On the character degree graphs of finite groups

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General notation

In this talk, the word "group" will always mean "finite group".

Given a group G, we denote by $\mathrm{Irr}(G)$ the set of *irreducible complex characters* of G, and by

$$\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(G)\}$$

the *set* of their degrees.

Question

(a) *What information is encoded in* $\mathrm{cd}(G)$ *?*

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Character Degrees

Theorem (Isaacs, Passman; 1968)

Assume $\operatorname{cd} G = \{1, m\}$, *with* $1 < m \in \mathbb{N}$. *Then :*

(a) (a1) *If* m *is no prime power, or*
(a2) *if* $m = p^a$, p *prime and* G *has abelian Sylow* p*-subgroups,*
then $\bar{G} = G/\mathbf{Z}(G) = \bar{K} \rtimes \bar{H}$ *is a Frobenius group, with* $|\bar{H}| = m$ *and both* K *and* H *abelian.*

(b) *If* $m = p^a$, p *prime, and* $P \in \mathrm{Syl}_p(G)$ *is non-abelian, then either* $G = A \times P$ *with* A *abelian or* $a = 1$ *and* G *has a normal abelian subgroup of index* p.

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$|\operatorname{cd}(G)|$ and derived length

Theorem (Taketa; 1930)

If G is monomial, then G is solvable and $\operatorname{dl}(G) \leq |\operatorname{cd} G|$.

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If $\mathrm{cd}(G) = \{1, p_1^{a_1}, p_2^{a_2}, \ldots, p_t^{a_t}\}$, *$p_i$ primes, $a_i > 0$, then*

(1) *G is solvable if and only if $|\{p_i | 1 \le i \le t\}| \le 2$ and $2 \le dl(G) \le 5$;*

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There is some interplay between the "arithmetical structure" of $\mathrm{cd}(G)$ and the group structure of G. Two celebrated instances:

Theorem (Ito 1951; Michler 1986)

Let p be prime number.
p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}\, G \Leftrightarrow$ if G has a normal abelian Sylow p-subgroup.

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The prime graph

One of the methods that have been devised in order to approach the degree-set is to introduce the *prime graph*.

Let X be a finite nonempty subset of $\mathbb{N}$. The prime graph on X is the simple undirected graph $\Delta(X)$ whose vertices are the primes that divide some number in X, and two (distinct) vertices p, q are adjacent if and only if there exists $x \in X$ such that $pq \mid x$.

Some significant sets of positive integers associated with a group G:

- $\mathrm{o}(G) = \{o(g) \ : \ g \in G\}$.
- $\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(G)\}$.
- $\mathrm{cs}(G) = \{|g^G| \ : \ g \in G\}$.

The prime graph can be attached to each of those sets.

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To what extent the group structure of G is reflected on and influenced by the structure of the relevant graph ?

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Character Degrees and Class Sizes: a connection

Theorem (Casolo, D.; 2009)

$\Delta(\mathrm{cd}(G))$ *is a subgraph of* $\Delta(\mathrm{cs}(G))$.

The character degree graph

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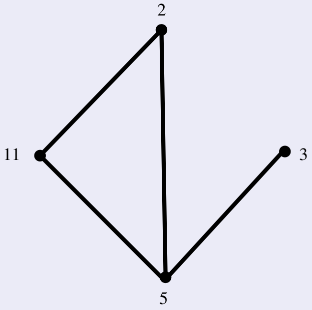
Example: $\Delta(M_{11})$



$$\mathrm{cd}\, M_{11} = \{1, 10, 11, 16, 44, 45, 55\}$$

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p prime, $P \in \mathrm{Syl}_p(G)$:
$p \notin \mathrm{V}(\Delta(G)) \Leftrightarrow P$ abelian and $P \triangleleft G$

Theorem (Pense; Zhang; 1996)

Assume G solvable. If $p, q \in \mathrm{V}(\Delta(G))$ are not adjacent in $\Delta(G)$ then $l_p(G) \leq 2$ and $l_q(G) \leq 2$.
If $l_p(G) + l_q(G) = 4$, then G has a normal section isomorphic to $(C_3 \times C_3) \rtimes \mathrm{GL}(2,3)$.

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Theorem (Manz, Staszewski, Willems; 1988)

- $\mathrm{n}(\Delta(G)) \leq 3$
- $\mathrm{n}(\Delta(G)) \leq 2$ *if* G *is solvable*

The groups G with disconnected graph $\Delta(G)$ have been classified

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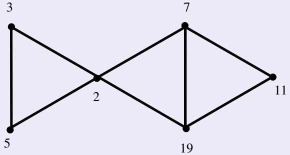
Example: $\Delta(J_1)$



The character degree graph for solvable groups

In the context of solvable groups, the following properties of the character degree graph have been proved.

- Let p, q and r be three vertices of $\Delta(G)$. Then two of them are adjacent in $\Delta(G)$ (Pálfy; 1998).
- $\Delta(G)$ has at most two connected components (Manz; 1985).
- If $\Delta(G)$ is disconnected, then each connected component induces a complete subgraph (Pálfy; 1998).
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The character degree graph for solvable groups

A significant property of $\Delta(G)$ in the disconnected case:

Theorem (Pálfy; 2001)

Let G be a solvable group such that $\Delta(G)$ is disconnected. If n_1 and n_2 are the sizes of the two connected components, then, assuming $n_1 \geq n_2$, we have $n_1 \geq 2^{n_2} - 1$.

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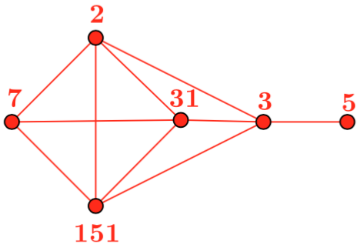

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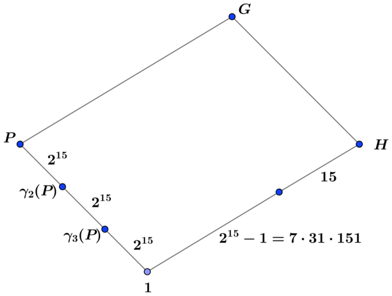

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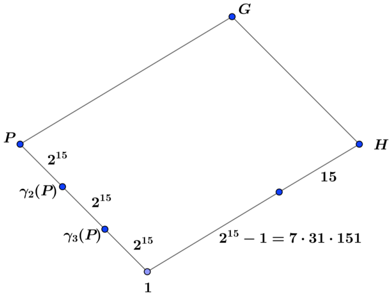

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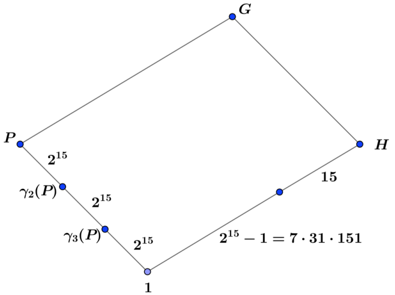

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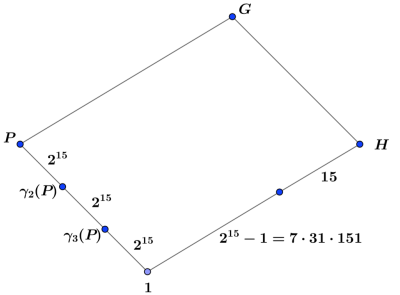

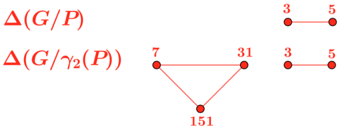

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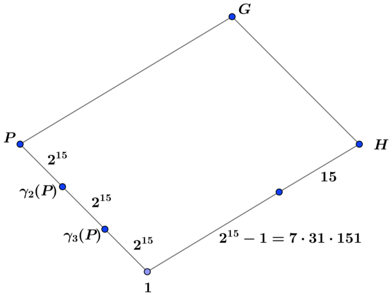

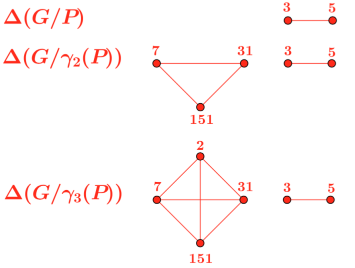

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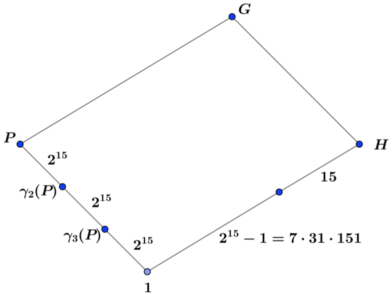
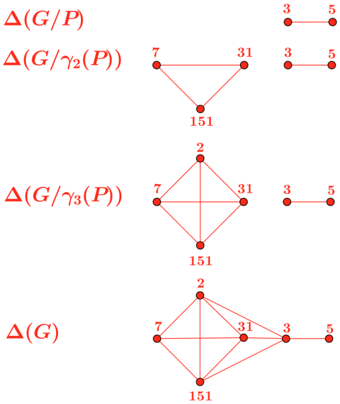
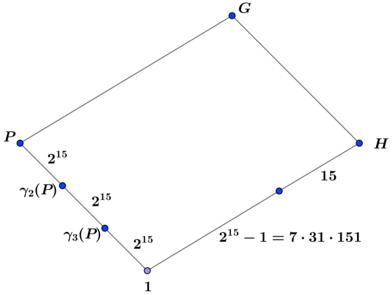




$$|G| = 2^{45} \cdot (2^{15} - 1) \cdot 15$$

$$cd(G) = \{1,\ 3,\ 5,\ 3 \cdot 5,\ 7 \cdot 31 \cdot 151,\ 2^{12} \cdot 31 \cdot 151,$$
$$2^a \cdot 7 \cdot 31 \cdot 151 \quad (a \in 7, 12, 13),$$
$$2^b \cdot 3 \cdot 31 \cdot 151 \quad (b \in 12, 15)\}$$

Some questions by Lewis



(a) Is this example "minimal"?

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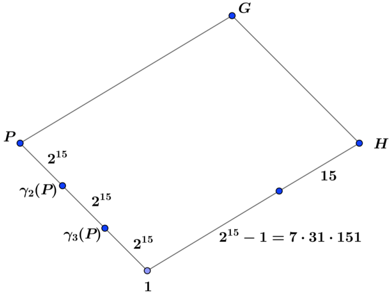

$$|G| = 2^{45} \cdot (2^{15} - 1) \cdot 15$$

(a) Is this example "minimal"?

(b) Let G be a solvable group such that $\Delta(G)$ is connected with diameter three. What can we say about the structure of G? For instance, what about $h(G)$?

Some questions by Lewis

(c) For G as above, is it true that there exists a normal subgroup N of G with

$$\mathrm{Vert}(\Delta(G/N)) = \mathrm{Vert}(\Delta(G))$$

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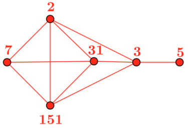

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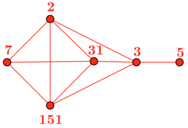
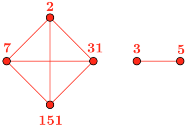


$\Delta(G/\gamma_3(P))$



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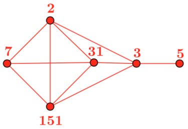
$$\mathrm{Vert}(\Delta(G/N)) = \mathrm{Vert}(\Delta(G))$$

and with $\Delta(G/N)$ disconnected?
Can $\mathrm{Vert}(\Delta(G))$ be partitioned into two subsets π_1 and π_2, both inducing complete subgraphs of $\Delta(G)$, such that $|\pi_1| \geq 2^{|\pi_2|}$?

(In Lewis' example:)

$\Delta(G)$



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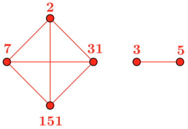
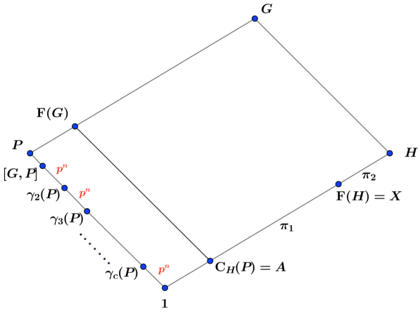


If $\Delta(G)$ is connected with diameter three then... [Casolo, D., Pacifici, Sanus; Sass (2016)]



(a) There exists a prime p such that $G = PH$, with P a normal nonabelian Sylow p-subgroup of G and H a p-complement.

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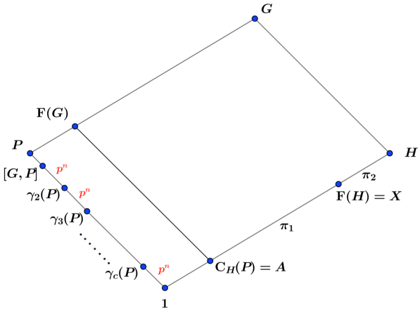

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(b) $\mathbf{F}(G) = P \times A$, where $A = \mathbf{C}_H(P) \leq \mathbf{Z}(G)$, H/A is not nilpotent and has cyclic Sylow subgroups.

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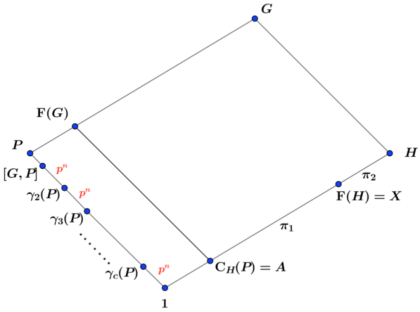

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(c)

$$h(G) = 3$$

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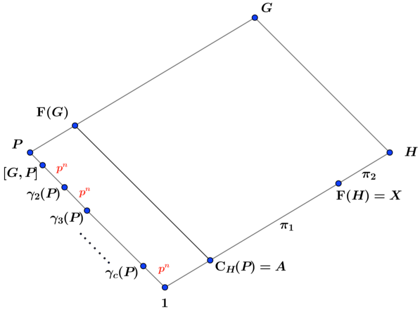

(c) $M_1 = [P, G]/P'$ and $M_i = \gamma_i(P)/\gamma_{i+1}(P)$, for $2 \leq i \leq c$ (where c is the nilpotency class of P) are chief factors of G of the same order p^n, with n divisible by at least two odd primes. $G/\mathbf{C}_G(M_j)$ embeds in $\Gamma(p^n)$ as an irreducible subgroup.

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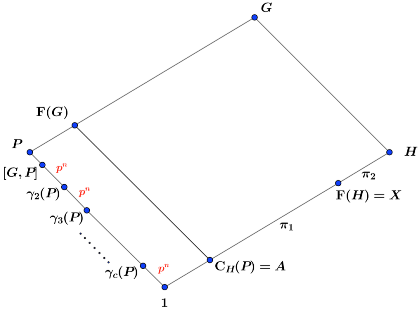

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$$\Gamma(p^n) = \{x \mapsto ax^\sigma \mid a, x \in \mathbb{K}, a \neq 0, \sigma \in \mathrm{Gal}(\mathbb{K})\} \text{ with } \mathbb{K} = \mathrm{GF}(p^n)$$

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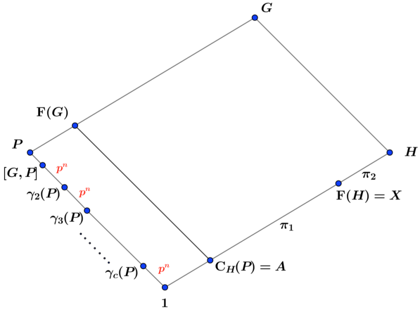
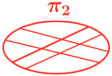


$\Delta(G/P)$



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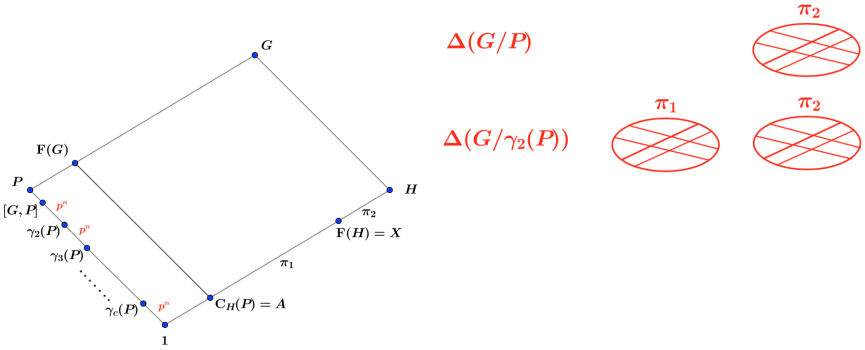

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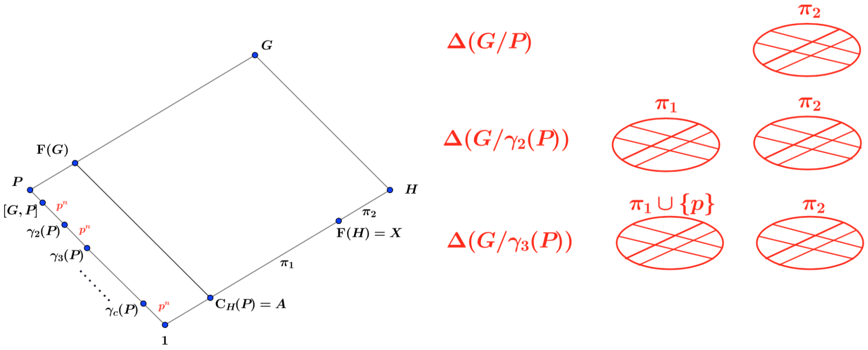

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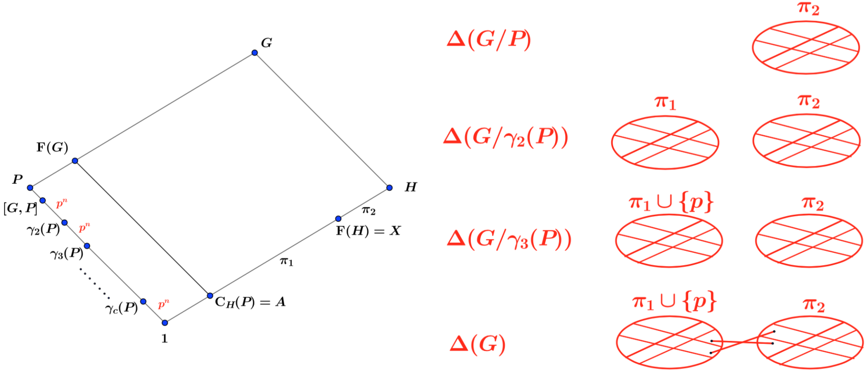

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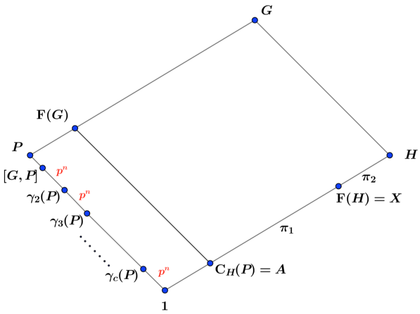

Finally, setting $d = |H/X|$, we have that $|X/A|$ is divisible by $(p^n - 1)/(p^{n/d} - 1)$. Since c must be at least 3, we get

$$|G| \geq p^{3n} \cdot \frac{p^n - 1}{p^{n/d} - 1} \cdot d \geq$$

$$\geq 2^{45} \cdot (2^{15} - 1) \cdot 15.$$

The complement graph

Let Δ be a graph. The *complement* of Δ is the graph $\overline{\Delta}$ whose vertices are those of Δ, and two vertices are adjacent in $\overline{\Delta}$ if and only if they are non-adjacent in Δ.

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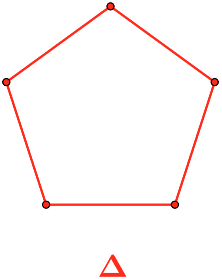

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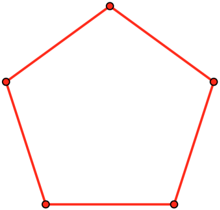
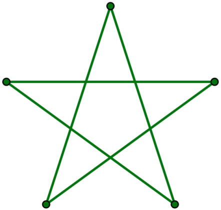


Δ



$\overline{\Delta}$

An extension of Pálfy's Theorem

Pálfy's "Three Vertex Theorem" can be rephrased as follows:

Theorem (Pálfy; 1998)

Let G be a solvable group. Then the graph $\overline{\Delta(G)}$ does not contain any cycle of length 3.

Recently, we obtained the following extension:

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But the graphs containing no cycles of odd length are precisely the bipartite graphs. Therefore the previous theorem asserts that, for any solvable group G, the graph $\overline{\Delta(G)}$ is bipartite. As an immediate consequence:

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Another remark:

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Let G be a solvable group. If n is the maximum size of a complete subgraph of $\Delta(G)$, then $\Delta(G)$ has at most $2n$ vertices.

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