More on p-groups of small breadth

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Nov 5-14, 2016

Summary

1. Breadth, isoclinism and central products
   - definitions
   - interplay
   - class-counting polynomials
   - bounds for the commutativity
2. Commutativity for small breadth
   - breadth 1
   - breadth 2
   - breadth 3

Breadth of a p-group

Let G be a finite p-group and $x \in G$.

$$\mathrm{b}(x) = \log_p \left| x^G \right| ,$$
$$\mathrm{b}(G) = \max_{x \in G} \{\mathrm{b}(x)\}.$$

$k(G)$: number of conjugacy classes of G

$k_i(G)$: number of conjugacy classes of G of length p^i (classes of elements of breadth i).

Isoclinism

We recall that (α, β) is an isoclinism between two groups G and H if $\alpha\colon G/Z(G) \to H/Z(H)$ and $\beta\colon G' \to H'$ are isomorphisms making the following diagram commutative

$$\begin{array}{ccc} G/Z(G) \times G/Z(G) & \xrightarrow{\alpha\times\alpha} & H/Z(H) \times H/Z(H) \\ {\scriptstyle [\cdot,\cdot]}\downarrow & & \downarrow{\scriptstyle [\cdot,\cdot]} \\ G' & \xrightarrow[\beta]{} & H' \end{array}$$

Isoclinism is an equivalence relation among isomorphism types of finite groups.
Groups S of minimal order in an isoclinism class are called *stem groups* and are characterized by the property:

$$Z(S) \leq S'$$

(P. Hall 1940).

Central products

A group G is the *central product* of its subgroups H and K if $G =< H, K >$ and H and K centralize each other.

Equivalently, given the groups H and K, and an isomorphism ϕ of a subgroup N of $Z(H)$ onto a subgroup of $Z(K)$, the central product of H and K with respect to ϕ will be

$$H \times_\phi K = \frac{H \times K}{< (u, \phi(u)^{-1}) | u \in N >}.$$

As an example, every extraspecial p-group is the (iterated) central product of non abelian p-groups of order p^3.

Breadth and isoclinism

Note that $C_G(g)$, $[g, G]$ and $\mathrm{b}(g)$ do not depend on the choice of g in a given coset $gZ(G)$, so that we can define

$$\mathrm{b}_G(gZ(G)) := \mathrm{b}(g).$$

If (α, β) is an isoclinism between G and H as above and $\alpha(gZ(G)) = hZ(H)$ then clearly

$$\alpha(C_G(g)/Z(G)) = C_H(h)/Z(H),$$
$$[h, H] = \beta[g, G],$$
$$\mathrm{b}_G(gZ(G)) = \mathrm{b}_H(hZ(H)).$$

In particular $\mathrm{b}(g) = \mathrm{b}(h)$.

P. Hall's isoclinic invariants

As a consequence we have

$$\begin{aligned} p^i\, k_i(G) &= \left|\{x \in G \mid b(x) = i\}\right| \\ &= |Z(G)| \cdot \left|\{gZ(G) \mid b_G(gZ(G)) = i\}\right| \end{aligned}$$

so that

$$\frac{k_i(G)}{|G|} = \frac{\left|\{gZ(G) \mid b_G(gZ(G)) = i\}\right|}{p^i \cdot |G : Z(G)|}$$

and (by adding up over i) the *commutativity* of G

$$\frac{k(G)}{|G|}$$

are invariant under isoclinism (P. Hall 1940).

Commutativity

The commutativity has been extensively studied by many authors, in very general settings. The "not Burnside's" lemma shows:

$$k(G) = \frac{1}{|G|}\sum_{g\in G}|C_G(g)| = \frac{1}{|G|}|\{(g,x)\in G\times G|gx=xg\}|$$

Then the commutativity

$$\frac{k(G)}{|G|} = \frac{|\{(g,x)\in G\times G|gx=xg\}|}{|G\times G|}$$

can be thought of as the probability that two randomly chosen elements of G commute.

Polynomials

The *class-counting polynomial* of G is defined as

$$k_G(x) = \sum_{i=0}^{b(G)} \frac{k_i(G)}{|G|} x^i.$$

We have easily

$$k_G(1) = \frac{k(G)}{|G|}, \tag{1}$$

$$k_G(0) = \frac{|Z(G)|}{|G|}, \tag{2}$$

$$k_G(p) = 1. \tag{3}$$

Let N be a minimal normal subgroup of G, $x \in G$.
Bar-notation for $\bar{G} = G/N$.
Say x^G is a *collapsing* conjugacy class if $|x^G| > |\bar{x}^{\bar{G}}|$. In other words x^G consists of a union of cosets of N.
Otherwise x^G is an *invariant* class.

$k_i^+(G, N)$: number of invariant G-conjugacy classes of breadth i
$k_i^-(G, N)$: number of collapsing G-conjugacy classes of breadth i.

Clearly we have that

$$k_i(\bar{G}) = \frac{k_i^+(G,N)}{p} + k_{i+1}^-(G,N) \tag{4}$$

Dividing by the order of $\bar{G}$ we obtain

$$\frac{k_i(\bar{G})}{|\bar{G}|} = \frac{k_i^+(G,N)}{|G|} + p\frac{k_{i+1}^-(G,N)}{|G|} \tag{5}$$

and adding up over i we easily get

$$\frac{k(\bar{G})}{|\bar{G}|} = p\frac{k(G)}{|G|} + (1-p)\frac{k^+(G,N)}{|G|} \tag{6}$$

This could be seen as a dual of the Burnside formula

$$\frac{k(M)}{|M|} = p^2\frac{k(G)}{|G|} + (1-p^2)\frac{s}{|M|} \tag{7}$$

which is more commonly written as

$$k(G) = sp + \frac{k(M)-s}{p} \tag{8}$$

relating $k(G)$ with $k(M)$ and the number s of G-invariant M-conjugacy classes of a maximal subgroup M.

More polynomials

Let again N be a minimal normal subgroup of G.
Define the collapsing and the invariant class-counting polynomials as follows:

$$k^{-}_{G,N}(x) = \sum_{i=0}^{b(G)} \frac{k_i^{-}(G,N)}{|G|} x^i = \sum_{i=1}^{b(G)} \frac{k_i^{-}(G,N)}{|G|} x^i$$

and

$$k^{+}_{G,N}(x) = \sum_{i=0}^{b(G)} \frac{k_i^{+}(G,N)}{|G|} x^i$$

All these polynomials are clearly invariant under isoclinism.

Polynomials and central products

Here are some of their easy properties:

$$k_{G/N}(x) = k^{+}_{G,N}(x) + \frac{p}{x} k^{-}_{G,N}(x), \tag{9}$$

$$k_{G \times H}(x) = k_G(x) k_H(x), \tag{10}$$

$$\begin{aligned} k_{G \times_\phi H}(x) = {} & k^{+}_{G,N}(x) k^{+}_{H,N}(x) + k^{+}_{G,N}(x) k^{-}_{H,N}(x) \\ & + k^{-}_{G,N}(x) k^{+}_{H,N}(x) + \frac{p}{x} k^{-}_{G,N}(x) k^{-}_{H,N}(x). \end{aligned} \tag{11}$$

In principle these formulae allow us to focus on stem p-groups which do not properly decompose as a central product, in our search for the values of isoclinic invariants.

Breadth and commutativity

In general if $p^n = |G|$ and $b = \mathrm{b}(G)$ then we have (N. Gavioli, A. Mann, V.Monti, A. Previtali, CMS 1998)

$$p^{n-b} + b(p-1) \leq k(G)$$

so that

$$\frac{1}{p^b} + b\frac{p-1}{|G|} \leq \frac{k(G)}{|G|}$$

In general, for G of breadth $b(G) > 0$, we have
$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \geq \frac{|G|}{|G'|} + \left(k(G) - \frac{|G|}{|G'|}\right) p^2$, so that

$$\frac{k(G)}{|G|} \leq \frac{1}{p^2} + \frac{p^2 - 1}{p^2|G'|} \leq \frac{1}{p^2} + \frac{1}{p^b} - \frac{1}{p^{b+2}}$$

where the inequality on the left is an equality iff the character degrees of G are 1 and p.

Well known (Isaacs' book):

a non-abelian p-group G has only character degrees equal to 1 and p iff it has a maximal subgroup A of index p which is abelian or $|G : Z(G)| \leq p^3$.

In the former case if $x \notin A$ then $\mathrm{b}(x) = b = \mathrm{b}(G)$ and $G' = [A, x]$ has order p^b (P. Hall 1940). Clearly $Z(G) = C_A(x)$, so that $|G : Z(G)| = p^{b+1}$.

$$\frac{k(G)}{|G|} = \frac{1}{p^2} + \frac{1}{p^b} - \frac{1}{p^{b+2}}$$

General bounds

For a p-group of breadth b we have "invariant" bounds:

$$\frac{1}{p^b} < \frac{k(G)}{|G|} \leq \frac{1}{p^2} + \frac{1}{p^b} - \frac{1}{p^{b+2}}$$

Actually the lower bound is an $\inf$ over all possible values of $|G|$, and the upper bound is attained when G has an abelian subgroup of index p independently of the value of $|G|$.

Not all the values between the two bounds are actually taken by $k(G)/|G|$, as we proved twenty years ago for groups of breadth up to 3. Can we do better in pinpointing the values that are actually taken?

Commutativity and breadth

As an example, the commutativity of any p-group of breadth b that has a maximal abelian subgroup has just been determined.

We can compute the commutativity and the class-counting polynomials of stem p-groups with no proper central factors, in order to use our techniques to compute the values of the same invariants for other p-groups.

Still far from real possibility to compute...

The case $\mathrm{b}(G) = 1$

Known: $|G'| = p$ (H. Knoche 1951, 1953).
Assume $Z(G) = G'$ (G stem).
In this case G is extraspecial of order p^n, where $n = 2m + 1$,
and G is the central product of m factors of order p^3.
We obtain

$$\frac{k(G)}{|G|} = \frac{1}{p} + \frac{1}{p^{2m}} - \frac{1}{p^{2m+1}}$$

(maximum value for $m = 1$)

$\mathrm{b}(G) = 2$

(G. Parmeggiani, B. Stellmacher 1999; also GMMPS 1998)

Theorem

$\mathrm{b}(G) = 2$ iff one of the following holds

1. $|G'| = p^3$ and $|G : Z(G)| = p^3$
2. $|G'| = p^2$

$\mathrm{b}(G) = 2$, $|G : Z(G)| = p^3$, $|G'| = p^3$

Well known (Isaacs' book):

a non-abelian p-group G has only character degrees equal to 1 and p iff it has a maximal subgroup A of index p which is abelian or $|G : Z(G)| \leq p^3$.

We have then
$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = \frac{|G|}{|G'|} + \left(k(G) - \frac{|G|}{|G'|}\right) p^2$, so that

$$\frac{k(G)}{|G|} = \frac{1}{p^2} + \frac{p^2 - 1}{p^2 |G'|} = \frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5}$$

We can also show that G does not have any elements of breadth 1.

$\mathrm{b}(G) = 2$, $|G'| = p^2$

We assume $Z(G) \leq G'$ and $|G| = p^n$.

1. G has class 2 and $Z(G) = G'$. We have infinitely many possibilities. Lower bound occurs when all the elements outside $Z(G)$ have breadth 2, whereas the upper bound is attained when G has an abelian maximal subgroup.

$$\frac{1}{p^2} + \frac{1}{p^{n-2}} - \frac{1}{p^n} \leq \frac{k(G)}{|G|} \leq \frac{2}{p^2} - \frac{1}{p^4}$$

The general pattern is

$$\frac{k(G)}{|G|} = \frac{1}{p^2} + (p-1) \sum_{i=1}^{p+1} \frac{1}{p^{n+2-k_i}}$$

where $2 \leq k_i \leq n-2$ for $i = 1, \ldots, p+1$.

$\mathrm{b}(G) = 2$, $|G'| = p^2$

We assume $Z(G) \leq G'$ and $|G| = p^n$.

1. G has class 2 and $Z(G) = G'$ ...
2. G has class 3 and $|Z(G)| = p$ and $|G'| = p^2$.
   In this case we could bound $|G|$, assuming that G does not have any proper central factor. We could explicitly compute the possible commutativity values:

$$\frac{k(G)}{|G|} = \frac{2}{p^2} - \frac{1}{p^4}$$

$$\frac{k(G)}{|G|} = \frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5}$$

When the index of the center is bounded

Proposition

If G is a stem group and $|G/Z(G)| \leq p^k$ then $|G| \leq p^{\binom{k+1}{2}}$

Proof.

We have that $\mathrm{b}(G) \leq k-1$ so that a well known result (Vaughan-Lee 1974) yields $|G'| \leq p^{\binom{k}{2}}$. Since G is stem, we have $Z(G) \leq G'$ so that the index of G' is at most p^k. Then trivially $|G| \leq p^{\binom{k}{2}} p^k = p^{\binom{k+1}{2}}$. □

It follows that if the index of the center is bounded then there are only finitely many values for $k(G)/|G|$. Note that equality can hold in the statement, e.g. for $G = F/\gamma_3(F)F^p$ where F is the free group on k generators.

breadth 3

[GMMPS] and [PS]

Theorem

Let p be odd and let G be a finite p-group. Then G has breadth 3 if and only if one of the following holds:

1. $|G'| = p^3$ and $|G : Z(G)| \geq p^4$,
2. $|G'| \geq p^4$ and $|G : Z(G)| = p^4$,
3. $|G'| = p^4$ and G has a quotient $\bar{G} = G/D$ with respect to a central subgroup D of order p such that $|\bar{G} : Z(\bar{G})| = p^3$.

Assuming again that $|G|$ has no proper central factors, can bound $|G|$ in cases 2 and 3, by the previous Proposition, a finite list of values for the commutativity.
In case 1 we have the same difficulties already encountered in case 1 for groups such that $|G'| = p^2$. We can write some patterns and give bounds, though.