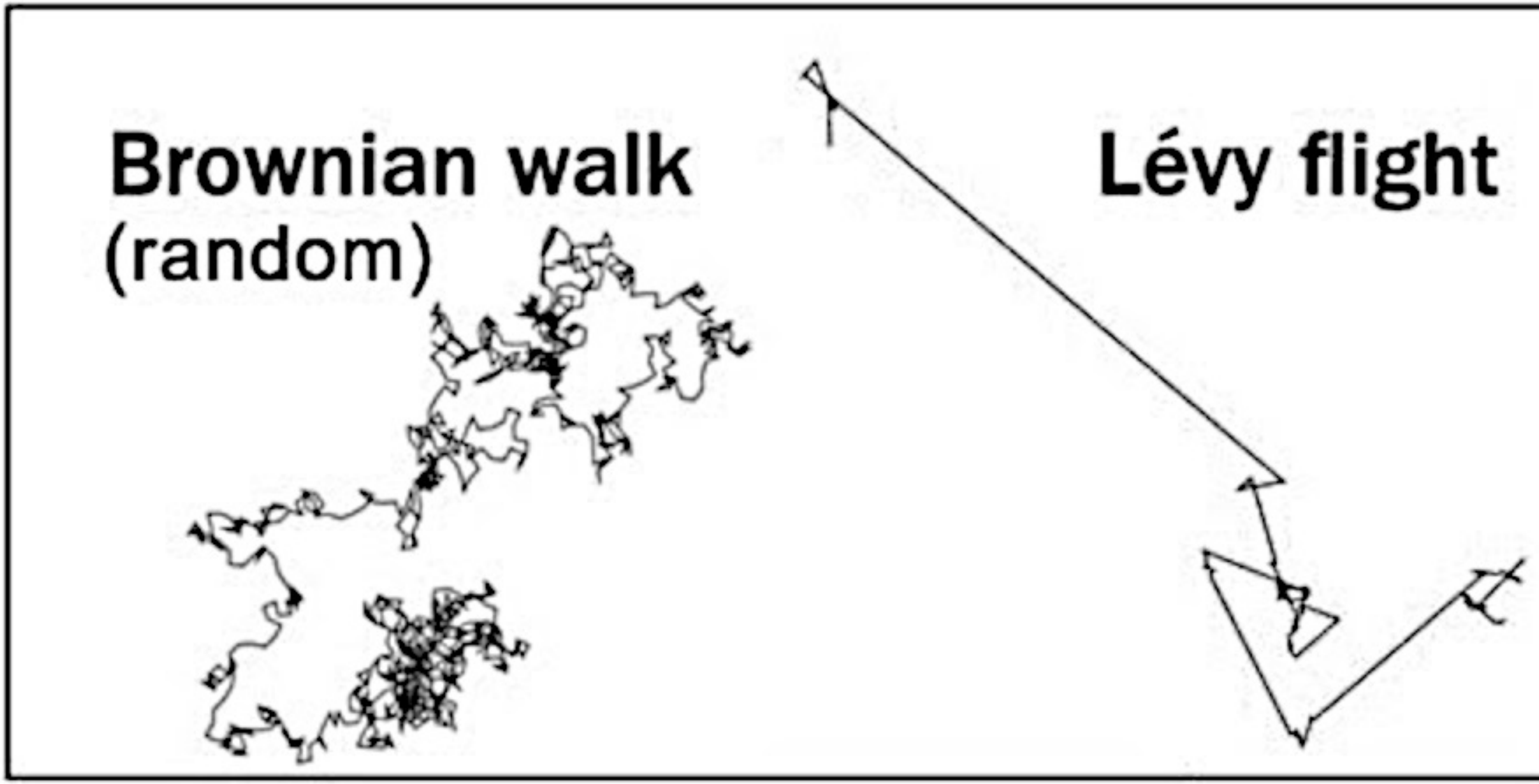
Random Walks



Random Walks



Random Walks



Normal Law for Grosswald-Schnitzer L-functions

$$X_N = \sum_{n=1}^{N} x_n$$

$$x_n = \cos(t \log p'_n - \theta(p'_n))$$

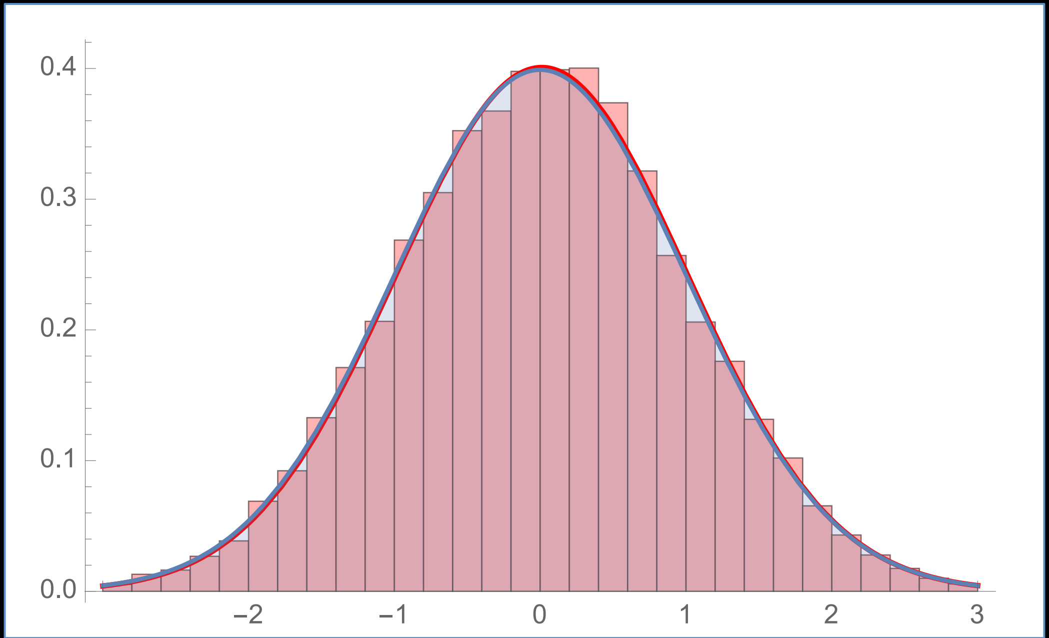
Random indipendent number

Normal Law for Grosswald-Schnitzer L-functions

$$X_N = \sum_{n=1}^{N} x_n$$

Lyapunov theorem
(Central limit th)

$$\frac{(X_N - m_N)}{s_N} \longrightarrow \mathcal{N}(0,1)$$



$$X_N = \sum_{n=1}^{N} \cos(t \log p'_n - \theta(p'_n))$$ GS L-function

$$m_N = \sum_{n=1}^{N} \cos(t \log p_n - \theta(p_n))$$ Dirichlet L-function

$$\boxed{(X_N - m_N) \simeq \mathcal{O}(\sqrt{N})}$$

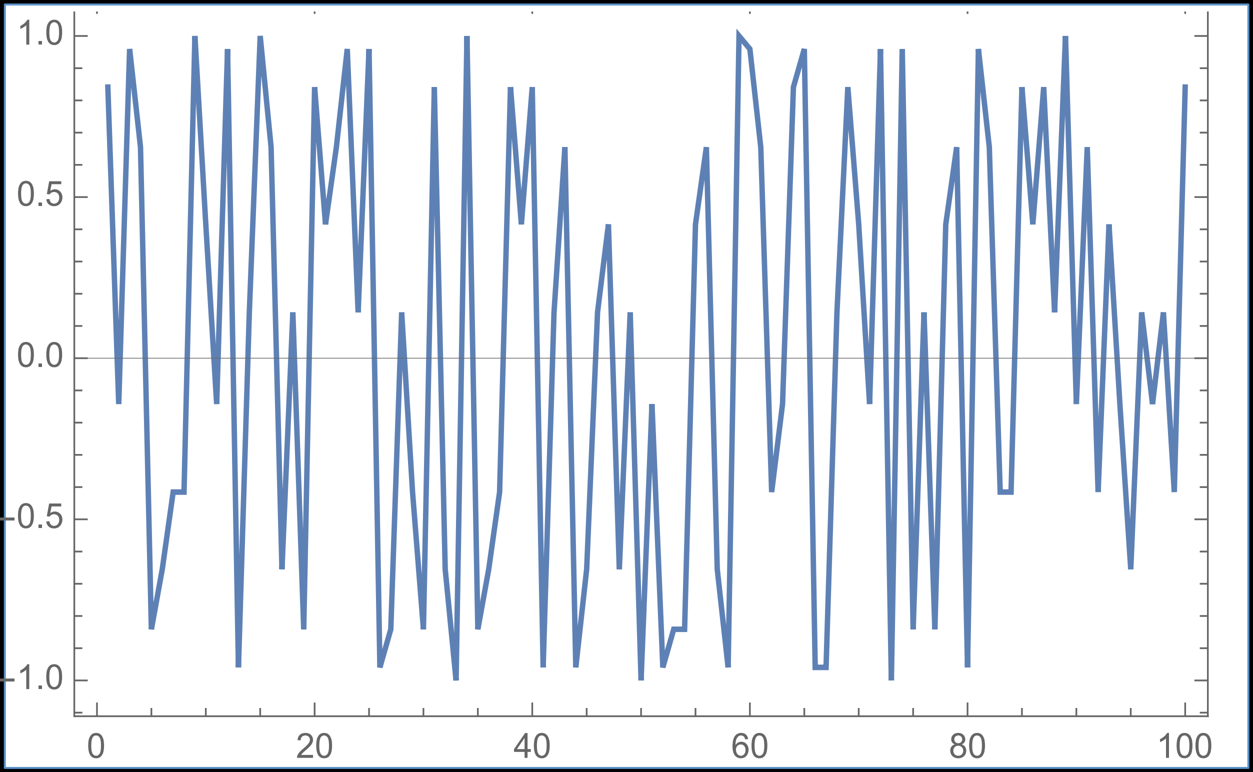
---

1. $X_N \simeq N^\alpha + A\sqrt{N}$ & $m_N \simeq N^\alpha + B\sqrt{N}$

2. $X_N \simeq A\sqrt{N}$ & $m_N \simeq B\sqrt{N}$

The hard case, t=0

$$B(N) = \sum_{n=1}^{N} c_n \quad , \quad c_n = \cos\theta(p_n) = b_n(0)$$



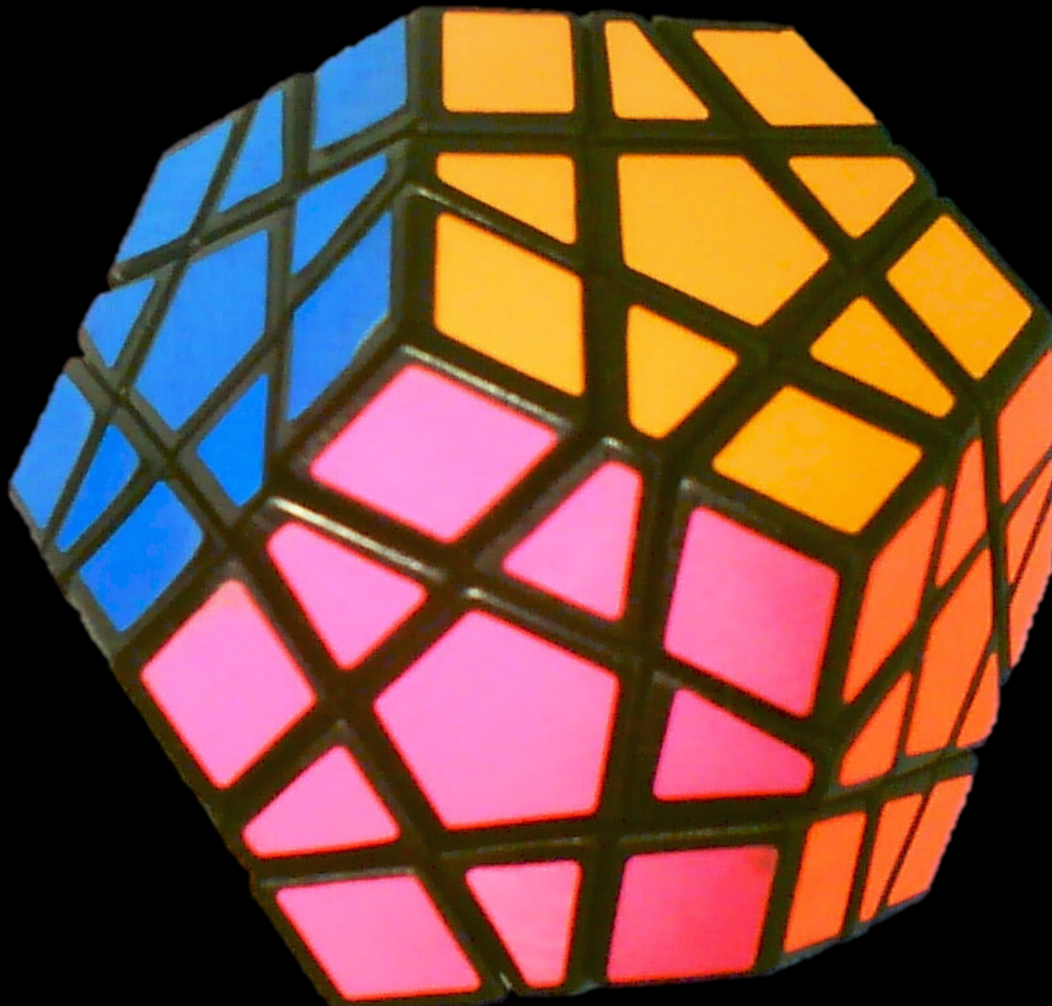
What do we know about the terms of this series?

$$c_n = \cos\theta(p_n)$$

Let’s nail it down

- All terms are of the same order (order 1)

- All the angles are equiprobable (Dirichlet theorem)



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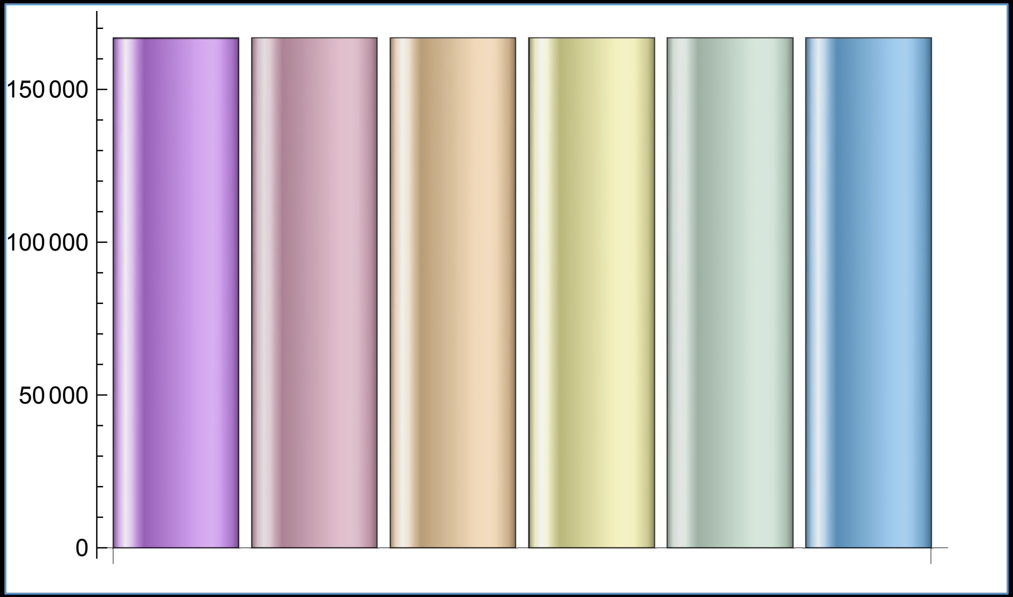


- All terms are of the same order (order 1)
- All the angles are equiprobable (Dirichlet theorem)

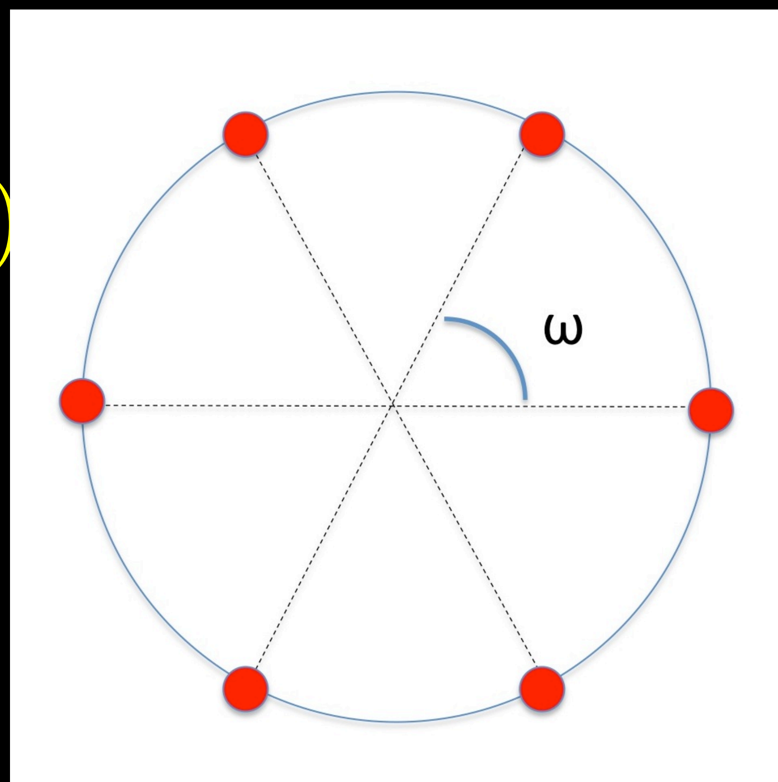
N	5^3	5^4	5^5	5^6	5^7	5^8	5^9	5^{10}
Pr_1	0.16000	0.16480	0.16544	0.16589	0.16644	0.16656	0.16664	0.16664
Pr_2	0.17600	0.16640	0.17024	0.16717	0.16696	0.16667	0.16670	0.16668
Pr_3	0.16800	0.16160	0.16416	0.16659	0.16652	0.16662	0.16664	0.16668
Pr_4	0.18400	0.17120	0.16640	0.16691	0.16698	0.16678	0.16668	0.16669
Pr_5	0.16000	0.16320	0.16512	0.16672	0.16646	0.16660	0.16662	0.16666
Pr_6	0.15200	0.17280	0.16864	0.16672	0.16664	0.16676	0.16673	0.16665

- All terms are of the same order (order 1)

- All the angles are equiprobable (Dirichlet theorem)

- As consequence, the mean of the series B(N) vanishes

$$\langle B(N) \qquad {}_{n} = 0$$



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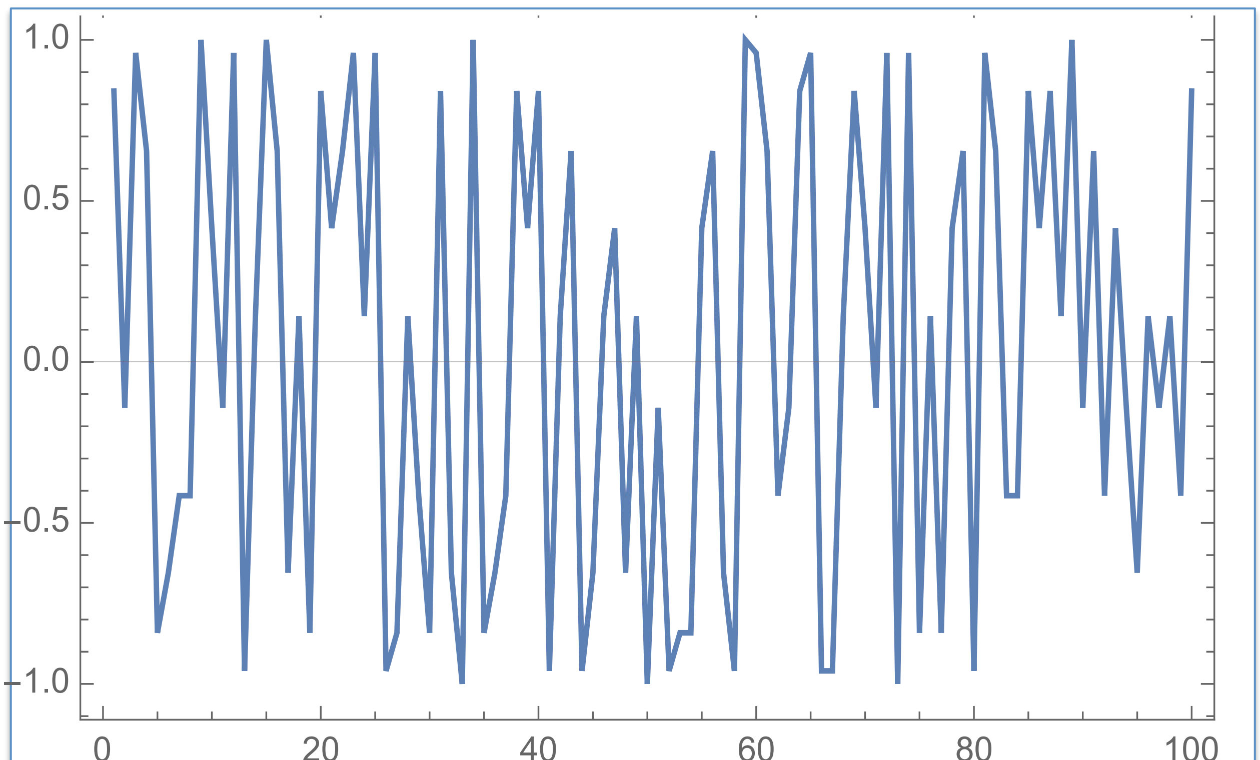
$$\langle B(N) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0$$

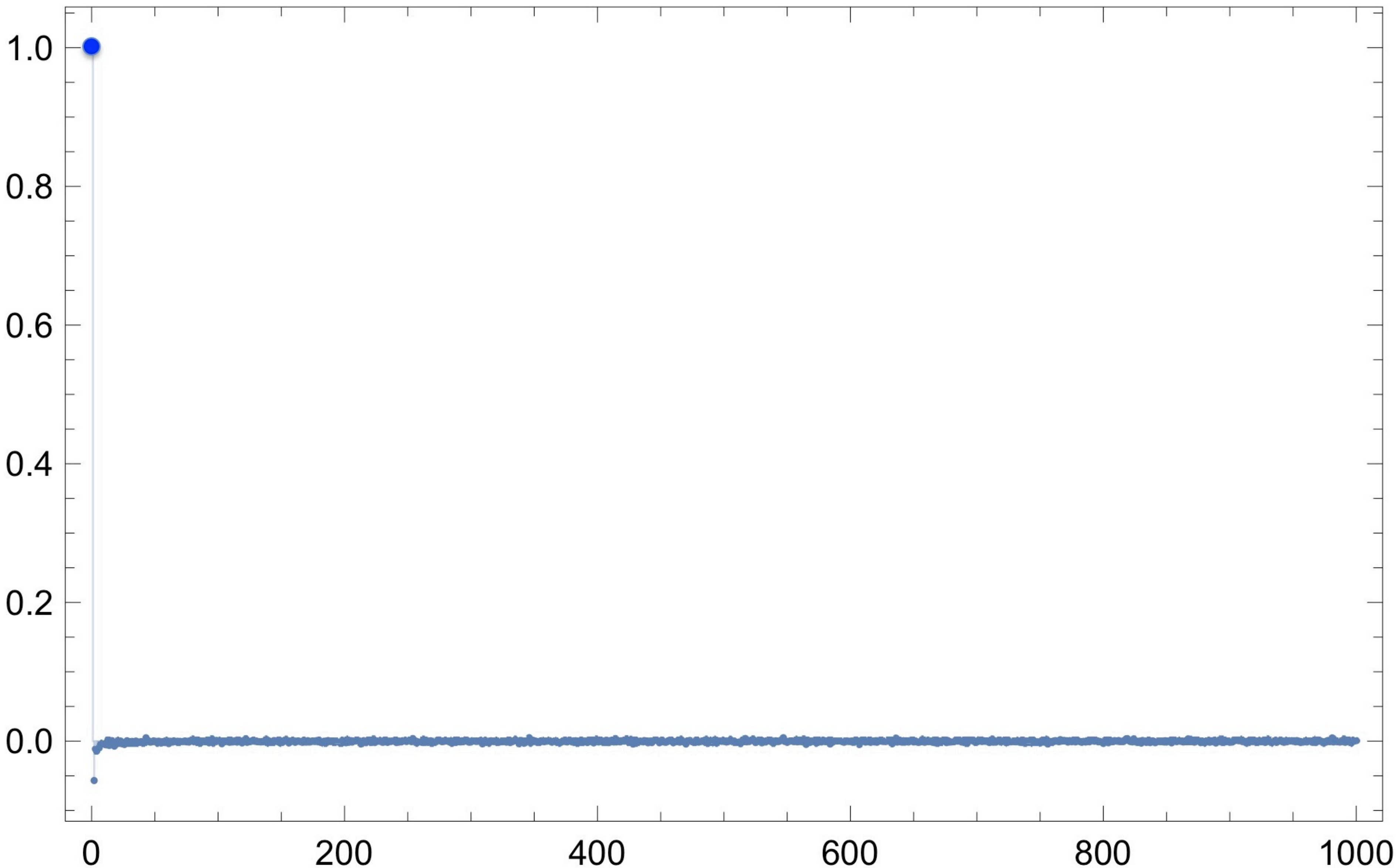
- The variable c_n are weakly correlated

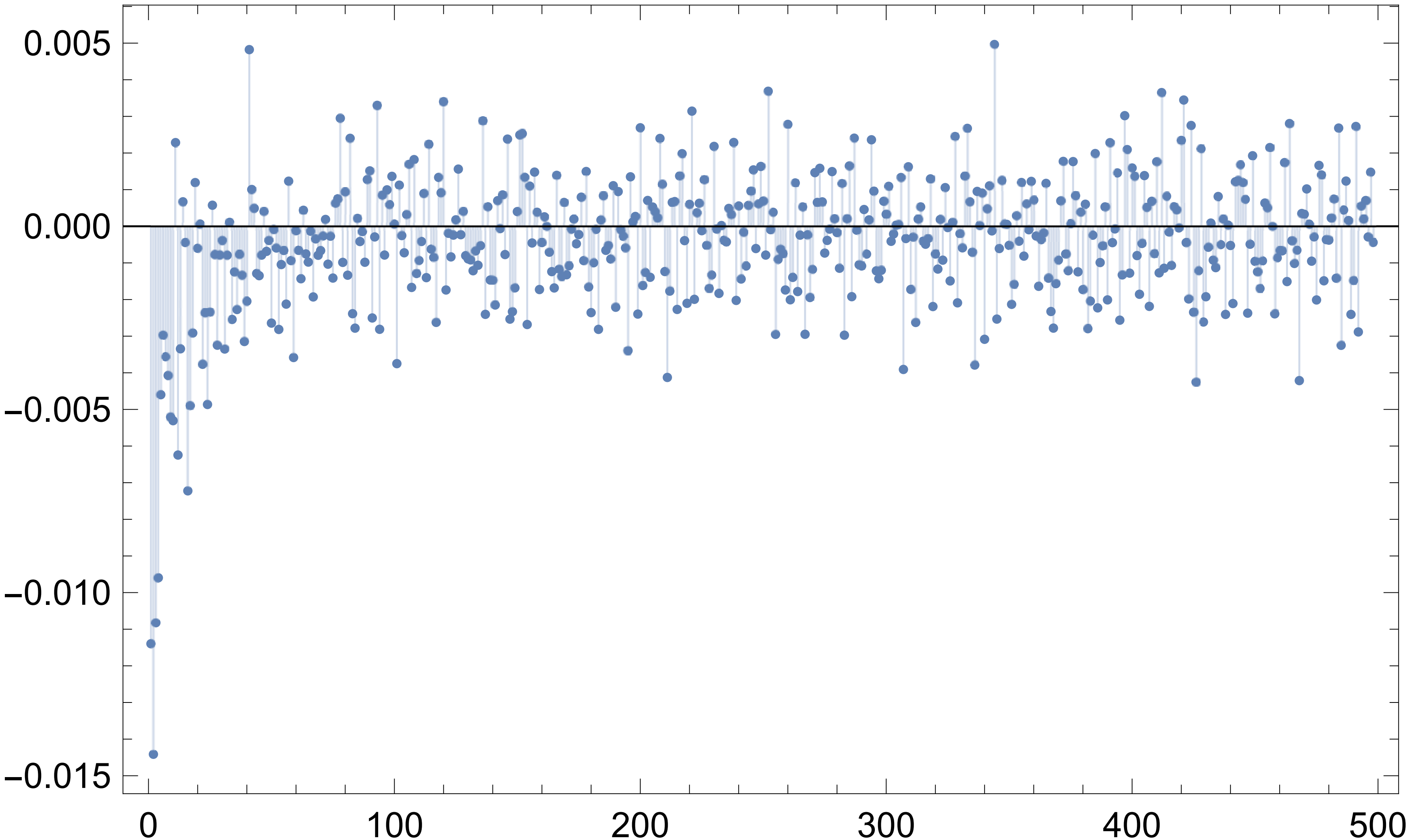
Correlation Function of lag j

$$G(j) = \frac{\sum_{n=1}^{N-1}(c_n - \mu)(c_{n+j} - \mu)}{\sum_{n=1}^{N}(c_n - \mu)^2}$$



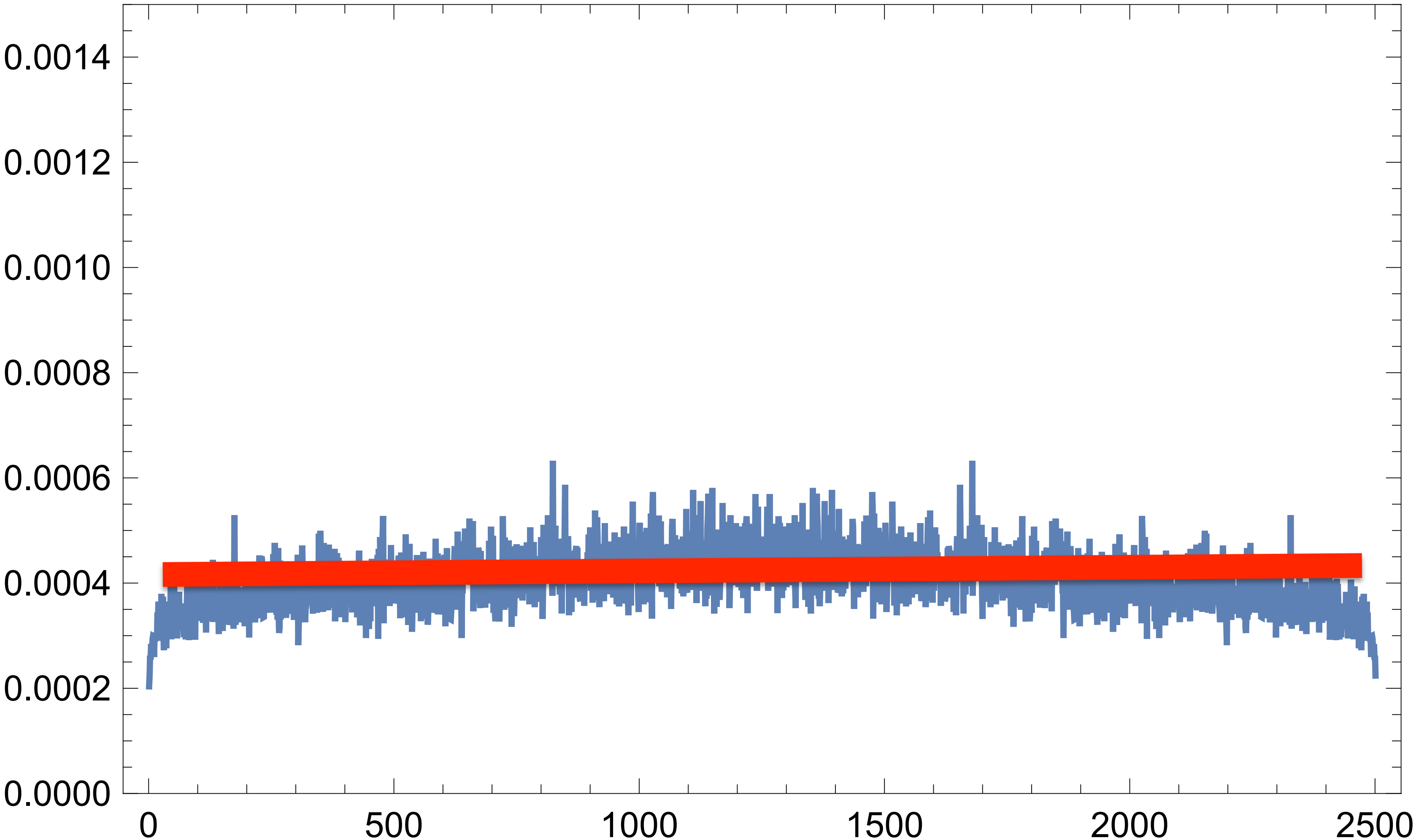
Correlation Function of lag j




0.005
0.000
−0.005
−0.010
−0.015
0
100
200
300
400
500

Power Spectrum

$$\mathcal{F}(k) = |\, \tilde{G}(k)\, |^2$$



As a matter of fact, we know all their correlations!

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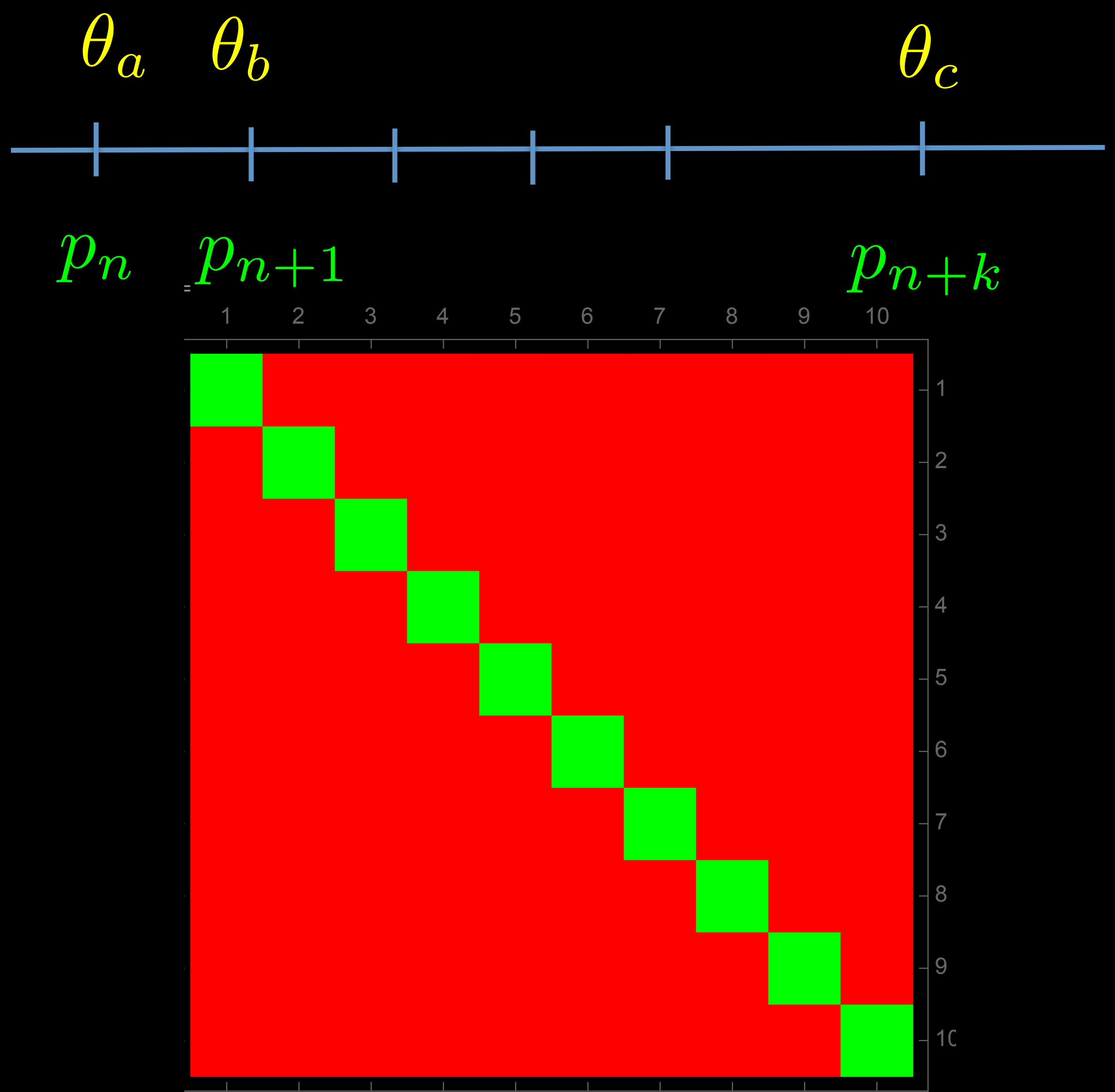
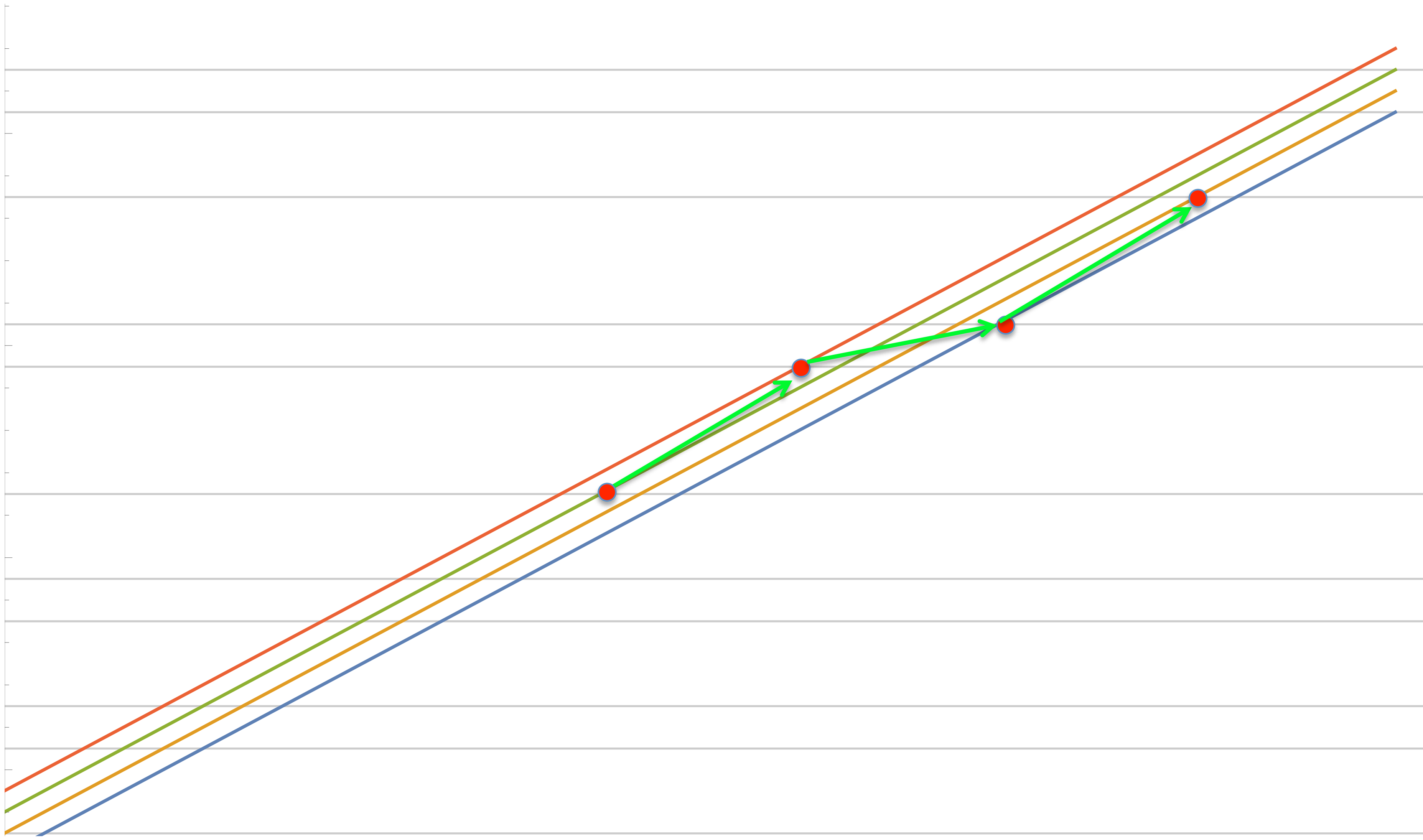
(Lemke Oliver-Soundararajan)





As a matter of fact, we know all their correlations!

(Lemke Oliver-Soundararajan)

$$f_{ab}(N,q,k) = \frac{\#\{p_n \le p_N : p_n \equiv h_a \pmod{q}\ ,\ p_{n+k} \equiv h_b \pmod{q}\}}{N}$$

$$f_{aa}(N,q,k) = \frac{1}{(\varphi(q))^2}\left(1 - \cancel{\frac{(\varphi(q)-1)}{2(k-1)}\frac{1}{\log N}} + O\left(\frac{1}{(\log N)^{7/4}}\right)\right)$$

$$f_{ab}(N,q,k) = \frac{1}{(\varphi(q))^2}\left(1 + \cancel{\frac{(1)}{2(k-1)}\frac{1}{\log N}} + O\left(\frac{1}{(\log N)^{7/4}}\right)\right)$$

Time Series

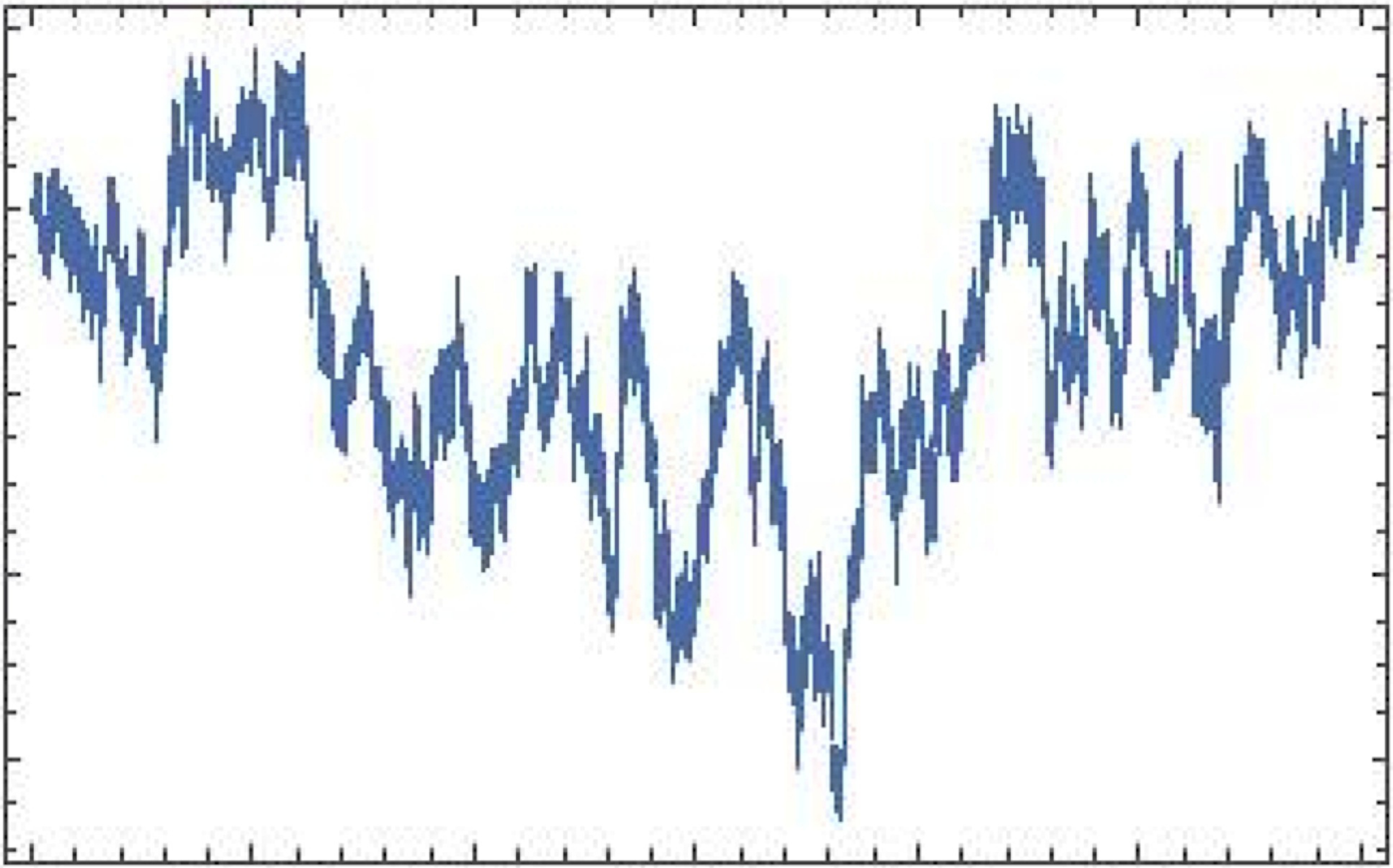
$$B(N) = \sum_{n=1}^{N} c_n$$

$$c_n = \cos\theta_{p_n}$$

B(N) grows as a brownian motion !

- $$\langle B(N)\rangle = 0$$
- $$\langle B^2(N)\rangle \simeq N$$

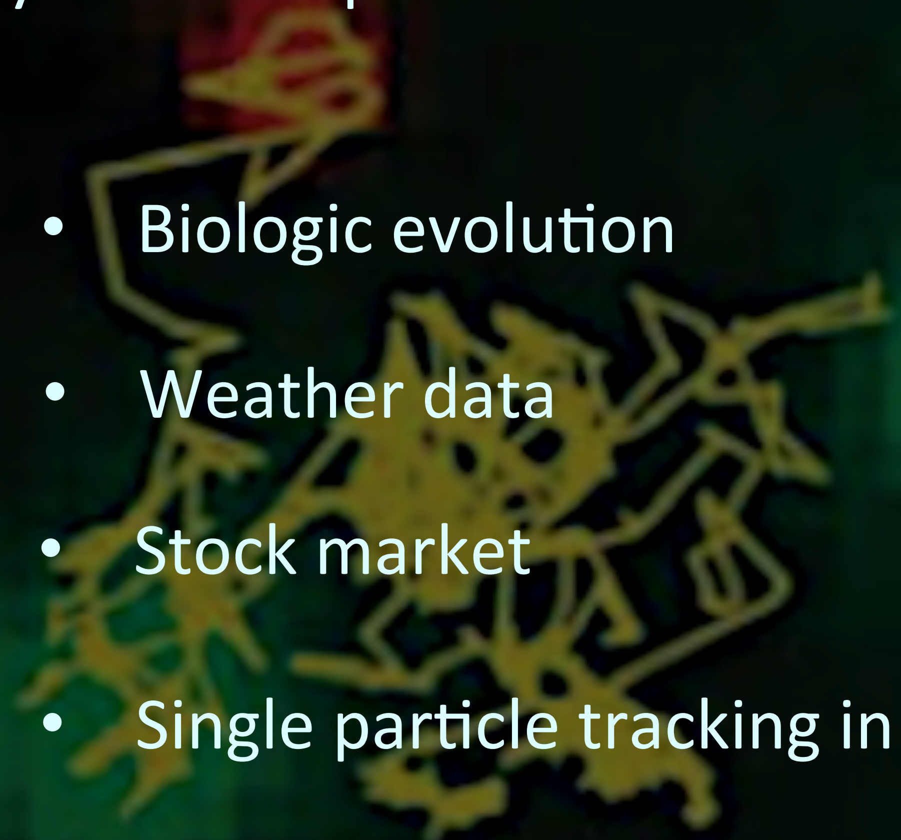
Time Series



N

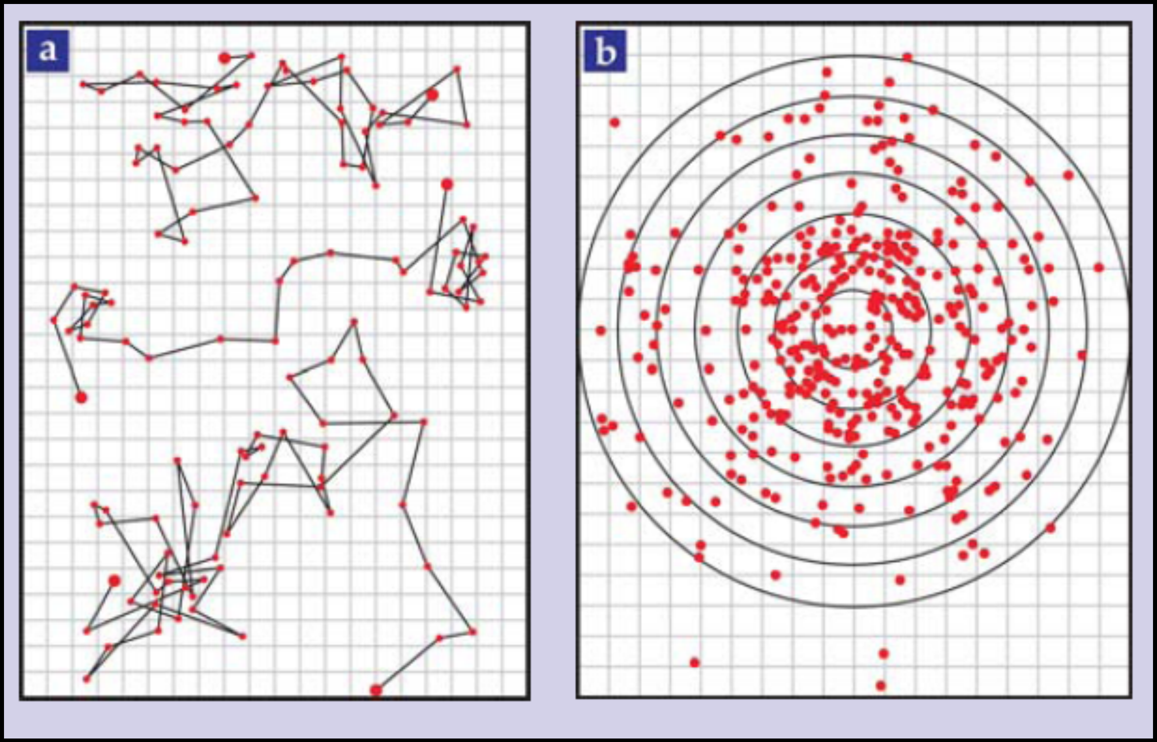
Problem of single Brownian trajectory

Very common problem of Data Analysis

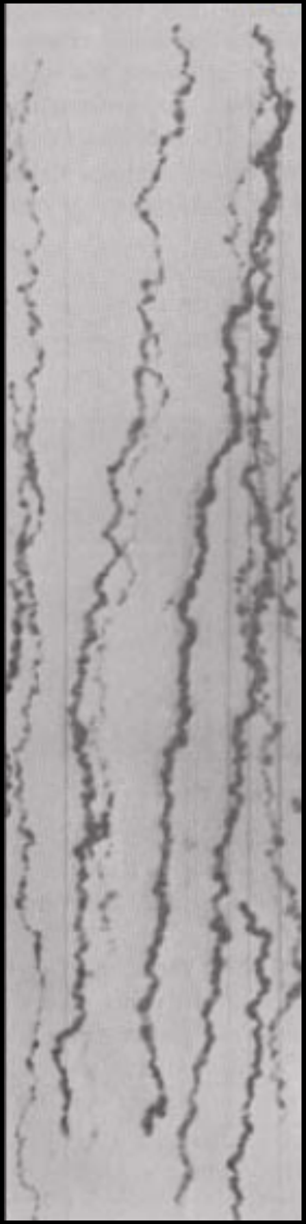


- Biologic evolution
- Weather data
- Stock market
- Single particle tracking in living cell

In all those cases where we cannot turn back time…



Jean Perrin (1908)



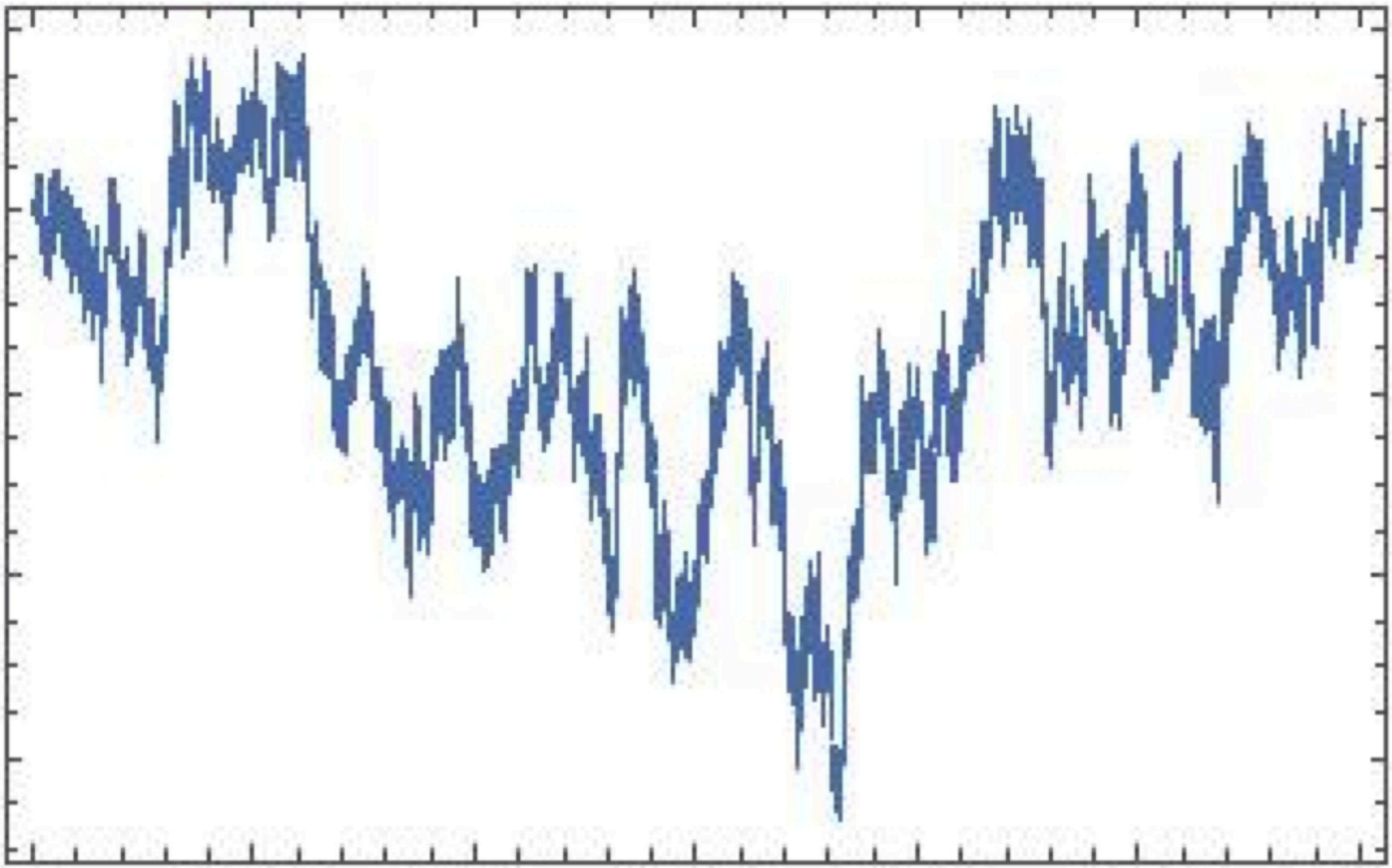
Ivar Nordlund (1914)

Problem of single Brownian trajectory

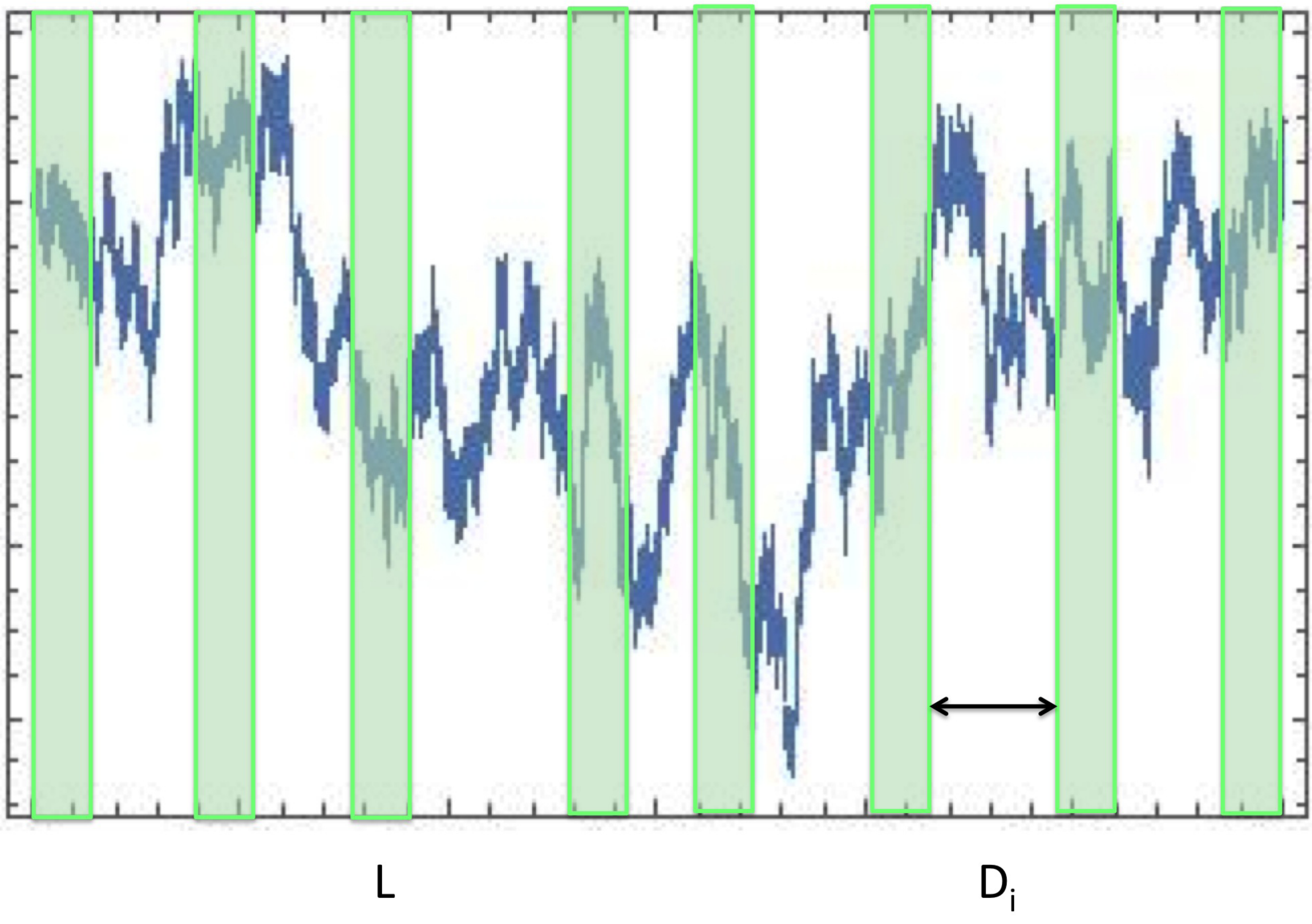
We can get around the impossibility of repeating the experiment many times, if we simply assume

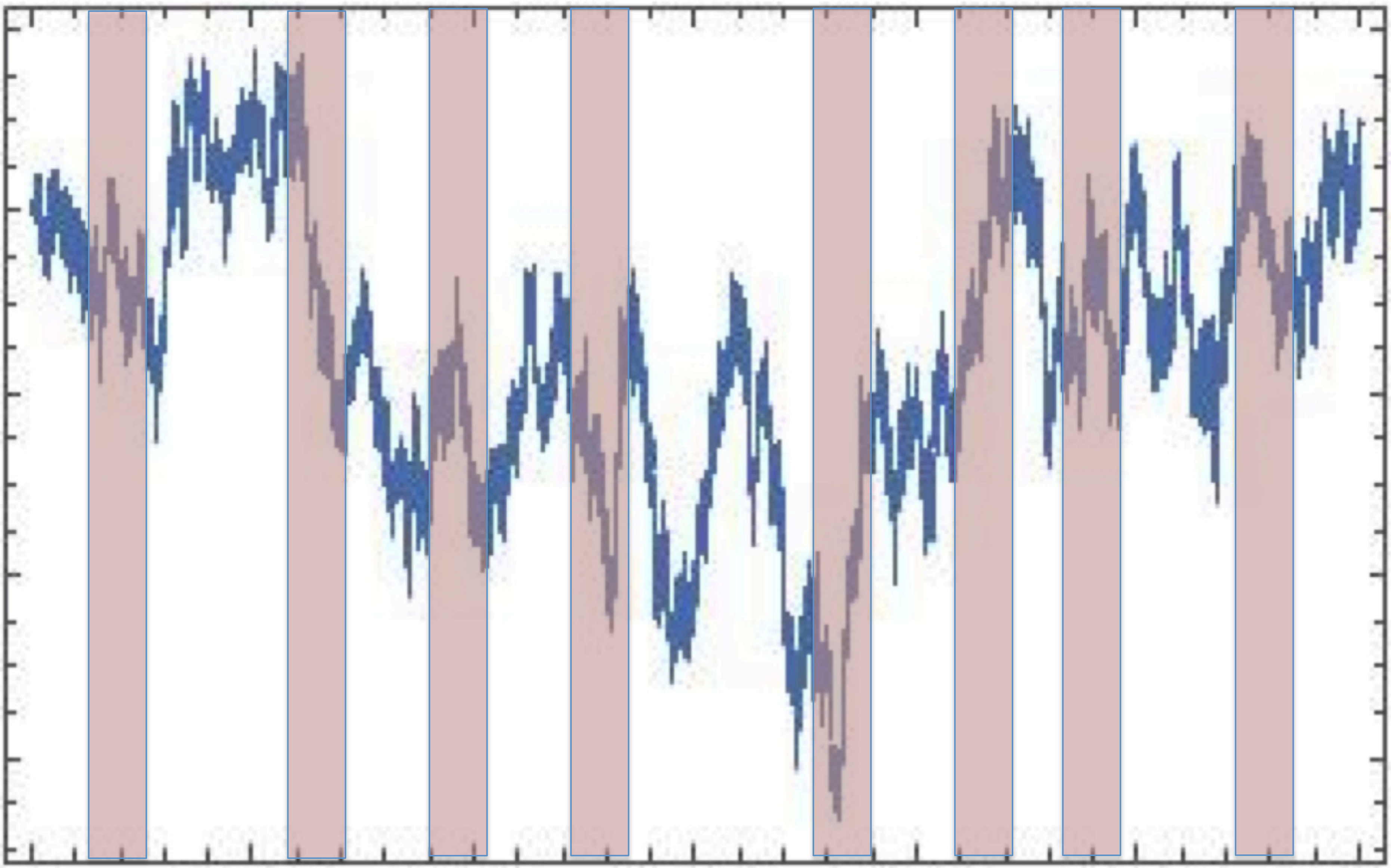
√ • The stationarity of the time series

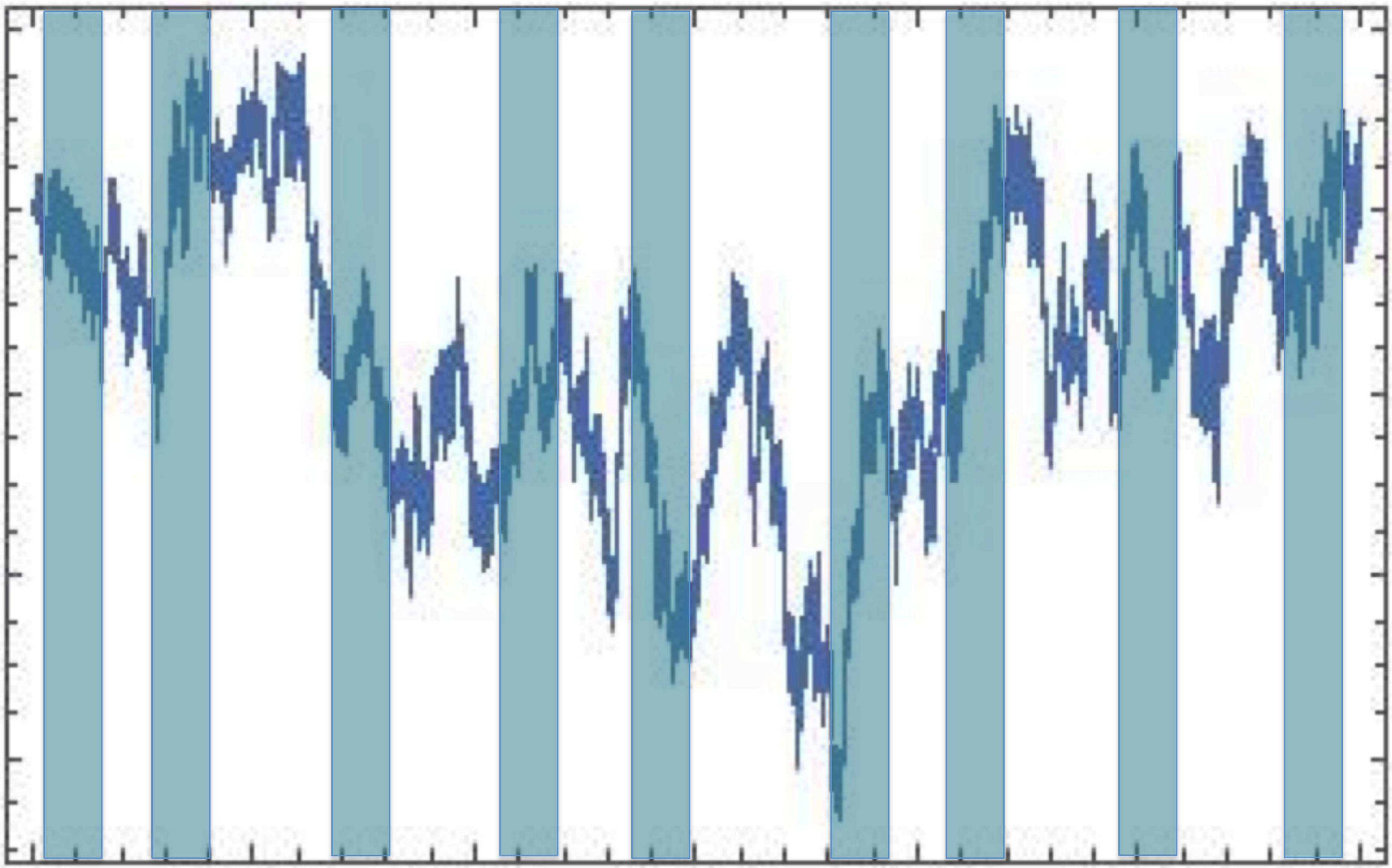
√ • The ergodicity of the time series

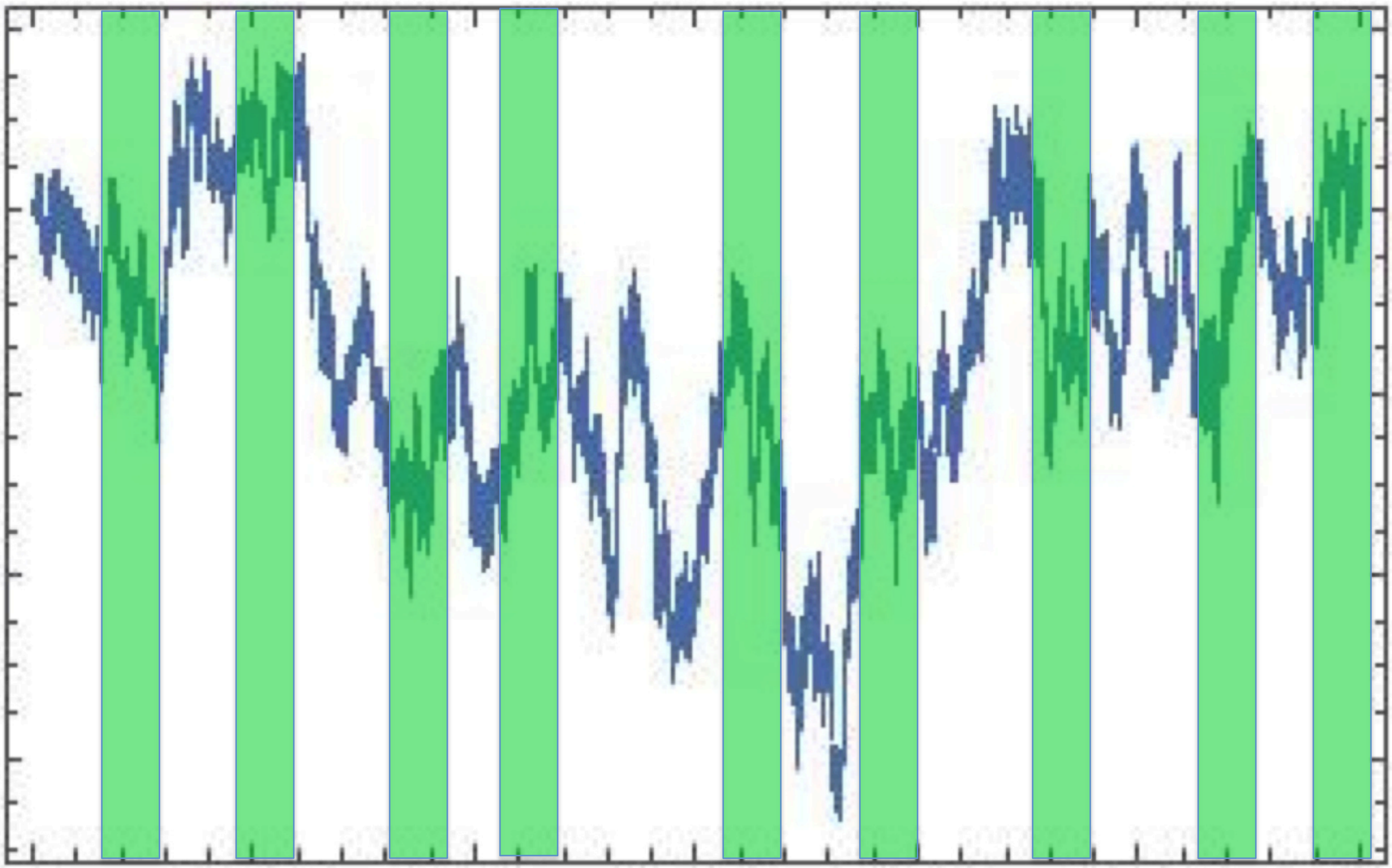


N

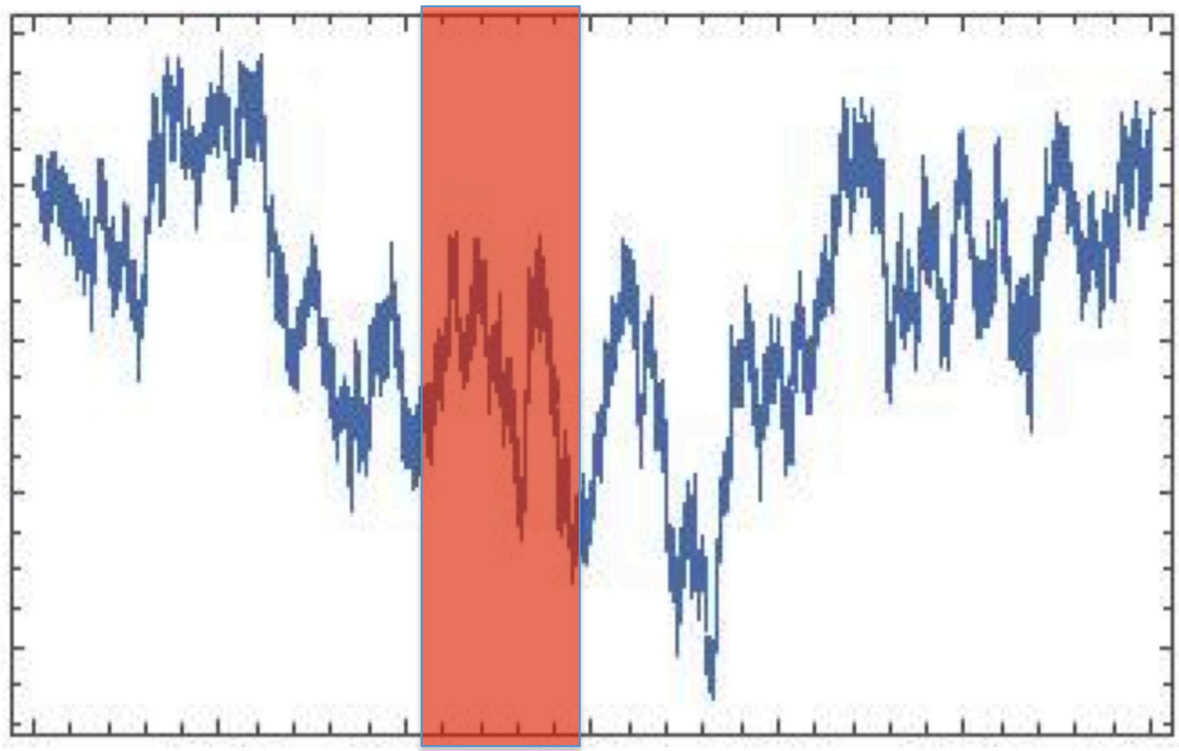

L
D_i







Block Variables



$$X_a(L) = \sum_{n=1}^{L} c_{a+n}$$

If the original series goes as

$$B_N \simeq \sqrt{N}$$

the same should happen for any of its subsets!

$$B'_L \simeq \sqrt{L}$$

Our result on the variance of the block variables

$$\sigma_L^2 = \boxed{L}\,\lambda(L)$$

$$\lambda(L) = b^2 \left[1 - \frac{1}{\log N}\boxed{(\log L}-\gamma_E - 1) + \right.$$
$$\left. - \frac{\log\log N}{\log N} + \mathcal{O}\left(\frac{1}{\log^{7/4} N}\right)\right]$$

$$b^2 \equiv \frac{1}{r}\sum_{k=1}^{r}\cos^2\phi_k = \begin{cases} 1 & ; \quad \text{if } \chi = \chi^* \\ 1/2 & ; \quad \text{if } \chi \neq \chi^* \end{cases}$$

Hence, if our analysis is correct we shall find for **ANY** ensemble made of subinterval of length L

$$\frac{\sigma_L^2}{\lambda(L)} = L$$

indipendent of the **modulus q**

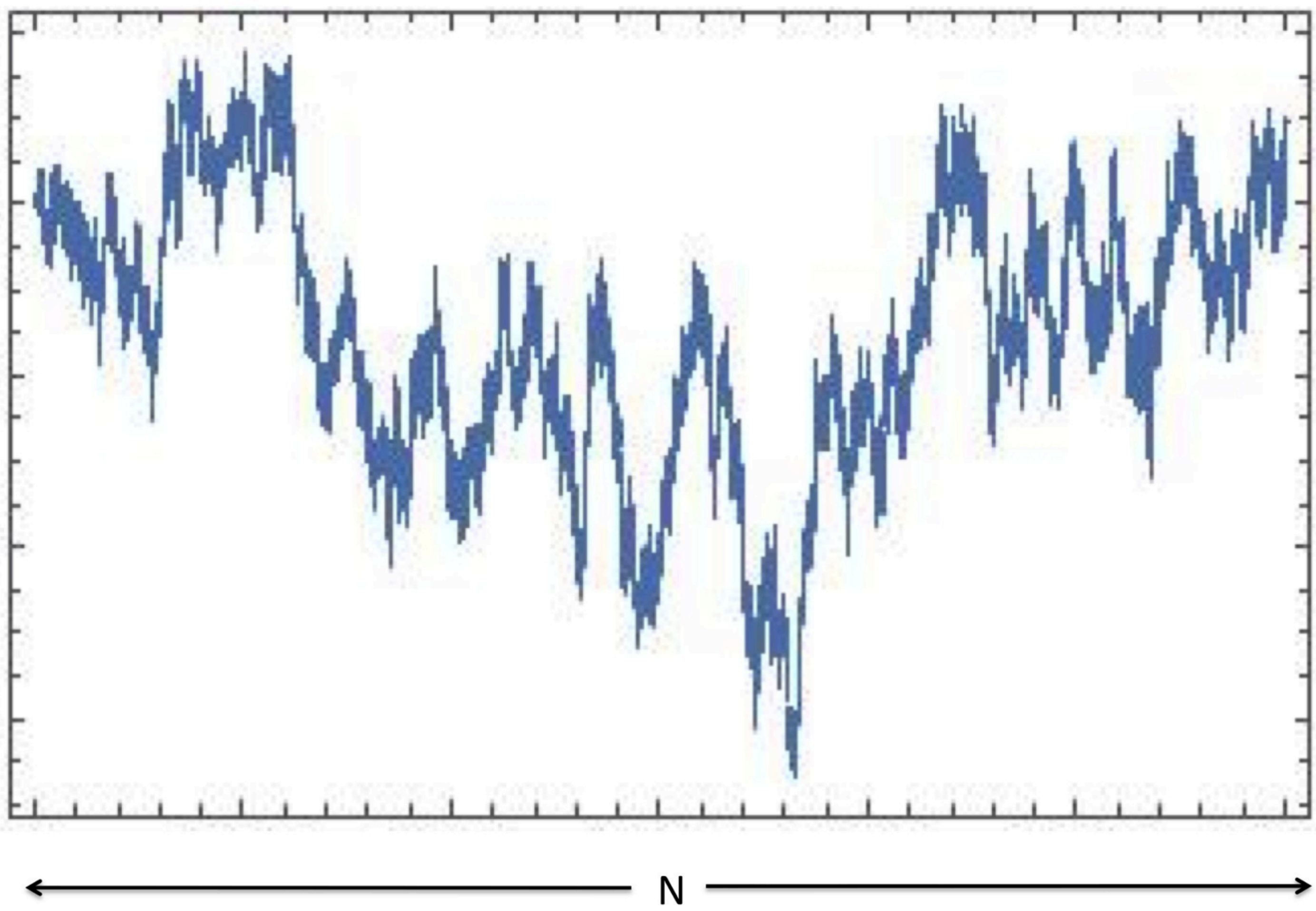
Hence, for **ANY** ensemble made of subinterval of length L

$$\frac{\sigma_L^2}{\lambda(L)} = L$$

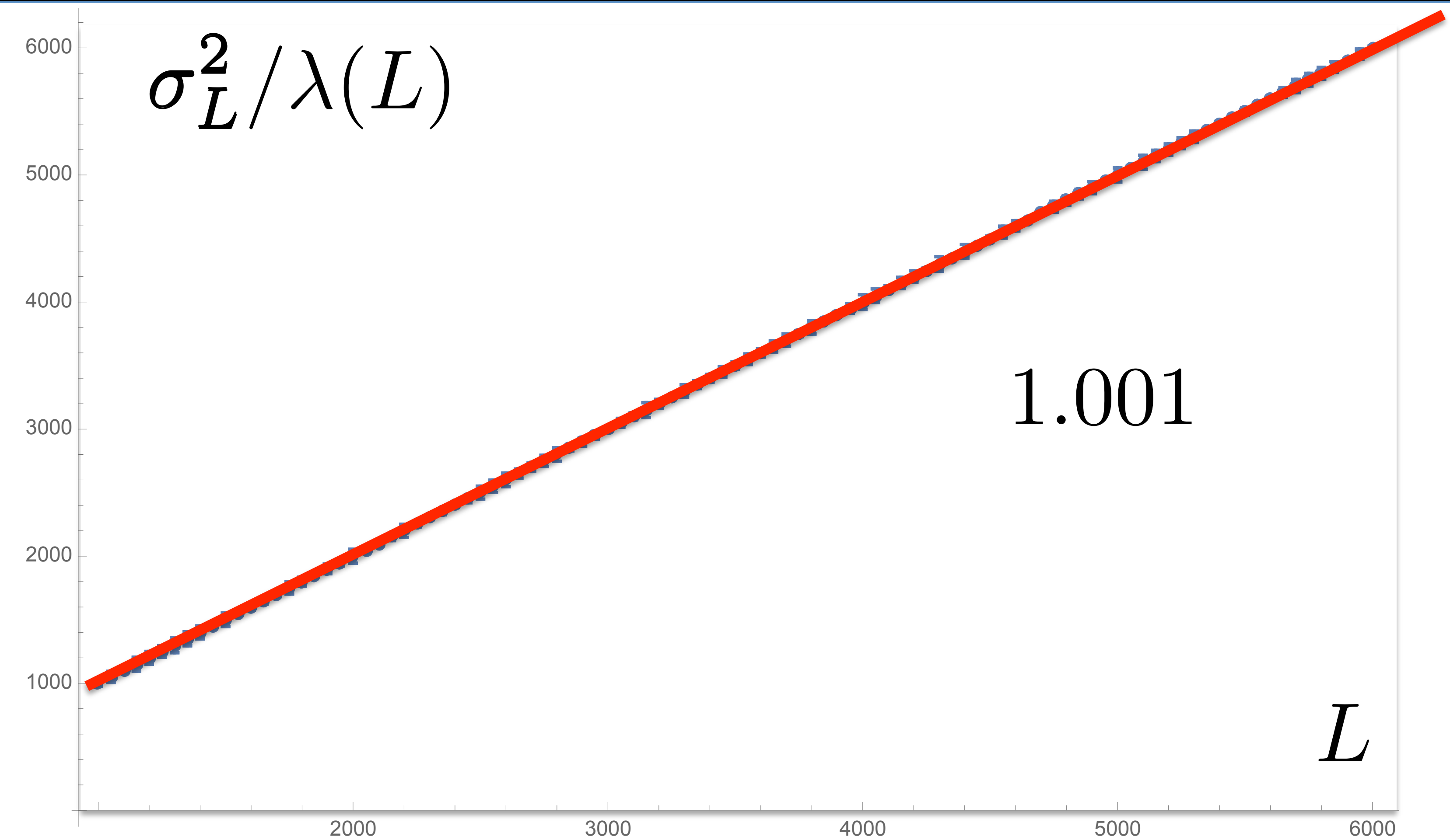
indipendent of the **modulus q**

$$N = 100.000.000 \quad \text{primes}$$

$$p_N = 2.038.074.743$$

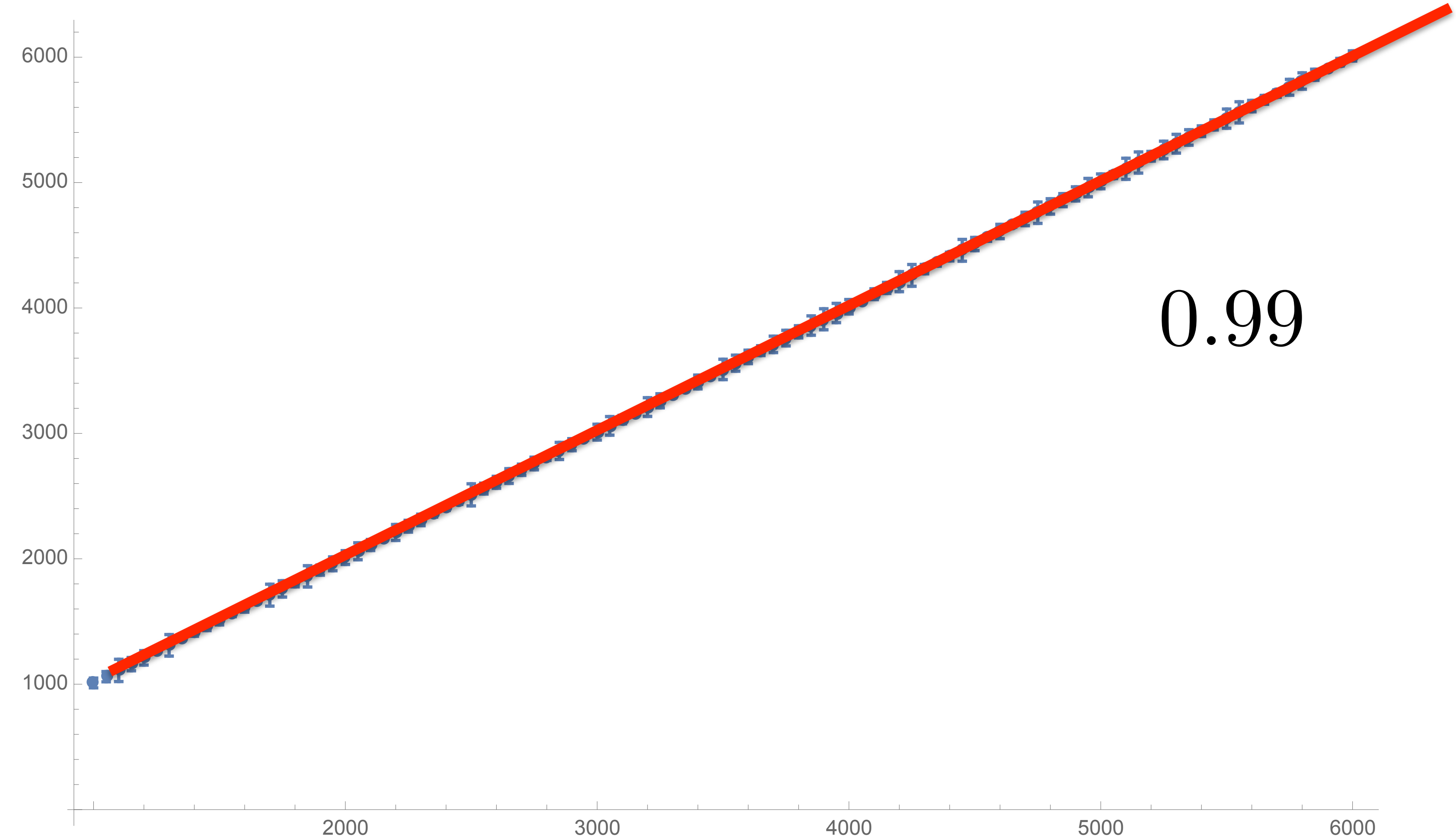


$\chi_2 \pmod 7$

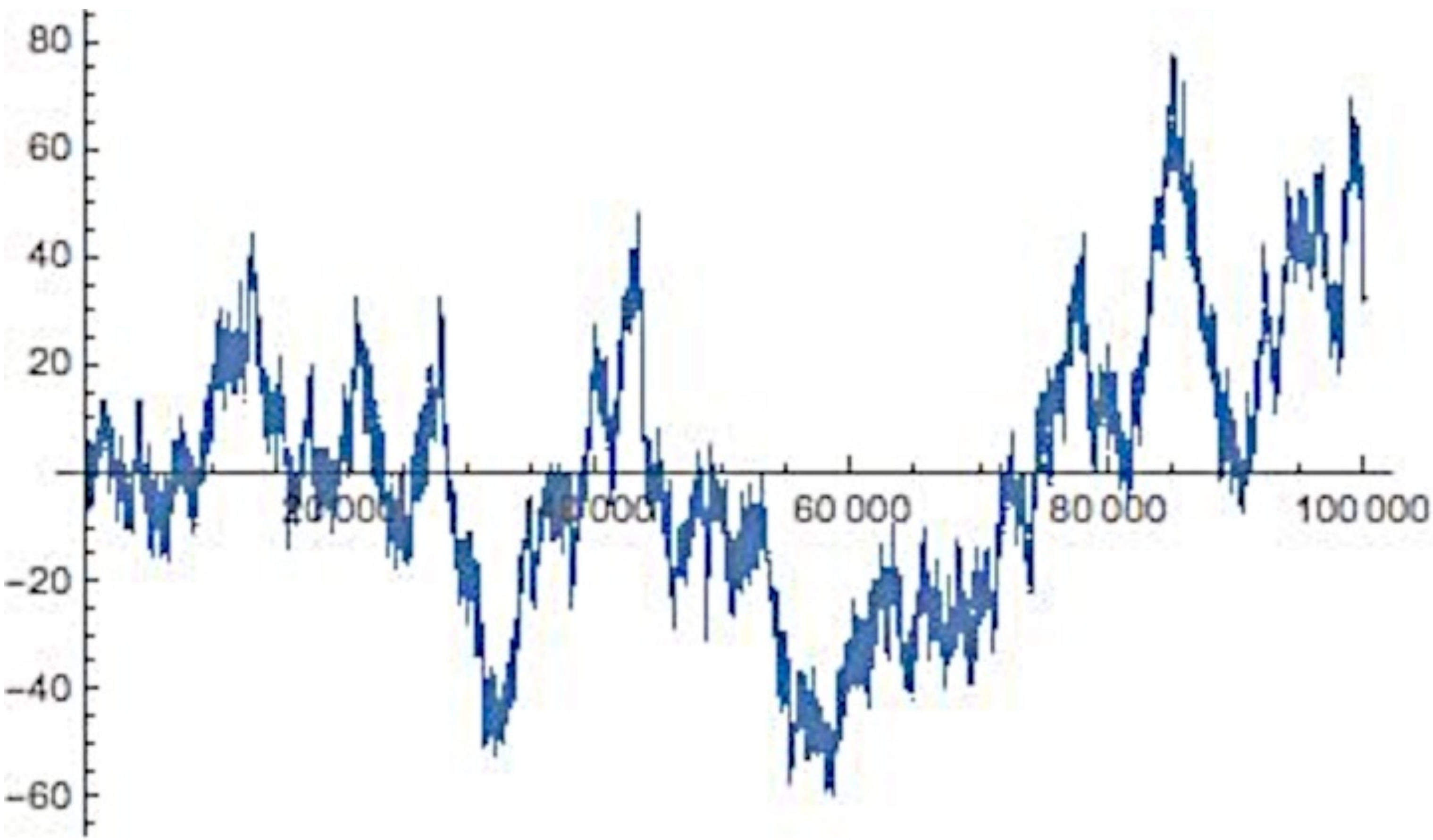


$\sigma_L^2/\lambda(L)$
6000
5000
4000
3000
2000
1000
1.001
L
2000
3000
4000
5000
6000

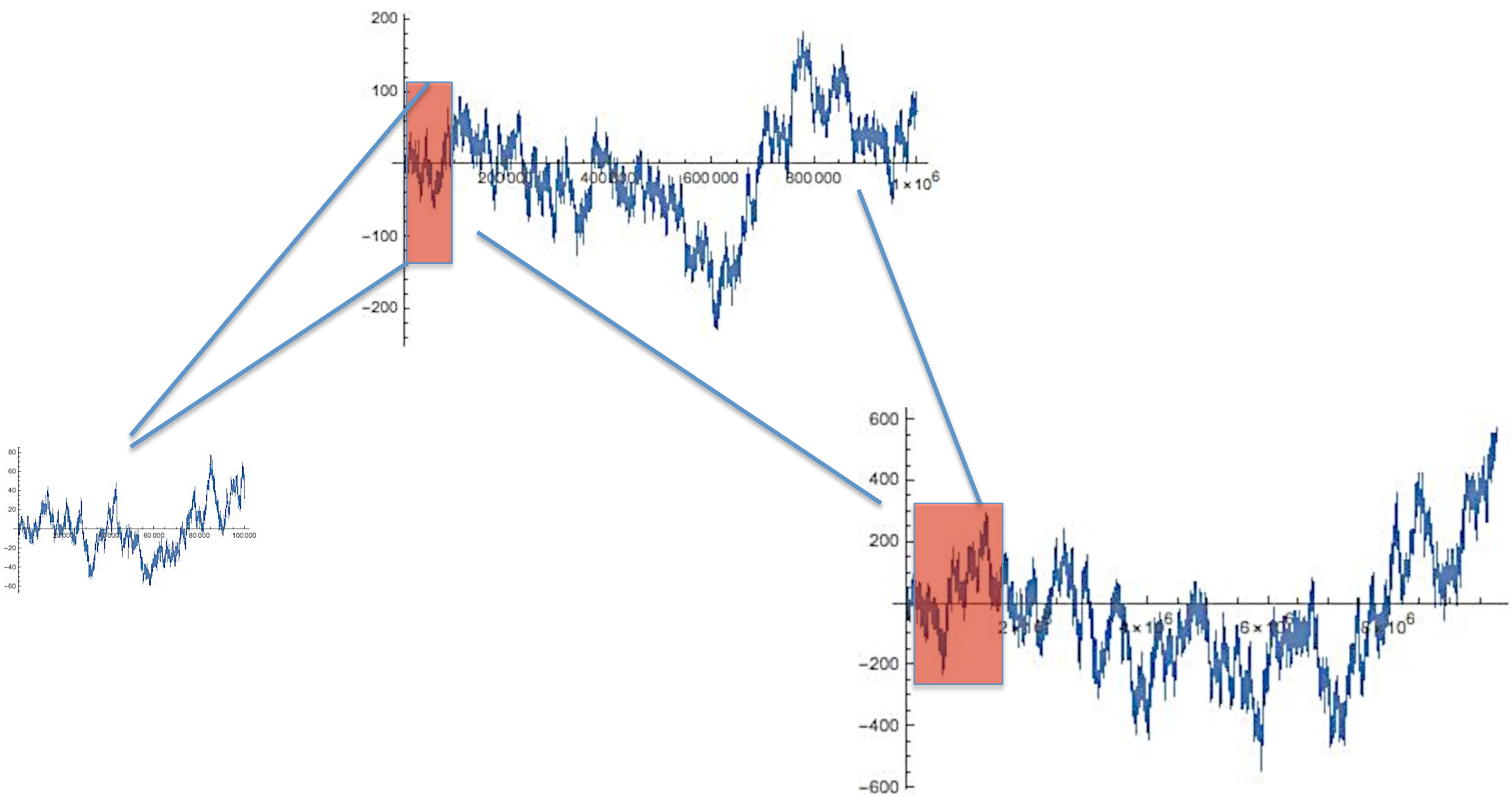
$\chi_3 \pmod{11}$



At large N, even better!



$$f(q,\mathbf{h}) \simeq \frac{1}{\varphi^r(q)}\left(1+\mathcal{O}\boxed{\left(\frac{1}{\log N}\right)}\right)$$



Analytic continuation of the infinite product

$$\log P(s) \quad = \sigma \int_2^\infty \frac{B(u)}{u^{\sigma+1}} du$$

$$B(N) = \sum_{n=1}^{N} \cos\theta_{p_n} \ \simeq N^{1/2+\epsilon}$$

The integral is then singular at σ =1/2

Namely, all zeros are on the critical line!

Conclusions

- There are infinitely many functions which, as the Riemann zeta function, has all their zeros along Re(s) = 1/2
- This properties is largely independent on the primes and can be traced instead to some of their random properties
- The validity of the GRH relies on probability theorems involving the residues and their (weak) correlations
- The "Random Walk theorem" presented here is based on the Dirichlet theorem on the residues and their correlations studied by Oliver-Soundararajan