On sums of element orders in finite groups

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Basic Problem

Let G be a periodic group.

Main Problem

Obtain information about the structure of G

by looking at the orders of its elements.

The function ω

Let G be a periodic group.

Definition

$$\omega(G) := \{\ o(x) \mid x \in G\ \}.$$

Problem

What can be said about the structure of G by looking at the set $\omega(G)$?

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Remark

$\omega(G) = \{1, 2\}$ if and only if G is an elementary abelian 2-group.

F. Levi, B.L. van der Waerden, 1932

If $\omega(G) = \{1, 3\}$, then G is nilpotent of class $\leqslant 3$.

B.H. Neumann, 1937

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Does $\omega(G)$ finite imply G locally finite?

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I. N. Sanov, 1940

If $\omega(G) \subseteq \{1, 2, 3, 4\}$, then G is locally finite.

D.V. Lytkina, 2007

If $\omega(G) = \{1, 2, 3, 4\}$, then either

- G is an extension of an (elementary abelian 3-group) by (a cyclic or a quaternion group), or
- G is an extension of a (nilpotent of class 2 2-group) by (a subgroup of $\mathcal{S}_3$).

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If $\omega(G) \subseteq \{1, 2, 3, 4, 5\}, \omega(G) \neq \{1, 5\}$, then G is locally finite.

E. Jabara, D. V. Lytkina, V. D. Mazurov, A. S. Mamontov, 2014

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Does $\omega(G) = \{1, 5\}$ imply G locally finite?

Still open!

M. C. Xu, W.J. Shi , 2003
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If G is a finite simple group, G_1 a finite group,

$$|G| = |G_1| \text{ and } \omega(G) = \omega(G_1),$$

then $G \simeq G_1$.

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The sets L_e

Let G be a finite group and let e be a divisor of the order of G.

Definition

$$L_e(G) := \{x \in G \mid x^e = 1\}.$$

Problem

Obtain information about the structure of G by looking at the orders of the sets $L_e(G)$.

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$|L_e(G)|$ divides $|G|$, for every e dividing $|G|$.

Remark

$|L_e(G)| = e$, for every e dividing $|G|$, if and only if G is cyclic.

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$$|L_e(G)| \leq 2e, \text{ for every } e \text{ dividing } |G|.$$

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Problem (J.G. Thompson)

Let G be a finite soluble group, G_1 a finite group.

Assume $|L_e(G)| = |L_e(G_1)|$, for any e dividing $|G|$.

Is it true that G_1 is soluble?

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What can be said about the structure of G

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The function ψ - some examples

Examples

$$\psi(\mathcal{S}_3) = 13.$$

For, $\psi(\mathcal{S}_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3$.

$$\psi(\mathcal{C}_6) = 21.$$

For, $\psi(\mathcal{C}_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6$.

$$\psi(\mathcal{C}_5) = 21.$$

For, $\psi(\mathcal{C}_5) = 1 \cdot 1 + 4 \cdot 5$.

where $\mathcal{C}_n$ is the cyclic group of order n and $\mathcal{S}_3$ is the symmetric group of degree 3.

The function ψ - some remarks

Remark

$\psi(G) = \psi(G_1)$ does not imply $G \simeq G_1$.

Example

Let $A = C_8 \times C_2$,
$B = C_2 \ltimes C_8$, where $C_2 = \langle a \rangle$, $C_8 = \langle b \rangle$, $b^a = b^5$.
Then
$\psi(A) = \psi(B) = 87$.

Remark

$|G| = |G_1|$ and $\psi(G) = \psi(G_1)$ do not imply $G \simeq G_1$.

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$\psi(G) = \psi(\mathcal{S}_3)$ implies $G \simeq \mathcal{S}_3$.

Problem

Find information about the structure of a finite group G from some inequalities on $\psi(G)$.

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The function ψ - a property

Proposition

If $G = G_1 \times G_2$, where $|G_1|$ and $|G_2|$ are coprime, then

$$\psi(G) = \psi(G_1)\psi(G_2).$$

Sum of the orders of the elements in a cyclic group

Remark

$$\psi(\mathcal{C}_n) = \sum_{d|n} d\varphi(d),$$

where φ is the Euler's function.

Proposition

Let p be a prime, $\alpha \geq 0$. Then:

$$\psi(\mathcal{C}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}.$$

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Proof. $\psi(C_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

$1 + p(p-1) + p^2(p^2-p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$

$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1} = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

Corollary

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_i's$ different primes, α_i's > 0. Then

$$\psi(C_n) = \prod_{i \in \{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

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Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, p_i's different primes, α_i's > 0, and let p be the largest prime divisor of n. Then

$$\psi(\mathcal{C}_n) = \prod_{i\in\{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1} > \frac{2}{p+1}n^2.$$

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Theorem [H. Amiri, S.M. Jafarian Amiri, M. Isaacs, Comm. Algebra 2009]

Let G be a finite group, $|G| = n$. Then

$$\psi(G) \leq \psi(C_n).$$

Moreover

$$\psi(G) = \psi(C_n) \text{ if and only if } G \simeq C_n.$$

Let G be a finite non-cyclic group, $|G| = n$. Then

$$\psi(G) < \psi(C_n).$$

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Theorem [M. Herzog, P. L., M. Maj]

Let G be a non-cyclic finite group, $|G| = n$.

Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Remark

This upper bound is the best possible:
for each $n = 4k$, k odd,
there exists a group G of order n satisfying
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Let k be an odd integer and let $n = 4k$. Then $\psi(\mathcal{C}_{2k} \times \mathcal{C}_2) = \frac{7}{11}\psi(\mathcal{C}_n)$.

Proof. We have

$$\psi(\mathcal{C}_n) = \psi(\mathcal{C}_{4k}) = \psi(\mathcal{C}_4)\psi(\mathcal{C}_k) = \frac{32+1}{2+1}\psi(\mathcal{C}_k) = 11\psi(\mathcal{C}_k).$$

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But for instance $\psi(\mathcal{S}_3) = 13 = \frac{13}{21}21 = \frac{13}{21}\psi(\mathcal{C}_6) < \frac{7}{11}\psi(\mathcal{C}_6)$.

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Theorem [M. Herzog, P. L., M. Maj]

Let G be a non-cyclic finite group, $|G| = n$ and let q be the smallest prime divisor of n.

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Corollary

Let G be a non-cyclic finite group of odd order n. Then
$$\psi(G) < \frac{1}{2}\psi(C_n).$$

Remark

In the Theorem it is not possible to substitute $q-1$ by q. For:
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Let G be a non-cyclic finite group of odd order n. Then $\psi(G) < \frac{1}{2}\psi(C_n)$.

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Problem

What can be said if $\psi(G) \geq \frac{1}{q}\psi(C_n)$?

Theorem [M. Herzog, P. L., M. Maj]

Let G be a finite group, $|G| = n$ and
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If $\psi(G) \geq \frac{1}{2(q-1)}\psi(C_n)$,

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Let G be a finite group, $|G| = n$.

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Remark

Notice that

$$\psi(C_2 \times C_2 \times C_2) = 15 < \frac{3}{5} \cdot 8 \cdot 4 = \frac{3}{5} n\varphi(n)$$

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Theorem [Ramanujan, 1913-1914]

Let $q_1, q_2, \ldots, q_s, \ldots$ be the sequence of all primes:

$$q_1 < q_2 < \cdots < q_s < \cdots.$$

Then

$$\prod_{i=1}^{\infty} \frac{q_i^2+1}{q_i^2-1} = \frac{5}{2}.$$

Lemma

Let $p_2, p_3, \ldots, p_s$ be primes satisfying $p_2 < p_3 < \cdots < p_s$.

If $p_2 > 3$ then

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Lemma 1 [H. Amiri, S.M. Jafarian Amiri, M. Isaacs, 2009]

Let G be a finite group, p a prime,
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Let G be a non-cyclic finite group, $|G| = n$ and let q be the smallest prime divisor of n. Then

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Assume $|G| = n$ and $\psi(G) \geq \frac{1}{q-1}\psi(C_n)$. We show that G is cyclic. Obviously

$$\psi(C_n) > n\varphi(n)$$

and, by the previous Lemma,

$$\varphi(n) \geq \frac{q-1}{p}n,$$

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Hence

$$\psi(G) > \frac{n^2}{p}$$

which implies that there exists $x \in G$ with $o(x) > n/p$.
Thus $[G : \langle x \rangle] < p$ and $\langle x \rangle$ contains a Sylow p-subgroup P of G. Since $\langle x \rangle \leq N_G(P)$, it follows that P is a cyclic normal subgroup of G and Lemma 1 implies that

$$\psi(P)\psi(G/P) \geq \psi(G) \geq \frac{1}{q-1}\psi(C_{p^r})\psi(C_{n/p^r}),$$

where $p^r = |P|$. Since $P \cong C_{p^r}$, cancellation yields

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A proof sketch - continued

If $n = p^r$, then from $o(x) > n/p$ it follows $o(x) = n$ and G is cyclic, as required.

So we may assume that n is divisible by exactly k different primes with $k > 1$. Apply induction with respect to k.

Then G/P is cyclic and $G = P \rtimes F$, with $F \cong G/P$ and $F \neq 1$.

Notice that $n = |P||F|$, P and F are both cyclic and $(|P|, |F|) = 1$.

Hence $\psi(C_n) = \psi(P)\psi(F)$.

If $C_F(P) = F$, then $G = P \times F$ and G is cyclic, as required.

So it suffices to prove that if $C_F(P) =: Z < F$, then $\psi(G) < \frac{1}{q-1}\psi(C_n)$, contrary to our assumptions.

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$$\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z) < \psi(P)\psi(Z) + |P|\psi(F).$$

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Since $\psi(C_n) > n\varphi(n)$ and by Lemma 2 $\varphi(n) \geq \frac{(q-1)n}{p}$, it follows by our assumptions that $\psi(G) > \frac{n^2}{2p}$. Hence there exists $x \in G$ such that $o(x) > \frac{n}{2p}$ and

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Since $\psi(C_n) > n\varphi(n)$ and by Lemma 2 $\varphi(n) \geq \frac{(q-1)n}{p}$, it follows by our assumptions that $\psi(G) > \frac{n^2}{2p}$. Hence there exists $x \in G$ such that $o(x) > \frac{n}{2p}$ and

$$[G : \langle x \rangle] < 2p.$$

A proof sketch - another lemma

Lemma 3

Let G be a finite group and suppose that there exists $x \in G$ such that

$$|G : \langle x \rangle| < 2p,$$

where p is the maximal prime divisor of $|G|$. Then one of the following holds:

- G has a normal cyclic Sylow p-subgroup,
- $\langle x \rangle$ is a maximal subgroup of G, and G is soluble.

Sum of the orders of the elements

Definition

Let n be a positive integer. Put

$$\mathcal{T}_n := \{\psi(H) \mid |H| = n\}$$

Recall that

$\psi(C_n)$ is the **maximum** of $\mathcal{T}_n$.

Problem

What is the structure of G if $\psi(G)$ is the **minimum** *of $\mathcal{T}_n$?*

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If $n = p^\alpha$ for some prime p and some $\alpha > 0$ and $|G| = p^\alpha$,
then obviously
$\psi(G)$ is minimum if and only if $expG = p$.

If $p = 2$ and $\psi(G)$ is minimum ,
then G is the elementary abelian group of order 2^α.

But there are non-isomorphic groups G and G_1 of order $p^\alpha > p^2$ ($p > 2$)
with $\psi(G) = \psi(G_1)$ minimum.

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What happens in the general case?

What happens for non-abelian simple groups?

Theorem [H. Amiri, S.M. Jafarian Amiri, 2011]

Let G be a finite nilpotent group.

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NO!

There are non-isomorphic simple groups S and S_1 such that
$|S| = |S_1|$ and $\psi(S) \neq \psi(S_1)$.

For instance, the groups $\mathcal{A}_8$ and $\mathcal{PSL}(3,4)$ are such that
$|\mathcal{A}_8| = 20160 = |\mathcal{PSL}(3,4)|$ and
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Conjecture [H. Amiri, S.M. Jafarian Amiri, 2011]

Let G be a finite non-simple group, S a finite simple group, $|G| = |S|$.

Then

$$\psi(S) < \psi(G).$$

Theorem [S.M. Jafarian Amiri, 2013]

Let G be a finite non-simple group.

If $|G| = 60$, then $\psi(\mathcal{A}_5) < \psi(G)$.

If $|G| = 168$, then $\psi(\mathcal{PSL}(2,7)) < \psi(G)$.

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Sum of the orders of the elements - minimum

Assume G is a finite non-simple group.

Using **GAP** it is possible to see that:

If $|G| = 360$, then $\psi(\mathcal{A}_6) < \psi(G)$.

If $|G| = 504$, then $\psi(\mathcal{PSL}(2,8)) < \psi(G)$.

If $|G| = 660$, then $\psi(\mathcal{PSL}(2,11)) < \psi(G)$.

If $|G| = 1092$, then $\psi(\mathcal{PSL}(2,13)) < \psi(G)$.

Sum of the orders of the elements - minimum

But the conjecture is **not** true.

Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]

Let $S = \mathcal{PSL}(2,64)$ and $G = 3^2 \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) \leq \psi(S)$.

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Let G be a finite soluble group, S a simple group, $|G| = |S|$. Then

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Some other functions

Let G be a finite group.

Definition

$$\mathcal{P}(G) := \prod_{x \in G} o(x).$$

Theorem [M. Garonzi, M. Patassini, 2015]

Let G be a finite group, $|G| = n$. Then

$$\mathcal{P}(G) \leq \mathcal{P}(C_n).$$

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Let G be a finite group, r, s real numbers.

Definition

$$\mathcal{R}_G(r,s) := \sum_{x\in G} \frac{o(x)^s}{\varphi(o(x))^r}.$$

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Remark

$$\mathcal{R}_G(0,1) = \psi(G).$$

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Theorem [M. Garonzi, M. Patassini, 2015]

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Then $\mathcal{R}_G(r) \geq \mathcal{R}_{\mathcal{C}_n}(r)$.

Moreover

$\mathcal{R}_G(r) = \mathcal{R}_{\mathcal{C}_n}(r)$ if and only if G is nilpotent.

Problem

Let G be a finite group, $|G| = n$, and r, s real numbers.

Does $\mathcal{R}_G(s, r) = \mathcal{R}_{\mathcal{C}_n}(s, r)$ imply G soluble?

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Thank you for the attention !

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